

## Stable bound states of one-dimensional autosolitons in a bistable laser

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Differential equations describing the interaction of two weakly overlapping autosolitons in the transverse section of a wide-aperture laser with a saturable absorber are derived and analyzed. The existence of in-phase and out-of-phase stable bound autosoliton states is predicted analytically and confirmed numerically.

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### I. INTRODUCTION

Localized structures (LS's) of laser radiation that can appear in the transverse section of wide-aperture laser systems are of interest from both fundamental and practical standpoints. Existence of such stable structures was predicted theoretically and observed experimentally in different active and passive optical schemes [1–6]. Recently optical LS's have attracted much attention due to their potential applications as ‘‘bits’’ for all-optical parallel information storage and processing. In connection with these applications the problem of the LS's interaction and bound states (BS's) formation becomes very important. Unlike the solitons of the nonlinear Schrödinger equation, localized solutions (autosolitons) of the dissipative equations, such as the complex Ginzburg-Landau equation (CGLE), can form different types of stationary BS's [1,7–17]. They arise as a result of interference between exponentially decaying autosoliton tails characterized by variable phase. In Refs. [8–12] the interaction of LS's was studied within the framework of the quintic CGLE. It was found that both inphase and antiphase BS's are unstable, while the states with the phase difference  $\varphi_- = \varphi_2 - \varphi_1 = \pm \pi/2$  between the autosolitons can be either weakly unstable or weakly stable. Note, however, that those latter BS's are restless [12].

Here we study the interaction of weakly overlapping autosolitons in the one-dimensional (1D) transverse section of a wide-aperture laser with a saturable absorber that is characterized by a purely dissipative type of nonlinearity. Another type of weak autosoliton interaction for which autosolitons travel with large relative velocity and, therefore, interact only at a short time interval is described in Ref. [18]. The model under consideration can also be applied to describe pulse propagation in optical fiber with saturable gain and absorption. We consider a situation typical of laser systems when the diffusion coefficient  $d$  is small. We show that in this model BS's with the phase difference  $\varphi_- = 0, \pi$  can be stable provided the diffusion coefficient exceeds certain very low threshold. Numerical evidence of the existence of such a threshold is presented. The BS's with  $\varphi_- = \pm \pi/2$  are found to be unstable for the parameter values typical of laser equations. In the diffusionless limit  $d=0$  numerical simulations indicate that the only stable BS is that characterized by the phase difference  $\varphi_- = \pi$  and corresponding to minimal distance between the autosolitons.

### II. MODEL EQUATIONS

We consider a model of a wide-aperture class A laser with a saturable absorber [1]

$$\partial_t E = (i + d) \partial_{xx} E + E f(|E|^2), \quad (1)$$

$$f(|E|^2) = -1 + \frac{g_0}{1 + |E|^2} - \frac{a_0}{1 + \beta|E|^2}. \quad (2)$$

Here the electric field envelope  $E$ , the transverse coordinate  $x$ , and the time  $t$  are dimensionless variables. The normalized diffusion (spatial filtering) coefficient  $d \ll 1$  can appear as a result of relaxation of the atomic polarization [19],  $g_0$  and  $a_0$  are the normalized linear gain and absorption coefficients,  $\beta$  is the ratio of the saturation intensities in absorbing and amplifying media.

The model equation (1) is invariant with respect to translations in space  $x \rightarrow x + c_1$  and phase shifts  $E \rightarrow E e^{ic_2}$  with arbitrary constant  $c_1$  and  $c_2$ . In the case when  $d=0$  it has an additional symmetry associated with Galilean transformation to a moving-coordinate frame. Due to this symmetry in the diffusionless limit  $d=0$  any stationary motionless localized (autosoliton) solution  $E(x, t) = A(x) e^{-iat}$  of Eq. (1) generates a whole family of uniformly moving autosoliton solutions. All solutions in the family have the same intensity profile but travel at different velocities  $v$ . They read as

$$E(x, t) = A(x - vt) e^{-iat + ivx/2 - iv^2 t/4}. \quad (3)$$

Here  $\alpha$  is the spectral parameter determining the frequency shift of the isolated motionless autosoliton. All autosoliton solutions belonging to the same family (3) have identical stability properties. Therefore, we can consider the motionless solution (3) with  $v=0$ . Substituting Eq. (3) into Eq. (1) for  $d=0$  we obtain ordinary differential equation for the autosoliton amplitude

$$i\alpha A(x) + i\partial_{xx} A(x) + A(x)f(|A(x)|^2) = 0,$$

with boundary conditions  $A(x) \rightarrow 0$  at  $x \rightarrow \pm \infty$ . We have calculated the amplitude  $A(x)$  numerically using the procedure described in Ref. [14].

Linear stability analysis of the autosoliton solution leads to eigenvalue problem  $\hat{L}\Psi = \lambda\Psi$  with linear operator

$$\hat{L}(A) = \begin{pmatrix} \hat{L}_{11} & \hat{L}_{12} \\ \hat{L}_{12}^* & \hat{L}_{11}^* \end{pmatrix}, \quad (4)$$

where

$$\begin{aligned} \hat{L}_{11}(A) &= i\alpha + i\partial_{xx} + f(|A(x)|^2) + |A(x)|^2 f'(|A(x)|^2), \\ \hat{L}_{12}(A) &= A^2 f'(|A(x)|^2). \end{aligned} \quad (5)$$

The stability properties of the isolated autosoliton solution (3) were analyzed in Ref. [14] by means of numerical calculation of the discrete spectrum of the operator  $\hat{L}$ . It was shown that a parameter domain exists in which this solution is stable. Further we study the autosoliton interaction in this parameter domain only.

The symmetry properties of Eq. (1) with  $d=0$  imply that the discrete spectrum of the operator  $\hat{L}$  contains a triply degenerate zero eigenvalue. This eigenvalue having geometrical multiplicity 2 corresponds to a pair of eigenvectors (neutral modes)  $\Psi_{1,2}$  and a root vector  $\Psi_3$ , which obey the relations

$$\hat{L}\Psi_{1,2} = 0, \quad \hat{L}\Psi_3 = \Psi_1. \quad (6)$$

Here  $\Psi_s = (\psi_s(x), \psi_s^*(x))^T$  with  $s=1,2,3$  and

$$\psi_1(x) = \partial_x A(x), \quad \psi_2(x) = iA(x), \quad \psi_3(x) = -\frac{ix}{2}A(x). \quad (7)$$

Since we consider the case when the amplitude of the isolated autosoliton solution is an even function of  $x$ ,  $A(-x) = A(x)$ , the functions  $\psi_s$  defined by Eq. (7) have the properties  $\psi_{1,3}(x) = -\psi_{1,3}(-x)$  and  $\psi_2(x) = \psi_2(-x)$ . Note, that the neutral modes  $\Psi_1$  and  $\Psi_2$  are associated with the invariance of Eq. (1) under translations in space and phase shifts respectively, while the root vector  $\Psi_3$  is related to the invariance with respect to Galilean transformation to a moving-coordinate frame.

In what follows along with the vectors  $\Psi_s$  we will use the vectors  $\Psi_s^\dagger = (\psi_s^\dagger(x), \psi_s^{\dagger*}(x))^T$  with  $s=1,2,3$ , which obey the relations  $\hat{L}^\dagger \Psi_{1,2}^\dagger = 0$  and  $\hat{L}^\dagger \Psi_3^\dagger = \Psi_1^\dagger$ . Here the adjoint operator  $\hat{L}^\dagger$  defined by the relation  $\langle \Phi, \hat{L}\Psi \rangle = \langle \hat{L}^\dagger \Phi, \Psi \rangle$  is obtained from Eq. (4) by transposition and complex conjugation and the scalar product is  $\langle \Phi, \Psi \rangle = 2 \int_{-\infty}^{\infty} \text{Re}[\phi^*(x)\psi(x)]dx$ . In particular, using the definition of  $\hat{L}^\dagger$  and Eqs. (6) we get  $\langle \Psi_1^\dagger, \Psi_1 \rangle = \langle \hat{L}^\dagger \Psi_3^\dagger, \Psi_1 \rangle = \langle \Psi_3^\dagger, \hat{L}\Psi_1 \rangle = 0$ .

Linear stability analysis of the trivial zero intensity solution of the ordinary differential equation yields the following asymptotic relations:

$$A(x) \sim a e^{\mp(\gamma-i\omega)x}, \quad \psi_s^\dagger(x) \sim (\pm 1)^s b_s e^{\mp(\gamma+i\omega)x}, \quad x \rightarrow \pm\infty. \quad (8)$$

Here the coefficients  $a$  and  $b_s$  can be calculated numerically. Since stable autosoliton solutions can appear only in the pa-

rameter domain where the zero-intensity steady state of Eq. (1) is stable, we have the inequality  $f(0) < 0$ . Using this inequality we obtain  $\gamma = \text{Im}\sqrt{\alpha - if(0)} > 0$  and  $\omega = \text{Re}\sqrt{\alpha - if(0)} > 0$ .

### III. INTERACTION OF AN AUTOSOLITON PAIR

In this section we construct equations governing autosoliton interaction using a perturbative approach similar to that in Ref. [20]. We consider weak interaction of two identical autosoliton solutions (3) each being stable when taken separately. It follows from Eq. (8) that if the autosoliton separation is large their overlap can be characterized by the quantity  $e^{-\gamma\zeta_-}$ , where  $\zeta_- = \zeta_2 - \zeta_1 > 0$  is the distance between autosoliton intensity maxima. Let the overlap be weak enough,  $e^{-\gamma\zeta_-} \sim \varepsilon^2$  ( $\varepsilon \ll 1$ ) and the diffusion coefficient be small,  $d = O(\varepsilon)$ . The latter condition assumes that the Galilean symmetry of Eq. (1) is only slightly broken. We write the solution of Eq. (1) describing the interacting autosoliton pair in the form

$$E(x,t) = e^{-i\alpha t} \left\{ \left[ \sum_{j=1}^2 u_j^{(0)}(x,t) \right] + u^{(1)}(x,t) + u^{(2)}(x,t) + O(\varepsilon^3) \right\}. \quad (9)$$

Here the functions  $u_j^{(0)}$  are the autosoliton solutions

$$u_j^{(0)}(x,t) = A_j e^{-i\varphi_j(t)}, \quad A_j = A[x - \zeta_j(t)], \quad (10)$$

with the coordinates  $\zeta_{1,2}$ , phases  $\varphi_{1,2}$ , and velocities  $\partial_t \zeta_{1,2}$  being slowly varying functions of time,  $\partial_t \zeta_j = O(\varepsilon)$ ,  $\partial_t \varphi_j, \partial_{tt} \zeta_j = O(\varepsilon^2)$ . The functions  $u^{(m)}(x,t)$  with  $m=1,2$  in Eq. (9) describe first- and second-order corrections to Eq. (10),  $u^{(m)} = O(\varepsilon^m)$ . Since the autosoliton velocities are assumed to be of order  $\varepsilon$ , we have  $\partial_t u^{(m)} = O(\varepsilon^{m+1})$ .

Let us introduce the notations  $\mathbf{u}^{(m)} = (u^{(m)}(x,t), u^{(m)*}(x,t))^T$  and  $\Psi_{sj} = (e^{-i\varphi_j} \psi_{sj}, e^{i\varphi_j} \psi_{sj}^*)^T$  with  $\psi_{sj} = \psi_s(x - \zeta_j)$  defined by Eq. (7). Then substituting Eq. (9) into Eq. (1) and collecting the first-order terms in  $\varepsilon$  we get the equation

$$\hat{L} \left( \sum_{j=1}^2 u_j^{(0)} \right) \mathbf{u}^{(1)} = - \sum_{j=1}^2 \Psi_{1j} \partial_t \zeta_j,$$

which has a solution

$$\mathbf{u}^{(1)} = - \sum_{j=1}^2 \Psi_{3j} \partial_t \zeta_j + O(\varepsilon^3).$$

Finally, equating the second-order terms in  $\varepsilon$  we obtain

$$\begin{aligned} \hat{L} \left( \sum_{j=1}^2 u_j^{(0)} \right) \mathbf{u}^{(2)} = & - \sum_{j=1}^2 [\Psi_{5j} (\partial_t \zeta_j)^2 + \Psi_{2j} \partial_t \varphi_j + \Psi_{3j} \partial_{tt} \zeta_j \\ & + d \Psi_{4j} \partial_t \zeta_j] - \mathbf{H}, \end{aligned} \quad (11)$$

where  $\Psi_{4j} = -\partial_{xx} \Psi_{3j}$  and  $\Psi_{5j} = \partial_x \Psi_{3j}$ .

Since the overlap between the autosolitons is weak, the vector  $\mathbf{H}=(h(x,t),h^*(x,t))^T$  in the right-hand side of Eq. (11) is small,  $h(x,t)=O(\varepsilon^2)$ . It is defined by

$$\begin{aligned} h(x,t) &= \left( \sum_{j=1}^2 A_j e^{-i\varphi_j} \right) f \left( \left| \sum_{j=1}^2 A_j e^{-i\varphi_j} \right|^2 \right) \\ &\quad - \sum_{j=1}^2 A_j e^{-i\varphi_j} f(|A_j|^2) + O(\varepsilon^3) \\ &= \sum_{j=1, j \neq k}^2 [A_j e^{-i\varphi_j} (f_{\Sigma} + |A_k|^2 f'_{\Sigma} - f_j) \\ &\quad + A_j^2 A_k e^{-i(2\varphi_j - \varphi_k)} f'_{\Sigma}] + O(\varepsilon^3), \end{aligned} \quad (12)$$

where  $f_{\Sigma}, f'_{\Sigma} = f, f'(|A_1|^2 + |A_2|^2)$  and  $f_j, f'_j = f, f'(|A_j|^2)$ .

The equations governing the slow-time evolution of the individual autosoliton parameters are obtained from the solvability conditions for Eq. (11). The right-hand side of Eq. (11) must be orthogonal to the solutions of the linear homogeneous equation  $\hat{L}^{\dagger}(\sum_{j=1}^2 u_j^{(0)})\Psi^{\dagger} = 0$ . These solutions can be estimated as  $\sum_{s,j=1}^2 C_{sj}\Psi_{sj}^{\dagger} + O(\varepsilon^2)$  with arbitrary constant  $C_{sj}$  and

$$\Psi_{sj}^{\dagger} = (e^{-i\varphi_j}\psi_{sj}^{\dagger}, e^{i\varphi_j}\psi_{sj}^{\dagger*})^T, \quad \psi_{sj}^{\dagger} = \psi_s^{\dagger}(x - \zeta_j) \quad (s, j = 1, 2). \quad (13)$$

Using the relations  $\langle \Psi_{1,2}^{\dagger}, \Psi_1 \rangle = 0$ , we obtain the following orthogonality conditions:

$$N_1 \partial_t v_{\pm} = -\langle \Psi_{12}^{\dagger} \pm \Psi_{11}^{\dagger}, \mathbf{H} \rangle - qv_{\pm}d, \quad (14)$$

$$N_2 \partial_t \varphi_{\pm} = -\langle \Psi_{22}^{\dagger} \pm \Psi_{21}^{\dagger}, \mathbf{H} \rangle + \frac{p}{4}[(v_+ + v_-)^2 \pm (v_+ - v_-)^2], \quad (15)$$

with  $\varphi_{\pm} = \varphi_2 \pm \varphi_1$ ,  $\zeta_{\pm} = \zeta_2 \pm \zeta_1$ , and  $v_{\pm} = \partial_t \zeta_{\pm}$ . The coefficients in Eqs. (14) and (15) are  $N_1 = \langle \Psi_3^{\dagger}, \Psi_1 \rangle = \langle \Psi_1^{\dagger}, \Psi_3 \rangle$ ,  $N_2 = \langle \Psi_2^{\dagger}, \Psi_2 \rangle$ ,  $p = \langle \Psi_2^{\dagger}, \Psi_5 \rangle$ , and  $q = \langle \Psi_1^{\dagger}, \Psi_4 \rangle$ . The scalar products  $\langle \Psi_{s2}^{\dagger} \pm \Psi_{s1}^{\dagger}, \mathbf{H} \rangle = 2 \operatorname{Re} \int_{-\infty}^{\infty} (\psi_{s2}^{\dagger*} h \pm \psi_{s1}^{\dagger*} h) dx$  from Eqs. (14) and (15) are evaluated in the Appendix. Using the expressions given in the Appendix we rewrite Eqs. (14) and (15) in the forms

$$\partial_{tt} \zeta_- + D \partial_t \zeta_- = r_1 e^{-\gamma \zeta_-} \cos \varphi_- \sin(\omega \zeta_- + \theta_1), \quad (16)$$

$$\partial_t \varphi_- = r_2 e^{-\gamma \zeta_-} \sin \varphi_- \cos(\omega \zeta_- + \theta_2) + P v_+ v_-, \quad (17)$$

$$\partial_t v_+ + D v_+ = r_1 e^{-\gamma \zeta_-} \sin \varphi_- \cos(\omega \zeta_- + \theta_1), \quad (18)$$

$$\partial_t \varphi_+ = r_2 e^{-\gamma \zeta_-} \cos \varphi_- \sin(\omega \zeta_- + \theta_2) + P(v_+^2 + v_-^2)/2, \quad (19)$$

with  $r_s = (-1)^s 4 N_s^{-1} |ab_s^*| \sqrt{\gamma^2 + \omega^2}$ ,  $\theta_s = \arg(ab_s^*) + \arg(\gamma - i\omega)$ ,  $D = qd/N_1$ , and  $P = p/N_2$ . The quantities  $r_{1,2}$ , and  $\theta_{1,2}$  can be

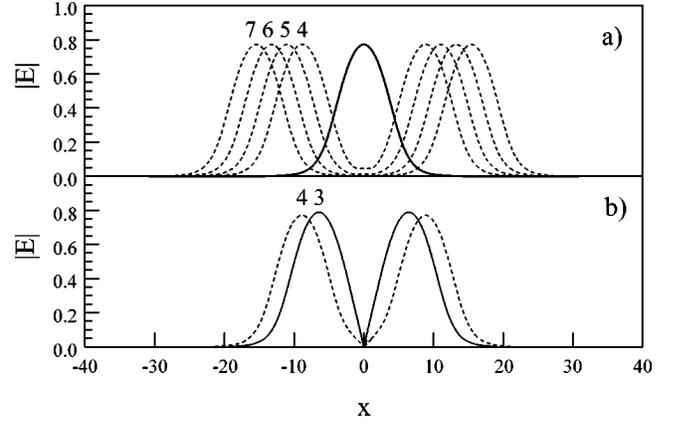


FIG. 1. Numerically calculated bound autosoliton states of Eq. (1) with  $d=0$ ,  $g_0=2.04$ ,  $a_0=2$ , and  $\beta=10$ . (a) Stable isolated autosoliton (solid line). Four in-phase BS's corresponding to smallest distances between the autosolitons (dotted lines). These states are unstable and correspond to  $n=4,5,6,7$  in Eq. (20). (b) Two antiphase bound states with smallest distances between the autosolitons. Stable (unstable) state corresponds to  $n=3$  ( $n=4$ ) and is shown by solid (dotted) line.

evaluated numerically. In particular, for  $d=0$ ,  $g_0=2.04$ ,  $a_0=2$ , and  $\beta=10$  stable isolated autosoliton solution exists which corresponds to the frequency shift  $\alpha=0.0672513$ . It is shown in Fig. 1(a) by solid line. The quantities determining the asymptotic behavior of this solution are given by  $\gamma=0.668992$  and  $\omega=0.717497$ . For the above given parameter values we have obtained the following numerical results  $q/N_1 > 0$ ,  $r_1 \approx -128$ ,  $r_2 = -169$ ,  $\theta_1 \approx -0.14$ , and  $\theta_2 \approx -0.02$ . Note that the damping terms in Eqs. (16) and (18) are proportional to the diffusion coefficient  $d$  (see also Ref. [21]). In the next section we show that these terms can stabilize certain inphase and antiphase BS's.

#### IV. BOUND AUTOSOLITON STATES

Equations (16) and (17) govern the time evolution of the distance between the autosoliton intensity maxima  $\zeta_-$  and the phase difference  $\varphi_-$ , whereas Eqs. (18) and (19) determine the center-of-mass velocity  $v_+/2$  and the mean frequency shift  $\partial_t \varphi_+/2$  of the autosoliton pair. The steady-state solutions of Eqs. (16)–(18) correspond to bound-autosoliton states. They are given by

$$BS_0: \quad \varphi_- = 0, \quad \zeta_- = \frac{\pi n - \theta_1}{\omega}, \quad v_+ = 0,$$

$$\partial_t \varphi_+ = (-1)^n r_2 e^{-\gamma \zeta_-} \sin \theta_-, \quad (20)$$

$$BS_{\pi}: \quad \varphi_- = \pi, \quad \zeta_- = \frac{\pi n - \theta_1}{\omega}, \quad v_+ = 0,$$

$$\partial_t \varphi_+ = (-1)^{n+1} r_2 e^{-\gamma \zeta_-} \sin \theta_-, \quad (21)$$

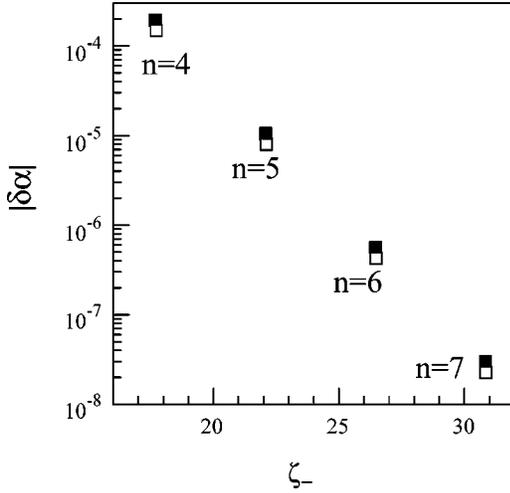


FIG. 2. Absolute value of the dimensionless frequency shift  $|\delta\alpha| = \alpha_n - \alpha_0$  vs dimensionless autosoliton intensity maxima spacing  $\zeta_-$  for the in-phase BS's. Filled (empty) squares correspond to BS's calculated numerically [obtained from Eqs. (20)]. Parameter values are the same as in Fig. 1.

$$BS_{\pm\pi/2}: \quad \varphi_- = \pm \frac{\pi}{2}, \quad \zeta_- = \frac{\pi(n+1/2) - \theta_2}{\omega},$$

$$v_+ = \pm (-1)^n \frac{r_1}{D} e^{-\gamma\zeta_-} \sin \theta_-, \quad \partial_t \varphi_+ = P v_+^2 / 2 \quad (22)$$

with  $\theta_- = \theta_2 - \theta_1$ . According to Eqs. (20)–(22) the BS's with the phase difference  $\varphi_- = 0, \pi$  between the autosolitons are motionless, while the states with  $\varphi_- = \pm \pi/2$  travel at the velocity  $v_+/2 = \pm [(-1)^n r_1 e^{-\gamma\zeta_-} \sin \theta_-] / (2D)$ .

The stability of the BS solutions can be analyzed in the framework of Eqs. (16)–(18). For  $D, \omega > 0$  the stability conditions for the steady states  $BS_0$  and  $BS_\pi$  are

$$r_1 r_2 \cos \theta_- > 0, \quad \mp (-1)^n r_1 > 0, \quad (23)$$

where the sign  $- (+)$  corresponds to the state  $BS_0$  ( $BS_\pi$ ). It follows from Eq. (23) that all the states  $BS_{0,\pi}$  are unstable when  $r_1 r_2 \cos \theta_- < 0$ . If, on the contrary,  $r_1 r_2 \cos \theta_- > 0$  the states  $BS_0$  ( $BS_\pi$ ) with  $\text{sgn}[(-1)^n r_1] < 0$  ( $\text{sgn}[(-1)^n r_1] > 0$ ) appear to be stable. According to Eqs. (16) and (17) for  $D=0$  and  $r_1 r_2 \cos \theta_- > 0$  these solutions, each having a pair of pure imaginary eigenvalues, are neutrally stable. Several bound autosoliton states calculated numerically for  $d=0$  using the procedure described in Ref. [14] are presented in Figs. 1(a),(b). We have found that all these BS's are weakly unstable except for the stable antiphase state  $BS_\pi$  corresponding to the steady-state solution (21) with  $n=3$ . The latter state exhibiting minimal possible distance between the autosolitons is shown in Fig. 1(b) by solid line. Figure 2 presents the comparison of the numerical results with those obtained from Eqs. (20). In this figure the quantity  $\delta\alpha$  is the frequency shift of the  $BS_0$  state with respect to the frequency of the isolated-autosoliton solution. Analytically it can be estimated as  $\delta\alpha \approx \partial_t \varphi_+ / 2$  with  $\varphi_+ / 2$  given by Eq. (20).

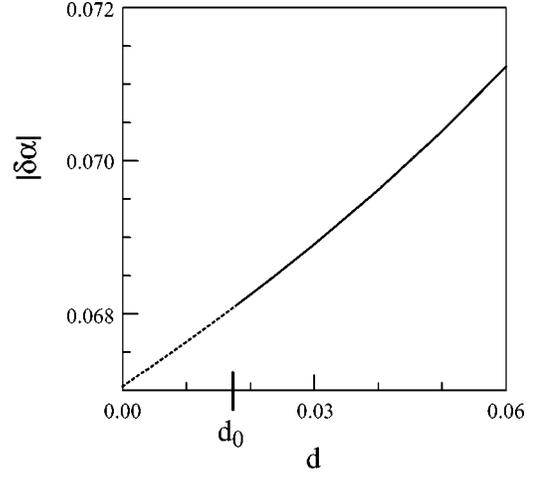


FIG. 3. Dimensionless frequency shift of the  $BS_0$  state with  $n=4$  vs normalized diffusion coefficient  $d$ . Solid (dashed) line corresponds to stable (unstable) BS.  $d_0$  indicates the threshold value of the diffusion coefficient above which the inphase  $BS_0$  state is stable. The parameter values are  $g_0=2.04$ ,  $a_0=2$ , and  $\beta=10$ .

Using the procedure proposed in Ref. [14] we have found numerically that for the parameter values of Figs. 1 and 2 the  $BS_0$  state with minimal distance between the autosolitons corresponds to  $n=4$  [see Fig. 1(a)]. For the values of the coefficients of Eqs. (16)–(19) given in the end of Sec. III we obtain  $r_1 r_2 \cos \theta_- \approx 21.5 \times 10^3 > 0$ . Since Figs. 1 and 2 correspond to the diffusionless limit  $d=0$ , according to Eq. (23) this solution is neutrally stable in the framework of Eqs. (16)–(19). As it was mentioned above, numerical results have indicated that it is weakly unstable. The instability could be related to the contribution of the higher order terms in  $\varepsilon$ , which were neglected in the derivation of Eqs. (16)–(19). However, as it follows from Eq. (23), for  $D > 0$  the in-phase BS can become stable when the contribution of the diffusion coefficient to the eigenvalues describing its stability is greater than that of the higher order terms. Hence, one could expect that there exists a threshold value of the diffusion coefficient  $d$  above which the state  $BS_0$  with  $n=4$  is stable. This conclusion is illustrated by Fig. 3 in which numerically calculated dependence of the frequency shift of the  $BS_0$  state is shown as a function of the diffusion coefficient  $d$ . Solid (dashed) line indicates stable (unstable) BS. It is seen from Fig. 3 and Fig. 4 that for  $d > d_0$  the state  $BS_0$  with  $n=4$  is stable. Similar threshold should exist for the inphase BS's corresponding to odd  $n > 4$  and the antiphase BS's corresponding to even  $n > 3$ . Since the contribution of the neglected higher order terms is smaller for greater  $n$ , the threshold value  $d_0$  is expected to decrease with the increase of the number  $n$ .

Now let us turn to the states  $BS_{\pm\pi/2}$  characterized by the phase shift  $\pm \pi/2$  between the autosolitons. For  $r_1 r_2 \cos \theta_- > 0$  these solutions are always unstable in the framework of Eqs. (16)–(19). Moreover, it follows from Eq. (18) that the bound states  $BS_{\pm\pi/2}$  travelling with constant velocity do not exist for  $D=0$  ( $d=0$ ). This conclusion is in agreement with the rigorous result obtained in Ref. [14] where it was shown using the bifurcation-theory methods that the only motion-

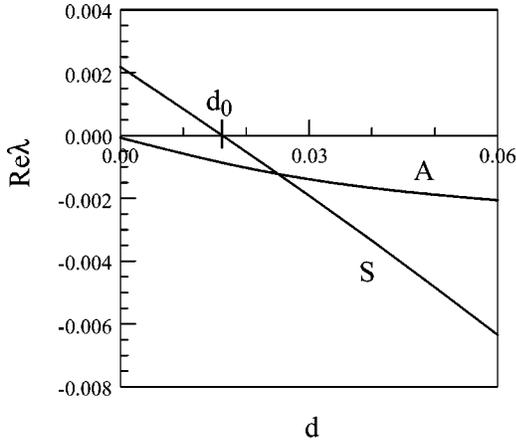


FIG. 4. Two nonzero eigenvalues with greatest real parts describing the stability of the  $BS_0$  branch shown in Fig. 3. S indicates the real part of the two complex conjugate eigenvalues associated with even perturbations invariant under  $x \rightarrow -x$ . The eigenvalue associated with odd perturbations is represented by the line A.  $d = d_0$  corresponds to a Hopf bifurcation that is the stability threshold for the BS solution.

less and uniformly moving BS's that can appear in the diffusionless limit  $d=0$  are in-phase and antiphase ones.

When the diffusion coefficient  $d$  increases the second term in the left-hand side of Eqs. (16) and (18) can become a dominating one. In this case we neglect the second derivative  $\partial_{tt}\zeta_-$  in Eq. (16). Then for  $v_{\pm} = 0$  Eqs. (16) and (17) take the form

$$\partial_t \zeta_- = D^{-1} r_1 e^{-\gamma \zeta_-} \cos \varphi_- \sin(\omega \zeta_- + \theta_1), \quad (24)$$

$$\partial_t \varphi_- = r_2 e^{-\gamma \zeta_-} \sin \varphi_- \cos(\omega \zeta_- + \theta_2), \quad (25)$$

The steady states of these equations coincide with those of Eqs. (16) and (17). In the framework of Eqs. (24) and (25) the stability conditions for the bound states  $BS_{0,\pi}$  are again given by Eq. (23). Note, however, that now these steady states cannot undergo a Hopf bifurcation since their eigenvalues are always real. The states  $BS_{\pm\pi/2}$  of Eqs. (16) and (17) are neutrally stable. Therefore, one could expect that the stability analysis of the latter states requires inclusion of the higher order terms into Eqs. (24) and (25). Note, that Eqs. (24) and (25) are similar to those derived in Refs. [8,9] where the interaction of localized solutions of the quintic CGLE was studied. The difference, however, is that in these publications the equations governing autosoliton interaction were derived from a single potential function with the assumption that dissipative parameters of the quintic CGLE can be considered as small perturbations. Under this assumption the eigenvectors of the adjoint operator can be approximated by the relation  $\Psi_s^\dagger = (i\psi_s(x), -i\psi_s^*(x))^T$  ( $s=1,2$ ), which implies that  $\theta_- = \pi - \arctan(\gamma/\omega)$ . For this particular value of  $\theta_-$  the states  $BS_{0,\pi}$  of the quintic CGLE were found to be unstable saddles. Numerical investigation of the BS's stability properties that was performed in Ref. [10] confirmed analytical conclusion about the instability of the  $BS_{0,\pi}$  states and demonstrated that the states  $BS_{\pm\pi/2}$  are

weakly unstable. Similar results concerning the instability of the  $BS_{0,\pi}$  states of the quintic CGLE were obtained by direct numerical simulation in Refs. [11,12] where it was shown that the bound states  $BS_{\pm\pi/2}$  can be either weakly unstable or weakly stable. Here we show that if the value diffusion coefficient  $d$  is above a certain very low threshold, stable  $BS_{0,\pi}$  states can exist in the model equation (1) that in contrast to the quintic CGLE considered in Refs. [8–12] is characterized by purely dissipative type of nonlinearity.

## V. CONCLUSION

We have derived asymptotic equations (16)–(19) governing the interaction of two weakly overlapping 1D autosoliton solutions in the transverse section of a bistable class A laser. Using these equations we have performed stability analysis of the autosoliton BS's in the parameter domain where the isolated-autosoliton solution is stable. Our analytical results concerning BS's stability are in a good agreement with those obtained numerically using the procedure described in [14] (see Fig. 3). According to Eqs. (16)–(19) for the parameter values of Figs. 1 and 2 the in-phase (antiphase) BS corresponding to even  $n$  in Eq. (20) [odd  $n$  in Eq. (21)] are neutrally stable for  $d=0$  and become stable for positive  $d$ . We have demonstrated numerically that these BS's (except for the antiphase BS with  $n=3$ ) are unstable below a certain very low threshold  $d < d_0$  and are stable for  $d > d_0$  (see Fig. 3). For  $d=0$  only the antiphase BS with  $n=3$  was found to be stable. The instability of the autosoliton BS that appears at  $d < d_0$  cannot be described in the framework of Eqs. (16)–(19). It could be related to the contribution of the higher order terms that we neglected in the derivation of Eqs. (16)–(19). However, when the diffusion coefficient  $d$  is greater than the threshold value  $d_0$ , so that the contribution of the diffusion into the eigenvalues that are pure imaginary in the framework of Eqs. (16)–(19) is greater than the contribution of the neglected higher order terms, the BS under consideration becomes stable. All the BS's with the phase difference  $\pm\pi/2$  between the autosolitons have been found to be unstable for the parameter values we used in our calculations.

Note that the results presented here differ from those of previous studies concerning the stability of BS's in the quintic CGLE [8–12] where all the in-phase and antiphase BS's were found to be unstable saddles. Here we have presented analytical and numerical evidence of the existence of such stable states in the model equations (1) and (2). Though in a certain limit situation our model (1) and (2) can be reduced to the quintic CGLE, the parameter domain we studied here is quite different from that considered in Refs. [8–12]. Analytical derivation of the interaction potential that was performed in [8–10] is based on the assumption that the dissipative terms in the quintic CGLE can be considered as small perturbations. This assumption does not hold in our case since we consider a purely dissipative saturable type of nonlinearity (2). Moreover, when dissipative terms are not small in general case the quantity  $\theta_-$  can take arbitrary values, and, hence, Eqs. (24) and (25) cannot be derived from a single potential function.

## APPENDIX

Taking into account Eqs. (12) and (13) the scalar products in Eqs. (14) and (15) can be expressed as

$$\begin{aligned}
\langle \Psi_{sj}^\dagger, \mathbf{H} \rangle &= \int_{-\infty}^{\infty} [\psi_{sj}^{\dagger*} A_k (f_\Sigma + |A_j|^2 f'_\Sigma - f_k) e^{i\varphi_{jk}} \\
&\quad + \psi_{sj}^{\dagger*} A_j^2 A_k^* f'_j e^{-i\varphi_{jk}}] dx + \text{c.c.} + O(\varepsilon^3) \\
&= \int_{-\infty}^{\infty} [A_k e^{i\varphi_{jk}} \{f_j + |A_j|^2 f'_j - f(0)\} \psi_{sj}^{\dagger*} \\
&\quad + A_j^2 A_k^* f'_j e^{-i\varphi_{jk}} \psi_{sj}^{\dagger*}] dx + \text{c.c.} + O(\varepsilon^3) \\
&= \langle \mathbf{u}_k^{(0)}, [\hat{L}^\dagger(A_j) - \hat{L}^\dagger(0)] \Psi_{sj}^\dagger \rangle + O(\varepsilon^2) \\
&= -\langle \mathbf{u}_k^{(0)}, \hat{L}(0) \Psi_{sj}^\dagger \rangle + O(\varepsilon^3) \\
&= -2 \operatorname{Re} \int_{-\infty}^{\infty} [e^{i\varphi_{jk}} A_k \hat{L}_{11}(0) \psi_{sj}^{\dagger*}] dx + O(\varepsilon^3),
\end{aligned}$$

with  $\hat{L}$ ,  $\hat{L}_{11}$ , and  $\mathbf{u}_k^{(0)} = (u_k^{(0)}, u_k^{(0)*})^T$  are defined by Eqs. (4), (5), and (10), respectively.  $\varphi_{jk} = \varphi_j - \varphi_k$  and  $s, j, k = 1, 2$ ;  $j \neq k$ . Here we have used the relations  $f_\Sigma = f_1 + f_2 - f(0) + O(\varepsilon^2)$  and  $\hat{L}^\dagger(A_j) \Psi_{sj}^\dagger = 0$ .

Thus, we get

$$\begin{aligned}
\langle \Psi_{s2}^\dagger \pm \Psi_{s1}^\dagger, \mathbf{H} \rangle &= -2 \operatorname{Re} \int_{-\infty}^{\infty} [e^{i\varphi_{21}} A_1 \hat{L}_{11}(0) \psi_{s2}^{\dagger*} \\
&\quad \pm e^{-i\varphi_{21}} A_2 \hat{L}_{11}(0) \psi_{s1}^{\dagger*}] dx + O(\varepsilon^3),
\end{aligned}$$

where  $\varphi_{21} = \varphi_2 - \varphi_1 \equiv \varphi_-$ .

Finally, using the relation

$$\begin{aligned}
&\int_{-\infty}^{\infty} [A_2 \hat{L}_{11}(0) \psi_{s1}^{\dagger*}] dx \\
&= (-1)^s \int_{-\infty}^{\infty} [A_1 \hat{L}_{11}(0) \psi_{s2}^{\dagger*}] dx \\
&\approx (-1)^s \int_{\xi_+/2}^{\infty} [A_1 \hat{L}_{11}(0) \psi_{s2}^{\dagger*}] dx
\end{aligned}$$

we obtain

$$\begin{aligned}
&-\langle \Psi_{s2}^\dagger \pm \Psi_{s1}^\dagger, \mathbf{H} \rangle \\
&= 2 \operatorname{Re} \left[ (e^{i\varphi_-} \pm (-1)^s e^{-i\varphi_-}) \int_{\xi_+/2}^{\infty} (A_1 \hat{L}_{11}(0) \psi_{s2}^{\dagger*}) dx \right] \\
&\quad + O(\varepsilon^3) \\
&= -2 \operatorname{Im} \{ [e^{i\varphi_-} \pm (-1)^s e^{-i\varphi_-}] \\
&\quad \times [\psi_{s2}^{\dagger*} \partial_x A_1 - A_1 \partial_x \psi_{s2}^{\dagger*}] |_{x=\xi_+/2} \} + O(\varepsilon^3) \\
&= -2 \operatorname{Im} \{ [e^{i\varphi_-} \pm (-1)^s e^{-i\varphi_-}] \partial_{\xi_-} (A_1 \psi_{s2}^{\dagger*}) |_{x=\xi_+/2} \} \\
&\quad + O(\varepsilon^3) \\
&= 2e^{-\gamma\xi_-} \operatorname{Im} \{ (-1)^s a b_s^* (\gamma - i\omega) \\
&\quad \times [e^{i\varphi_-} \pm (-1)^s e^{-i\varphi_-}] e^{i\omega\xi_-} \} + O(\varepsilon^3),
\end{aligned}$$

where the coefficients  $a$  and  $b_s$  are defined by Eq. (8).

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