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Stability of Weakly Nonparaxial Spatial Optical Solitons in a Medium with a Kerr Nonlinearity

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Abstract—A variant of perturbation theory is developed to determine the characteristics and stability of transversely two-dimensional spatial solitons in a Kerr medium under conditions of small deviations from paraxial. Distributions of the transverse and longitudinal components of the soliton electric and magnetic fields are obtained. It is shown that the power of a nonparaxial soliton in a Kerr medium increases as the propagation constant increases. A linear analysis is made of soliton stability. In addition to confirming stability, this analysis revealed "internal modes" of nonparaxial solitons and their characteristics were determined. *© 2000 MAIK "Nauka/Interperiodica".*

1. INTRODUCTION

The final stage of the self-focusing of high-intensity radiation in a transparent medium with a Kerr nonlinearity is of considerable interest and has been studied on many occasions (see the book [1] and the literature cited therein). Using the standard approximation of slowly varying amplitudes (quasi-optic or paraxial equation for the electric field envelope), the theory predicts the absence of stable spatial solitons (beams having a constant transverse field profile) and collapse of radiation beams having powers exceeding the critical self-focusing power. The limitation of collapse and formation of spatial solitons as predicted in [2] may occur for various reasons. The most common of these is the nonparaxial nature of narrow (width comparable to the wavelength) beams [1] which precludes us from using the approximation of the quasi-optic equation.

Although nonparaxial effects also arise for the scalar nonlinear wave equation [3–6], for electromagnetic radiation it is important to allow for its polarization. Thus, nonparaxial self-focusing theory should be based on a complete system of Maxwell nonlinear vector equations. Previously, vector self-focusing theory was preferentially developed for special cases of the polarization of radiation having an axisymmetric intensity distribution and unit nonzero electric field component [7–10], i.e., in fact for a scalar variant. The spatial solitons which may appear in this case have a power considerably higher than the critical self-focusing power, which serves as an indication of their instability [11]. For a transversely one-dimensional geometry it is possible to construct a fairly comprehensive classification of an infinite set of localized structures [12, 13] although in a continuous nonlinear medium all these structures are unstable with respect to decay along the other transverse coordinate. As far as we are aware, the existence of stable spatial solitons of electromagnetic radiation in a medium with a Kerr nonlinearity has not yet been proven. As will be shown subsequently, using numerical calculations of the type [14, 15] to solve this problem may not yield the correct result because establishment is extremely slow under weakly nonparaxial conditions.

The task for the present study is to make an analytic investigation of the characteristics and properties of weakly nonparaxial spatial transversely two-dimensional solitons of electromagnetic radiation in a medium with a Kerr nonlinearity. The analysis is based on perturbation theory with a small nonparaxial parameter which is used to find nonparaxial corrections to the soliton shape and to determine its stability. The initial (zeroth) approximation is the well-studied nonlinear Schrödinger equation and beams having an axisymmetric intensity distribution and linearly polarized radiation ("Townes mode"). Following [16] (see also [14]), in Section 2 we give a derivation of the control equation for the envelope of a weakly nonparaxial soliton field. We then obtain its approximate solution in Section 3, i.e., we determine the transverse distribution of the electron and magnetic field intensities. In Sections 4 and 5 we analyze soliton stability using a method proposed in [17] (see also [18]) for paraxial solitons in a medium with saturation of the nonlinearity. For this we use a linearized control equation whose properties are analyzed in Section 4. The final conclusion on the stability of a weakly nonparaxial soliton is formulated in Section 5 and calculations of various matrix elements are presented in the Appendix. The results are discussed in the Conclusions.

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2. EVOLUTION EQUATION

The initial equations are the Maxwell equations for monochromatic radiation of frequency ω [in complex notation the factor $\exp(-i\omega t)$ is omitted, t is the time) in a nonmagnetic medium (with unit magnetic permeability):

$$\mathrm{rot}\mathbf{E} = i\frac{\omega}{c}\mathbf{H}, \quad \mathrm{rot}\mathbf{H} = -i\frac{\omega}{c}\mathbf{D}, \qquad (2.1)$$
$$\mathrm{div}\mathbf{H} = 0, \quad \mathrm{div}\mathbf{D} = 0.$$

Here $\mathbf{E}$ and $\mathbf{H}$ are the intensities of the electric and magnetic fields, c is the speed of light in vacuum, and $\mathbf{D}$ is the electric induction which has the form (Kerr striction nonlinearity)

$$\mathbf{D} = (\varepsilon_0 + \varepsilon_{nl})\mathbf{E}, \quad \varepsilon_{nl} = \varepsilon_2|\mathbf{E}|^2, \quad \varepsilon_2 > 0. \qquad (2.2)$$

Here ε_0 is the linear permittivity. The form (2.2) allows only for self-interaction effects whereas the generation of third and high-order harmonics is considered to be ineffective (phase matching conditions are not satisfied for these).

Eliminating the magnetic field intensity from the Maxwell equation, we obtain the generalized Helmholtz equation

$$\frac{\partial^2\mathbf{E}}{\partial z^2} + \Delta_\perp\mathbf{E} + \frac{\omega^2}{c^2}\mathbf{D} - \mathrm{grad}\,\mathrm{div}\mathbf{E} = 0,$$

$$\Delta_\perp = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

From this vector equation it follows that

$$\frac{\partial^2\mathbf{E}_\perp}{\partial z^2} + \Delta_\perp\mathbf{E}_\perp + \frac{\omega^2}{c^2}\mathbf{D}_\perp - \mathrm{grad}_\perp\mathrm{div}\mathbf{E} = 0. \qquad (2.3)$$

Here we introduce the transverse components of the electric field $\mathbf{E}_\perp = (E_x, E_y)$ and the induction $\mathbf{D}_\perp = (D_x, D_y)$. We transform the last term on the left-hand side of equation (2.3) as follows. From the final Maxwell equation (2.1) it follows that

$$\mathrm{div}\mathbf{E} = -\frac{1}{\varepsilon_0 + \varepsilon_{nl}}\mathbf{E}\cdot\mathrm{grad}\varepsilon_{nl}$$
$$= -\frac{1}{\varepsilon_0 + \varepsilon_{nl}}\left(\mathbf{E}_\perp\cdot\mathrm{grad}_\perp\varepsilon_{nl} + E_z\frac{\partial\varepsilon_{nl}}{\partial z}\right). \qquad (2.4)$$

In the lowest approximation (weak and continuously varying nonlinearity) we have div $\mathbf{E} = 0$ whence

$$E_z \approx \frac{i}{k}\mathrm{div}_\perp\mathbf{E}_\perp, \qquad (2.5)$$

where $k = (\omega/c)\sqrt{\varepsilon_0}$ is the wave number in the linear medium. For a more accurate estimate we have

$$\mathrm{div}\mathbf{E} \approx -\frac{1}{\varepsilon_0}\mathbf{E}_\perp\cdot\mathrm{grad}_\perp\varepsilon_{nl}.$$

We then arrive at a closed equation for the transverse field components:

$$\frac{\partial^2\mathbf{E}}{\partial z^2} + \Delta_\perp\mathbf{E}_\perp + \frac{\omega^2}{c^2}\varepsilon_0\mathbf{E}_\perp + \frac{\omega^2}{c^2}\varepsilon_2|\mathbf{E}_\perp|^2\mathbf{E}_\perp = \mathbf{Q}_s(\mathbf{E}_\perp), \qquad (2.6)$$

where

$$\mathbf{Q}_s(\mathbf{E}_\perp) = -\frac{\varepsilon_2}{\varepsilon_0} \qquad (2.7)$$
$$\times[|\mathrm{div}_\perp\mathbf{E}_\perp|^2\mathbf{E}_\perp + \mathrm{grad}_\perp(\mathbf{E}_\perp\cdot\mathrm{grad}_\perp|\mathbf{E}_\perp|^2)].$$

For a steady-state soliton we have

$$\mathbf{E}_\perp = \mathbf{A}_s(\mathbf{r}_\perp)e^{i\Gamma z}, \qquad (2.8)$$

where Γ is the real propagation constant. The solitons have a continuous spectrum with respect to Γ and the condition $\Gamma^2 > k^2$ must be satisfied for the field to decrease (tend to zero) at the soliton edge, in accordance with (2.6). A measure of the nonparaxial property is given by

$$\mu^2 = \frac{\Gamma^2 - k^2}{k^2} \approx 2\frac{\Gamma - k}{k} \ll 1. \qquad (2.9)$$

Inequality (2.9) implies that the propagation constant Γ for the soliton is close to the wave number k in the linear medium. This occurs if the soliton width is considerably greater than the wavelength of light, the maximum amplitude of the field is extremely small, and the power is close to the critical self-focusing power (see below). In this limit the order of the derivative with respect to z can be reduced [14]. We shall assume that the field is close to a steady-state soliton so that

$$\mathbf{E}_\perp = \mathbf{A}_\perp(\mathbf{r}_\perp, z)e^{i\Gamma z}, \qquad (2.10)$$

where the dependence of the amplitude $\mathbf{A}_\perp(\mathbf{r}_\perp, z)$ on the longitudinal coordinate z is slow (on a scale of the order of the wavelength of light). Then, retaining terms of the lowest order of smallness in the transformations of $\partial^2E/\partial r^2$, instead of (2.6) we obtain the evolution equation

$$2i\Gamma\frac{\partial\mathbf{A}_\perp}{\partial z} + \Delta_\perp\mathbf{A}_\perp - (\Gamma^2 - k^2)\mathbf{A}_\perp \qquad (2.11)$$
$$+ k^2\frac{\varepsilon_2}{\varepsilon_0}|\mathbf{A}_\perp|^2\mathbf{A}_\perp = \mathbf{Q}_s(\mathbf{A}_\perp) + \mathbf{Q}_z(\mathbf{A}_\perp),$$

where

$$\mathbf{Q}_z(\mathbf{A}_\perp) = \frac{1}{4k^2}[\Delta_\perp - (\Gamma^2 - k^2)]$$
$$\times\left[\Delta_\perp\mathbf{A}_\perp - (\Gamma^2 - k^2)\mathbf{A}_\perp + k^2\frac{\varepsilon_2}{\varepsilon_0}|\mathbf{A}_\perp|^2\mathbf{A}_\perp\right]$$
$$+\frac{\varepsilon_2}{4\varepsilon_0}\Big\{(\Delta_\perp\mathbf{A}_\perp\cdot\mathbf{A}_\perp^*)\mathbf{A}_\perp - (\Delta_\perp\mathbf{A}_\perp^*\cdot\mathbf{A}_\perp)\mathbf{A}_\perp \quad (2.12)$$
$$+|\mathbf{A}_\perp|^2\left[\Delta_\perp\mathbf{A}_\perp - (\Gamma^2 - k^2)\mathbf{A}_\perp + k^2\frac{\varepsilon_2}{\varepsilon_0}|\mathbf{A}_\perp|^2\mathbf{A}_\perp\right]\Big\}.$$

Note that for a steady-state spatial soliton we have $\mathbf{Q}_z(\mathbf{A}_s) = 0$. Equation (2.11) can be used not only to find a weakly nonparaxial steady-state soliton but also to investigate its stability. In order to isolate the nonparaxial parameter in explicit form in (2.11), we convert to dimensionless coordinates and amplitude:

$$z' = \frac{\Gamma^2 - k^2}{\Gamma}z, \quad (x', y') = \sqrt{\Gamma^2 - k^2}(x, y),$$
$$\mathbf{A}'_\perp = \frac{k}{\sqrt{\Gamma^2 - k^2}}\sqrt{\frac{\varepsilon_2}{\varepsilon_0}}\mathbf{A}_\perp, \quad \Delta'_\perp = \frac{1}{\Gamma^2 - k^2}\Delta_\perp. \quad (2.13)$$

Then (2.11) has the form

$$2i\frac{\partial\mathbf{A}'_\perp}{\partial z'} + \Delta'_\perp\mathbf{A}'_\perp - \mathbf{A}'_\perp + |\mathbf{A}'_\perp|^2\mathbf{A}'_\perp$$
$$= \mu^2[\mathbf{Q}'_s(\mathbf{A}'_\perp) + \mathbf{Q}'_z(\mathbf{A}'_\perp)], \quad (2.14)$$

where

$$\mathbf{Q}'_s(\mathbf{A}'_\perp) = -[|\mathrm{div}'_\perp\mathbf{A}'_\perp|^2\mathbf{A}'_\perp + \mathrm{grad}'_\perp(A'_\perp\cdot\mathrm{grad}'_\perp|\mathbf{A}'_\perp|^2)],$$
$$\mathbf{Q}'_z(\mathbf{A}'_\perp) = \frac{1}{4}[\Delta'_\perp - 1][\Delta'_\perp\mathbf{A}'_\perp - \mathbf{A}'_\perp + |\mathbf{A}'_\perp|^2\mathbf{A}'_\perp]$$
$$+\frac{1}{4}[(\Delta'_\perp\mathbf{A}'_\perp\cdot\mathbf{A}'^*_\perp)\mathbf{A}'_\perp - (\Delta'_\perp\mathbf{A}'^*_\perp\cdot\mathbf{A}'_\perp)\mathbf{A}'_\perp \quad (2.15)$$
$$+|\mathbf{A}'_\perp|^2(\Delta'_\perp\mathbf{A}'_\perp - \mathbf{A}'_\perp + |\mathbf{A}'_\perp|^2\mathbf{A}'_\perp)].$$

The dimensionless form (2.14) is convenient for determining the corrections to the shape of a steady-state soliton while the dimensional form (2.11) is convenient for analyzing its stability, containing derivatives of the amplitudes with respect to the propagation constant Γ.

3, NONPARAXIAL CORRECTIONS TO SOLITON SHAPE

The right-hand side of equation (2.14) serves as a correction (as a result of the nonparaxial property) to the nonlinear Schrödinger equation for which $\mu = 0$. Note that this correction is nonlocal since it not only depends on the transverse components of the field intensity but also on their derivatives in the transverse direction. We find the field distribution for a weakly nonparaxial steady-state [of the form (2.8)] spatial soliton by solving equation (2.14) using perturbation theory with the small parameter (2.9). In the lowest approximation this equation gives the standard vector nonlinear Schrödinger equation (we omit the primes in this section):

$$\Delta_\perp A_{s0x} - A_{s0x} + (|A_{s0x}|^2 + |A_{s0y}|^2)A_{s0x} = 0,$$
$$\Delta_\perp A_{s0y} - A_{s0y} + (|A_{s0x}|^2 + |A_{s0y}|^2)A_{s0y} = 0. \quad (3.1)$$

For the main (fundamental) soliton the functions A_{s0x} and A_{s0y} can be considered to be real. Generally speaking, the equations (3.1) are written for the particular case of a soliton with a common propagation constant for both polarizations but a difference between these values is only possible in the paraxial approximation [1].

We now introduce a small correction to the steady-state soliton:

$$\mathbf{A}_s(\mathbf{r}_\perp) = \mathbf{A}_{s0}(\mathbf{r}_\perp) + \mu^2\delta\mathbf{A}_s(\mathbf{r}_\perp). \quad (3.2)$$

Equation (2.14) linearized with respect to the perturbation $\delta\mathbf{A}_s$ is written in the form

$$\Delta_\perp\delta\mathbf{A}_s - \delta\mathbf{A}_s + [|A_{s0}|^2\delta\mathbf{A}_s$$
$$+(\mathbf{A}_{s0}^*\cdot\delta\mathbf{A}_s)\mathbf{A}_{s0} + (\mathbf{A}_{s0}\cdot\delta\mathbf{A}_s^*)\mathbf{A}_{s0}] = \mathbf{Q}_s(\mathbf{A}_{s0}). \quad (3.3)$$

In terms of Cartesian components we have

$$\Delta_\perp\delta A_{sx} - \delta A_{sx} + [(A_{s0x}^2 + A_{s0y}^2)\delta A_{sx}$$
$$+ A_{s0x}^2(\delta A_{sx} + \delta A_{sx}^*)$$
$$+ A_{s0x}A_{s0y}(\delta A_{sy} + \delta A_{sy}^*)] = Q_{sx}(A_{s0x}, A_{s0y}),$$
$$\Delta_\perp\delta A_{sy} - \delta A_{sy} + [(A_{s0x}^2 + A_{s0y}^2)\delta A_{sy} \quad (3.4)$$
$$+ A_{s0x}A_{s0y}(\delta A_{sx} + \delta A_{sx}^*)$$
$$+ A_{s0y}^2(\delta A_{sy} + \delta A_{sy}^*)] = Q_{sy}(A_{s0x}, A_{s0y}),$$

$$Q_{sx}(A_{s0x}, A_{s0y}) = -\left\{\left|\frac{\partial A_{s0x}}{\partial x} + \frac{\partial A_{s0y}}{\partial x}\right|^2 A_{s0x}\right.$$
$$\left.+\frac{\partial}{\partial x}\left[A_{s0x}\frac{\partial(A_{s0x}^2 + A_{s0y}^2)}{\partial x} + A_{s0y}\frac{\partial(A_{s0x}^2 + A_{s0y}^2)}{\partial y}\right]\right\}, \quad (3.5)$$
$$Q_{sy}(A_{s0x}, A_{s0y}) = -\left\{\left|\frac{\partial A_{s0x}}{\partial x} + \frac{\partial A_{s0y}}{\partial x}\right|^2 A_{s0y}\right.$$

$$+\frac{\partial}{\partial y}\left[A_{s0x}\frac{\partial(A_{s0x}^2+A_{s0y}^2)}{\partial x}+A_{s0y}\frac{\partial(A_{s0x}^2+A_{s0y}^2)}{\partial y}\right]\Bigg\}.$$

As the zeroth approximation we take the linearly polarized Townes mode having an axisymmetric field distribution (polar coordinates r, φ, $r = \sqrt{x^2+y^2}$):

$$A_{s0x} = F_0(r), \quad A_{s0y} = 0. \tag{3.6}$$

The function $F_0(r)$ is defined as finite over the entire range $0 < r < \infty$ and the solution of the equation which tends to zero as $r \longrightarrow \infty$ is

$$L_0F_0 = 0, \quad L_0 = \Delta_\perp - 1 + F_0^2. \tag{3.7}$$

Bearing in mind the axial symmetry of the Townes mode, equation (3.7) has the form

$$\frac{d^2F_0}{dr^2}+\frac{1}{r}\frac{dF_0}{dr}-F_0+F_0^3 = 0. \tag{3.8}$$

We write the equations (3.4) for the corrections to the soliton shape in the form

$$\begin{gathered} L_1\delta A'_{sx} = Q_{x0}(r)+Q_{x2}(r)\cos 2\varphi, \\ L_0\delta A''_{sx} = 0, \\ L_0\delta A_{sy} = Q_{y2}(r)\sin 2\varphi, \end{gathered} \tag{3.9}$$

where

$$L_1 = L_0+2F_0^2, \quad \delta A_{sx} = \delta A'_{sx}+i\delta A''_{sx}, \tag{3.10}$$

$$\begin{gathered} Q_{x0}(r) = -\left[\frac{1}{2}F_0\left(\frac{dF_0}{dr}\right)^2+\left(\frac{d}{dr}+\frac{1}{r}\right)\left(F_0^2\frac{dF_0}{dr}\right)\right], \\ Q_{x2}(r) = -\left[\frac{1}{2}F_0\left(\frac{dF_0}{dr}\right)^2+\left(\frac{d}{dr}-\frac{1}{r}\right)\left(F_0^2\frac{dF_0}{dr}\right)\right], \\ Q_{y2}(r) = -\left(\frac{d}{dr}-\frac{1}{r}\right)\left(F_0^2\frac{dF_0}{dr}\right). \end{gathered} \tag{3.11}$$

The second of the linear equations (3.9) (for $\delta A''_{sx}$) has the solution [see (3.7)]

$$\delta A''_{sx} = C''A_{s0}, \quad A_{s0}\equiv A_{s0x} = F_0, \tag{3.12}$$

which corresponds to a phase shift of the initial soliton. Since we are not interested in this shift, we can set $C'' = 0$ and accordingly $\delta A''_{sx} = 0$. The two remaining inhomogeneous equations (3.9) can be solved provided that their right-hand sides are orthogonal to the solutions of the corresponding homogeneous equations with the boundary conditions specified above. For the last of the equations (3.9) orthogonality follows from the angular dependence of the right-hand side,

$$\int_0^{2\pi} d\varphi \sin 2\varphi = 0.$$

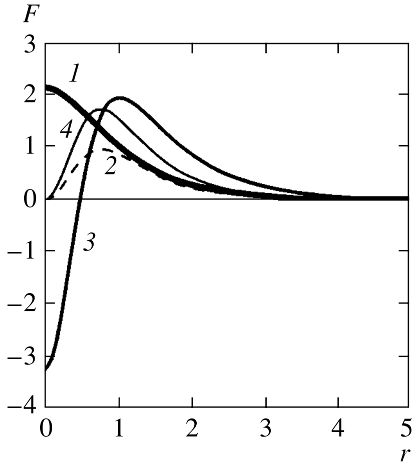

Fig. 1. Radial profiles of the amplitudes F_0 of a fundamental soliton (Townes modes, curve *1*) and its nonparaxial distortions F_y (curve *2*), F_1 (curve *3*), and F_2 (curve *4*).

Thus, omitting the solution of the homogeneous equation (symmetry with respect to rotation of the axes x, y), we obtain the correction δA_{sy} in the form

$$\delta A_{sy} = F_y(r)\sin 2\varphi. \tag{3.13}$$

Then $F_y(r)$ is defined as the only finite axisymmetric solution of the equation

$$L_2F_y = Q_{y2}(r), \quad L_2 = L_1-\frac{4}{r^2}. \tag{3.14}$$

The solution of the homogeneous equation corresponding to the first of the equations (3.9) corresponds to a shift of the initial soliton along x and y:

$$\begin{aligned} \delta A'_{sx1} &= C_1\frac{\partial F_0}{\partial x} = C_1\frac{dF_0}{dr}\cos\varphi, \\ \delta A'_{sx2} &= C_2\frac{\partial F_0}{\partial y} = C_2\frac{dF_0}{dr}\sin\varphi. \end{aligned} \tag{3.15}$$

The orthogonality condition is again satisfied as a result of the angular dependence of these solutions and the right-hand side of this equation. Also omitting the solution of the homogeneous equation (3.14) ($C_1 = C_2 = 0$) we obtain $\delta A'_{sx}$ in the form

$$\delta A'_{sx} = F_1(r)+F_2(r)\cos 2\varphi. \tag{3.16}$$

The radial functions appearing in (3.16) are obtained as (unique) finite axisymmetric solutions of the equations

$$L_1F_1 = Q_{x0}, \quad L_2F_2 = Q_{x2}. \tag{3.17}$$

Graphs of these functions obtained by solving numerically axisymmetric variants of equations (3.8), (3.14), and (3.17) with the conditions of finiteness for $r = 0$ and which decrease as $r \longrightarrow \infty$ specified above are plotted in Fig. 1.

We shall now return to dimensional quantities. For the electric field intensity of a steady-state soliton we have

$$E_x = \sqrt{\frac{\varepsilon_0}{\varepsilon_2}}\{\mu F_0(r) + \mu^3[F_1(r) + F_2(r)\cos 2\varphi]\},$$

$$\begin{aligned} E_y &= \sqrt{\frac{\varepsilon_0}{\varepsilon_2}}\mu^3 F_y(r)\sin 2\varphi, \\ E_z &= i\mu^2\sqrt{\frac{\varepsilon_0}{\varepsilon_2}}\frac{dF_0(r)}{dr}\cos\varphi, \end{aligned} \quad (3.18)$$

$$r = \sqrt{\Gamma^2 - k^2}\rho, \quad \rho = \sqrt{x^2 + y^2}.$$

In accordance with (3.18), the field exhibits weak axial symmetry. The main correction to the initial linear polarization occurs as a result of the longitudinal component of the field $E_z \propto \mu^2$. For this correction the phase of the field is shifted relative to the main component by $\pi/2$ since the polarization becomes elliptic. Time oscillations of the electric intensity vector are described by a prolate ellipse in the plane x_z. When additional allowance is made for corrections proportional to μ^3, it is found that the slope of the plane in which the ellipse is located varies over the beam cross section.

The components of the magnetic intensity are expressed in terms of the electric field intensity using the Maxwell equations (2.1):

$$H_x = -\frac{c}{\omega}\left(\Gamma E_y + i\frac{\partial E_z}{\partial y}\right) = -\mu^3\frac{\varepsilon_0}{2\sqrt{\varepsilon_2}} \times\left(\frac{2}{r}\frac{dF_0}{dr} - F_0 + F_0^3 + 2F_y\right)\sin 2\varphi,$$

$$H_y = \frac{c}{\omega}\left(\Gamma E_x + i\frac{\partial E_z}{\partial x}\right) = H_{y0} + \mu^3\frac{\varepsilon_0}{2\sqrt{\varepsilon_2}} \quad (3.19)$$

$$\times\left[(2F_1 + F_0^3) + \left(2F_2 + F_0^3 - F_0 + \frac{2}{r}\frac{dF_0}{dr}\right)\cos 2\varphi\right],$$

$$H_z = \frac{i}{k}\sqrt{\varepsilon_0}\frac{\partial E_x}{\partial y} = i\mu^2\frac{\varepsilon_0}{\sqrt{\varepsilon_2}}\frac{dF_0}{dr}\sin\varphi,$$

$$H_{y0} = \frac{\varepsilon_0}{\sqrt{\varepsilon_2}}\mu F_0(r).$$

The radiation power P is defined as the integral of the longitudinal component of the time-averaged Poynting vector over the transverse coordinates:

$$P = \frac{c}{8\pi}\iint \mathrm{Re}(E_x^* H_y - E_y^* H_x)dxdy. \quad (3.20)$$

Substituting into (3.20) the approximate expressions for the soliton field intensities obtained above, we obtain the power

$$P = \frac{c^2}{8\pi\omega k}\frac{\varepsilon_0}{\varepsilon_2}(P_0 + \mu^2 p_1). \quad (3.21)$$

The constants P_0 and p_1 are defined as follows:

$$\begin{aligned} P_0 &= \int_0^\infty F_0^2(r)rdr = \frac{11.7}{2\pi} = 1.862, \\ p_1 &= \int_0^\infty\left[2F_0(r)F_1(r) + \frac{1}{2}F_0^4(r)\right]rdr = 5.990. \end{aligned} \quad (3.22)$$

In (3.21) the term containing P_0 corresponds to the critical self-focusing power which agrees with that obtained in the paraxial limit ($\mu \longrightarrow 0$) where it does not depend on the propagation constant [1]. The term containing p_1 is the nonparaxial correction to the power which depends on the propagation constant. Note that the increase in soliton power with increasing propagation constant is consistent with the Vakhitov–Kolokolov criterion [19] for soliton stability. However, this criterion was obtained in the paraxial approximation and thus we still need to demonstrate the stability of a nonparaxial soliton.

4. LINEARIZED EQUATIONS

For a linear analysis of stability we shall set [see (3.2)]

$$\mathbf{A}_\perp(\mathbf{r}_\perp, z) = \mathbf{A}_s(\mathbf{r}_\perp) + \delta\mathbf{A}(\mathbf{r}_\perp, z). \quad (4.1)$$

Substituting (4.1) into (2.11) and linearizing this with respect to the small perturbation $\delta\mathbf{A}$, we find

$$2i\Gamma\frac{\partial\delta\mathbf{A}}{\partial z} + \Delta_\perp\delta\mathbf{A} - (\Gamma^2 - k^2)\delta\mathbf{A} + k^2\frac{\varepsilon_2}{\varepsilon_0} \quad (4.2)$$

$$\times[A_{s0}^2\delta\mathbf{A} + \mathbf{A}_{s0}(\mathbf{A}_{s0}\cdot\delta\mathbf{A}) + \mathbf{A}_{s0}(\mathbf{A}_{s0}\cdot\delta\mathbf{A}^*)] = \delta\mathbf{Q}.$$

Here we have

$$\delta\mathbf{Q} = \delta\mathbf{Q}_s + \delta\mathbf{Q}_z + \delta\mathbf{Q}_k,$$

$$\delta\mathbf{Q}_k = -k^2\frac{\varepsilon_2}{\varepsilon_0}[(\mathbf{A}_s^2 - \mathbf{A}_{s0}^2)\delta\mathbf{A} + \delta\mathbf{A}_s(\mathbf{A}_{s0}\cdot\delta\mathbf{A})$$

$$+ \mathbf{A}_{s0}(\delta\mathbf{A}_s\cdot\delta\mathbf{A}) + \delta\mathbf{A}_s(\delta\mathbf{A}_s\cdot\delta\mathbf{A}) + \delta\mathbf{A}_s(\mathbf{A}_{s0}\cdot\delta\mathbf{A}^*)$$

$$+ \mathbf{A}_{s0}(\delta\mathbf{A}_s\cdot\delta\mathbf{A}^*) + \delta\mathbf{A}_s(\delta\mathbf{A}_s\cdot\delta\mathbf{A}^*)],$$

$$\delta\mathbf{Q}_s = -\frac{\varepsilon_2}{\varepsilon_0}\{|\mathrm{div}_\perp\mathbf{A}_s|^2\delta\mathbf{A}$$

$$+ [\mathrm{div}_\perp\mathbf{A}_s^*\mathrm{div}_\perp\delta\mathbf{A} + \mathrm{div}_\perp\mathbf{A}_s\mathrm{div}_\perp\delta\mathbf{A}^*]\mathbf{A}_s$$

$$+ \mathrm{grad}_\perp[\mathbf{A}_s\cdot\mathrm{grad}_\perp(\mathbf{A}_s^*\cdot\delta\mathbf{A} + \mathbf{A}_s\cdot\delta\mathbf{A}^*)$$

$$+ \delta\mathbf{A}\cdot \mathrm{grad}_\perp|\mathbf{A}_s|^2]\},$$

$$\delta\mathbf{Q}_z = \frac{1}{4k^2}[\Delta_\perp - (\Gamma^2 - k^2)]^2\delta\mathbf{A} \tag{4.3}$$

$$+ \frac{\varepsilon_2}{4\varepsilon_0}\{\Delta_\perp[|\mathbf{A}_s|^2\delta\mathbf{A} + \mathbf{A}_s(\mathbf{A}_s^*\cdot\delta\mathbf{A}) + \mathbf{A}_s(\mathbf{A}_s\cdot\delta\mathbf{A}^*)]$$

$$+ |\mathbf{A}_s|^2\Delta_\perp\delta\mathbf{A} + (\mathbf{A}_s^*\cdot\delta\mathbf{A} + \mathbf{A}_s\cdot\delta\mathbf{A}^*)\Delta_\perp\mathbf{A}_s$$

$$- 2(\Gamma^2 - k^2)[|\mathbf{A}_s|^2\delta\mathbf{A} + \mathbf{A}_s(\mathbf{A}_s^*\cdot\delta\mathbf{A}) + \mathbf{A}_s(\mathbf{A}_s\cdot\delta\mathbf{A}^*)]$$

$$+ k^2\frac{\varepsilon_2}{\varepsilon_0}[|\mathbf{A}_s|^4\delta\mathbf{A} + 2\mathbf{A}_s|\mathbf{A}_s|^2(\mathbf{A}_s^*\cdot\delta\mathbf{A} + \mathbf{A}_s\cdot\delta\mathbf{A}^*)]$$

$$+ (\Delta_\perp\delta\mathbf{A}\cdot\mathbf{A}_s^*)\mathbf{A}_s + (\Delta_\perp\mathbf{A}_s\cdot\delta\mathbf{A}^*)\mathbf{A}_s$$

$$+ (\Delta_\perp\mathbf{A}_s\cdot\mathbf{A}_s^*)\delta\mathbf{A} - (\Delta_\perp\delta\mathbf{A}^*\cdot\mathbf{A}_s)\mathbf{A}_s$$

$$- (\Delta_\perp\mathbf{A}_s^*\cdot\delta\mathbf{A})\mathbf{A}_s - (\Delta_\perp\mathbf{A}_s^*\cdot\mathbf{A}_s)\delta\mathbf{A}\}.$$

Introducing the real and imaginary parts of the x-components of the perturbation $\delta A_x = \delta A_r + i\delta A_i$ and the inhomogeneity $\delta Q_x = \delta Q_r + i\delta Q_i$, we write the linear equation (4.2) in the form

$$2\Gamma\frac{\partial\delta A_r}{\partial z} + L_0\delta A_i = \delta Q_i,$$

$$-2\Gamma\frac{\partial\delta A_i}{\partial z} + L_1\delta A_r = \delta Q_r, \tag{4.4}$$

$$2i\Gamma\frac{\partial\delta A_y}{\partial z} + L_0\delta A_y = \delta Q_y,$$

and also in the matrix form

$$2\Gamma\frac{\partial\delta\mathbf{A}}{\partial z} = \mathbf{M}\delta\mathbf{A}. \tag{4.5}$$

Here

$$\delta\mathbf{A} = \begin{pmatrix}\delta A_r\\ \delta A_i\\ \delta A_y\end{pmatrix},$$

we have

$$\mathbf{M}\delta\mathbf{A} = \begin{pmatrix} -L_0\delta A_i + \delta Q_i\\ L_1\delta A_r - \delta Q_r\\ iL_0\delta A_y - i\delta Q_y\end{pmatrix},$$

$$A_{s0} = \mu\sqrt{\frac{\varepsilon_0}{\varepsilon_2}}F_0(\sqrt{\Gamma^2 - k^2}\rho), \tag{4.6}$$

$$L_0 = \Delta_\perp - (\Gamma^2 - k^2) + k^2\frac{\varepsilon_2}{\varepsilon_0}A_{s0}^2,$$

$$L_1 = L_0 + 2k^2\frac{\varepsilon_2}{\varepsilon_0}A_{s0}^2.$$

The form of the operators $L_{0,1}$ in (4.6) corresponds to the dimensional form of the relationships (3.7) and (3.10) with a choice of linearly polarized unperturbed soliton (3.6) when the control equations for the perturbations are simplified considerably. The properties of solutions of the equations corresponding to the linearized nonlinear Schrödinger equation ($\delta\mathbf{Q} \longrightarrow 0$) are also important:

$$2\Gamma\frac{\partial\delta A_{r0}}{\partial z} + L_0\delta A_{i0} = 0,$$

$$-2\Gamma\frac{\partial\delta A_{i0}}{\delta z} + L_1\delta A_{r0} = 0, \tag{4.7}$$

$$2i\Gamma\frac{\partial\delta A_{y0}}{\partial z} + L_0\delta A_{y0} = 0,$$

or in matrix form

$$2\Gamma\frac{\partial\delta\mathbf{A}_0}{\partial z} = \mathbf{M}_0\delta\mathbf{A}_0,$$

$$\mathbf{M}_0 = \begin{pmatrix} 0 & -L_0 & 0\\ L_1 & 0 & 0\\ 0 & 0 & iL_0\end{pmatrix}. \tag{4.8}$$

We write the simplest solutions of the system (4.4) [or (4.5)] and (4.7) [or (4.8)]. First, it follows from the well-known symmetry with respect to the phase shift and propagation constant of a steady-state soliton that the linearized equations have two solutions. The first of these

$$\delta\mathbf{A}_\varphi = \begin{pmatrix}0\\ A_{sx}\\ iA_{sy}\end{pmatrix},\quad \delta\mathbf{A}_\varphi^{(0)} = \begin{pmatrix}0\\ A_{s0}\\ 0\end{pmatrix}, \tag{4.9}$$

corresponds to the eigenvector of the matrix $\mathbf{M}$ ($\mathbf{M}_0$) with zero eigenvalue

$$\mathbf{M}\delta\mathbf{A}_\varphi = 0,\quad \mathbf{M}_0\delta\mathbf{A}_\varphi^{(0)} = 0. \tag{4.10}$$

The second solution

$$\delta\mathbf{A}_\Gamma = \begin{pmatrix}\frac{1}{2\Gamma}\frac{\partial A_{sx}}{\partial\Gamma}\\ 0\\ \frac{i}{2\Gamma}\frac{\partial A_{sy}}{\partial\Gamma}\end{pmatrix},\quad \delta\mathbf{A}_\Gamma^{(0)} = \begin{pmatrix}\frac{1}{2\Gamma}\frac{\partial A_{s0}}{\partial\Gamma}\\ 0\\ 0\end{pmatrix}, \tag{4.11}$$

is not an eigenvalue but a root:

$$\mathbf{M}\delta\mathbf{A}_\Gamma = \delta\mathbf{A}_\varphi,\quad \mathbf{M}^2\delta\mathbf{A}_\Gamma = 0,$$
$$\mathbf{M}_0\delta\mathbf{A}_\Gamma^{(0)} = \delta\mathbf{A}_\varphi^{(0)},\quad \mathbf{M}_0^2\delta\mathbf{A}_\Gamma^{(0)} = 0. \tag{4.12}$$

Finally, the symmetry with respect to rotation of the x, y axes yields an eigenvector with zero eigenvalue:

$$\delta\mathbf{A}_{\text{rot}} = C_{\text{rot}}\begin{pmatrix} -A_{sy} \\ 0 \\ A_{sx} \end{pmatrix}, \quad \delta A_{\text{rot}}^{(0)} = C_{\text{rot}}\begin{pmatrix} 0 \\ 0 \\ A_{sx0} \end{pmatrix}, \tag{4.13}$$
$$\mathbf{M}\delta\mathbf{A}_{\text{rot}} = 0, \quad \mathbf{M}_0\delta\mathbf{A}_{\text{rot}}^{(0)} = 0.$$

Subsequently, disregarding the rotational transformation of the axes, we set $C_{\text{rot}} = 0$.

The following two solutions are specific to the linearized nonlinear Schrödinger equation [equations (4.7) or (4.8)] whereas they are absent for the more general form of the equations (4.4) or (4.5). The invariance of the nonlinear Schrödinger equation to a focusing transformation determined by Talanov [20] thus yields the solution (4.7) [21]:

$$\delta A_{r0} = \frac{\partial A_{s0}}{\partial \Gamma}z, \quad \delta A_{i0} = \frac{1}{2\Gamma}\left(-\frac{\rho^2}{4} + \Gamma z^2\right)A_{s0}, \tag{4.14}$$
$$\delta A_{y0} = 0.$$

The corresponding vector

$$\delta\mathbf{A}_f = \begin{pmatrix} 0 \\ \dfrac{1}{8(k^2-\Gamma^2)}A_{s0}\rho^2 \\ 0 \end{pmatrix} \tag{4.15}$$

is also a root:

$$\mathbf{M}_0\delta\mathbf{A}_f = \delta\mathbf{A}_{\Gamma 0}, \quad \mathbf{M}_0^3\delta\mathbf{A}_f = 0. \tag{4.16}$$

The last of the solutions of the linearized equations (4.7) required for the following analysis has the form [18, 22]

$$\delta A_{r0} = a(\rho) + b(\rho)z^2, \quad \delta A_{i0} = c(\rho)z + d(\rho)z^3, \tag{4.17}$$
$$\delta A_{y0} = 0.$$

After substituting (4.17) into (4.7), we find

$$d = A_{s0}, \quad c = -\frac{3}{\Gamma^2 - k^2}A_{s0}\rho^2, \quad b = 3\frac{\partial A_{s0}}{\partial \Gamma}. \tag{4.18}$$

The function $a(\rho)$ is defined as the only finite axisymmetric solution of the equation [the conditions for solubility are satisfied because of the axial symmetry of the right-hand side (4.19)]

$$L_1 a = -\frac{6\Gamma^3}{\Gamma^2 - k^2}A_{s0}\rho^2, \tag{4.19}$$

or

$$\frac{d^2a}{d\rho^2} + \frac{1}{\rho}\frac{da}{d\rho} - (\Gamma^2 - k^2)a$$
$$+ 3k^2\frac{\varepsilon_2}{\varepsilon_0}A_{s0}^2 a = -\frac{6\Gamma^3}{\Gamma^2 - k^2}A_{s0}\rho^2. \tag{4.20}$$

We do not require the specific form of the function $a(\rho)$. The perturbation vector corresponding to this solution is also a root:

$$\mathbf{M}_0\delta\mathbf{A}_a = \delta\mathbf{A}_f, \quad \mathbf{M}_0^4\delta\mathbf{A}_a = 0,$$
$$\delta\mathbf{A}_a = \begin{pmatrix} \dfrac{1}{48\Gamma^3}a(\rho) \\ 0 \\ 0 \end{pmatrix}. \tag{4.21}$$

These solutions exhaust the family of localized axisymmetric solutions of the linearized nonlinear Schrödinger equation with a zero eigenvalue.

Using the small nonparaxial parameter (2.9), we can write expansions of the matrix $\mathbf{M}$, the eigenvector $\delta\mathbf{A}_\varphi$ and the root vector $\delta\mathbf{A}_\Gamma$ in the form

$$\mathbf{M} = \mathbf{M}_0 + \mu^2\mathbf{M}_2 + \mu^4\mathbf{M}_4 + \ldots,$$
$$\delta\mathbf{A}_\varphi = \delta\mathbf{A}_\varphi^{(0)} + \mu^2\delta\mathbf{A}_\varphi^{(2)} + \ldots, \tag{4.22}$$
$$\delta\mathbf{A}_\Gamma = \delta\mathbf{A}_\Gamma^{(0)} + \mu^2\delta\mathbf{A}_\Gamma^{(2)} + \ldots$$

Terms with a zero index are determined by the nonlinear Schrödinger equation and are given in relationships (4.8), (4.9), and (4.11). The remaining terms of the expansion are obtained using the expansion for the field of a steady-state soliton determined in Section 3.

5. STABILITY AND OSCILLATIONS OF PERTURBED SOLITONS

We shall now find the eigenvalue of the matrix operator $\mathbf{M}$ which goes to zero in the limit $\mu \longrightarrow 0$ (nonparaxial soliton limit). For this we shall seek the eigensolution of the linearized equation (4.5) in the form

$$\delta\mathbf{A} = \mathbf{\Psi}(x, y)e^{\mu\gamma z/2\Gamma}. \tag{5.1}$$

Here we introduce the unknown eigenvalue

$$\mu\gamma = \mu\gamma_1 + \mu^2\gamma_2 + \mu^3\gamma_3 + \ldots \tag{5.2}$$

and the eigenvector $\mathbf{\Psi}$ which obey the following equation derived from (4.5)

$$\mathbf{M\Psi} = \mu\gamma\mathbf{\Psi}. \tag{5.3}$$

The expansion of the eigenvector $\mathbf{\Psi}$ can be conveniently expressed in the following form:

$$\mathbf{\Psi} = \delta\mathbf{A}_\varphi + \mu\gamma\delta\mathbf{A}_\Gamma + \mu^2\mathbf{\Psi}_2 + \mu^3\mathbf{\Psi}_3 + \mu^4\mathbf{\Psi}_4 + \ldots \tag{5.4}$$

The first two terms on the right-hand side (5.4) are determined by equations (4.9) and (4.11). Substituting into (5.3) the expansions of the corresponding quantities in powers of the small nonparaxial parameter μ and equating terms of the same order with respect to this parameter, we find in the second order in μ [equations of lower orders are automatically satisfied given the choice made in (5.4)]

$$\mathbf{M}_0\boldsymbol{\Psi}_2 = \gamma_1^2\delta\mathbf{A}_{\Gamma 0}. \tag{5.5}$$

Taking into account (4.16) it then follows that

$$\boldsymbol{\Psi}_2 = \gamma_1^2\delta\mathbf{A}_f. \tag{5.6}$$

In the third order we have

$$\begin{aligned}\mathbf{M}_0\boldsymbol{\Psi}_3 &= 2\gamma_1\gamma_2\delta\mathbf{A}_{\Gamma 0} + \gamma_1\boldsymbol{\Psi}_2\\ &= 2\gamma_1\gamma_2\delta\mathbf{A}_{\Gamma 0} + \gamma_1^3\delta\mathbf{A}_f,\end{aligned} \tag{5.7}$$

so that taking into account (4.16) and (4.21) we have

$$\boldsymbol{\Psi}_3 = 2\gamma_1\gamma_2\delta\mathbf{A}_f + \gamma_1^3\delta\mathbf{A}_a. \tag{5.8}$$

Finally, in the fourth order we have

$$\begin{aligned}\mathbf{M}_0\boldsymbol{\Psi}_4 + \mathbf{M}_2\boldsymbol{\Psi}_2 &= \gamma_1^2\delta\mathbf{A}_{\Gamma 2}\\ + (\gamma_2^2 + 2\gamma_1\gamma_3)&\delta\mathbf{A}_{\Gamma 0} + \gamma_1\boldsymbol{\Psi}_3 + \gamma_2\boldsymbol{\Psi}_2,\end{aligned} \tag{5.9}$$

or allowing or (5.6) and (5.8)

$$\begin{aligned}\mathbf{M}_0\boldsymbol{\Psi}_4 + \gamma_1^2\mathbf{M}_2\delta\mathbf{A}_f &= \gamma_1^2\delta\mathbf{A}_{\Gamma 2}\\ + (\gamma_2^2 + 2\gamma_1\gamma_3)\delta\mathbf{A}_{\Gamma 0} &+ 3\gamma_1^2\gamma_2\delta\mathbf{A}_f + \gamma_1^4\delta\mathbf{A}_a.\end{aligned} \tag{5.10}$$

We now introduce the transposed operator matrix $\mathbf{M}_0^\dagger$ and the conjugate three-dimensional vectors $\mathbf{A}_{\varphi 0}^+$ and $\mathbf{A}_{\Gamma 0}^+$ using the relationships

$$\mathbf{M}_0^\dagger = \begin{pmatrix} 0 & L_1 & 0\\ -L_0 & 0 & 0\\ 0 & 0 & iL_0\end{pmatrix}, \tag{5.11}$$

$$\mathbf{M}_0^\dagger\delta\mathbf{A}_{\varphi 0}^+ = 0, \quad \mathbf{M}_0^\dagger\delta\mathbf{A}_{\Gamma 0}^+ = \delta\mathbf{A}_{\varphi 0}^+.$$

In explicit form we have

$$\delta\mathbf{A}_{\varphi 0}^+ = \begin{pmatrix} A_{s0}\\ 0\\ 0\end{pmatrix}, \quad \delta\mathbf{A}_{\Gamma 0}^+ = \begin{pmatrix} 0\\ \dfrac{\partial A_{s0}}{\partial\Gamma}\\ 0\end{pmatrix}. \tag{5.12}$$

We introduce the scalar product of the three-dimensional vectors (4.6) using the relationships

$$\begin{aligned}\langle\mathbf{U}, \mathbf{V}\rangle &= \frac{1}{2\pi}\\ &\times\int_0^{2\pi} d\varphi\int_0^\infty d\rho\rho(U_1V_1 + U_2V_2 + U_3V_3),\end{aligned} \tag{5.13}$$

where we have for the complex (third) components

$$U_3V_3 = \mathrm{Re}U_3\mathrm{Re}V_3 + \mathrm{Im}U_3\mathrm{Im}V_3. \tag{5.14}$$

Then, as a result of the self-adjoint property of the Laplace operator for arbitrary vectors $\mathbf{U}$ and $\mathbf{V}$ the following identity is satisfied

$$\langle\mathbf{U}, \mathbf{M}_0\mathbf{V}\rangle = \langle\mathbf{M}_0^\dagger\mathbf{U}, \mathbf{V}\rangle. \tag{5.15}$$

We now multiply the scalar three-dimensional vector $\boldsymbol{\Psi}_{\varphi 0}^+$ by the left- and right-hand sides of equation (5.10) and equate the products. Here we use the following relationships for the matrix elements:

$$\langle\delta\mathbf{A}_{\varphi 0}^+, \mathbf{M}_0\boldsymbol{\Psi}_4\rangle = \langle\mathbf{M}_0^\dagger\delta\mathbf{A}_{\varphi 0}^+, \boldsymbol{\Psi}_4\rangle = 0, \tag{5.16}$$

$$\begin{aligned}\langle\delta\mathbf{A}_{\varphi 0}^+, \delta\mathbf{A}_{\Gamma 0}\rangle &= \left\langle\begin{pmatrix} A_{s0}\\ 0\\ 0\end{pmatrix}, \begin{pmatrix} \dfrac{1}{2\Gamma}\dfrac{\delta A_{s0}}{\partial\Gamma}\\ 0\\ 0\end{pmatrix}\right\rangle\\ &= \frac{1}{2\Gamma}\int_0^\infty A_{s0}\frac{\partial A_{s0}}{\partial\Gamma}\rho d\rho = 0,\end{aligned} \tag{5.17}$$

$$\begin{aligned}&\langle\delta\mathbf{A}_{\varphi 0}^+, \delta\mathbf{A}_f\rangle\\ &= \left\langle\begin{pmatrix} A_{s0}\\ 0\\ 0\end{pmatrix}, \begin{pmatrix} 0\\ \dfrac{1}{8(k^2-\Gamma^2)}A_{s0}\rho^2\\ 0\end{pmatrix}\right\rangle = 0,\end{aligned} \tag{5.18}$$

$$\begin{aligned}m_{\varphi, a} &= \langle\delta\mathbf{A}_{\varphi 0}^+, \delta\mathbf{A}_a\rangle = \langle\mathbf{M}_0^\dagger\delta\mathbf{A}_{\Gamma 0}^+, \delta\mathbf{A}_a\rangle\\ &= \langle\delta\mathbf{A}_{\Gamma 0}^+, \mathbf{M}_0\delta\mathbf{A}_a\rangle = \langle\delta\mathbf{A}_{\Gamma 0}^+, \delta\mathbf{A}_f\rangle\\ &= -\frac{1}{32\Gamma(\Gamma^2-k^2)}\int_0^\infty\frac{\partial A_{s0}^2}{\partial\Gamma}\rho^3 d\rho,\end{aligned} \tag{5.19}$$

$$m_{\varphi, \Gamma} = \langle\delta\mathbf{A}_{\varphi 0}^+, \delta\mathbf{A}_{\Gamma 2}\rangle = \frac{1}{2\Gamma}\int_0^\infty A_{s0}\frac{\partial\delta A_{s0}}{\partial\Gamma}\rho d\rho, \tag{5.20}$$

and we also introduce the notation

$$m_2 = \langle \delta\mathbf{A}^{+}_{\varphi 0}, \mathbf{M}_2\delta\mathbf{A}_f \rangle. \tag{5.21}$$

We then obtain an equation to determine the square of the eigenvalue in the lowest approximation (see also [17])

$$\gamma_1^2[(m_2 - m_{\varphi,\Gamma}) - m_{\varphi,a}\gamma_1^2] = 0. \tag{5.22}$$

The two zero roots of this equation correspond to the symmetries with respect to the phase shift and the shift of the propagation constant conserved when allowance is made for the nonparaxial property. In accordance with (5.22) two nonzero eigenvalues are split off from these (as a result of a shift of the eigenvalue corresponding to the vectors $\delta\mathbf{A}_f$ and $\delta\mathbf{A}_a$ for the nonlinear Schrödinger equation). In order to determine these it is convenient to calculate the matrix elements appearing in (5.22) by going over to the functions $F_{0,1}(r)$ introduced earlier in the integrand expressions (see Fig. 1 and Appendix). We then finally obtain

$$\left(\frac{\tilde{\gamma}}{k}\right)^2 = \frac{1}{k^2}\left(\frac{\gamma_1}{2\Gamma}\right)^2 = -43.35\left(\frac{\Gamma-k}{k}\right)^3. \tag{5.23}$$

In accordance with (5.23), a weakly nonparaxial soliton is stable ($\Gamma > k$). However, the imaginary nature of the eigenvalue γ implies that an "internal mode" occurs whose field distribution as given by (5.4) is close to the soliton field (and is phase shifted by $\pi/2$). The longitudinal period of the oscillations of the perturbed field $2\pi/\tilde{\gamma}$ increases without bound in the paraxial soliton limit $\mu \longrightarrow 0$ ($\Gamma \longrightarrow k$). As a result of the weak radiation damping of these internal modes [23–27] it is difficult for a steady-state soliton to be established under these conditions (an anomalously long nonlinear medium is required).

6. CONCLUSIONS

We have therefore demonstrated for the first time that nonparaxial solitons of electromagnetic radiation are stable in a medium with a Kerr nonlinearity. These solitons may occur at the final stage of self-focusing of supercritical-power radiation. Since the maximum intensity of these weakly nonparaxial solitons is low, the Kerr nonlinearity will be the dominant mechanism (no competing mechanisms of nonlinearity exist at low intensities). We reemphasize that the fields of these solitons do not possess axial symmetry and the polarization structure of the radiation strictly corresponds to elliptic polarization which varies over the cross section. It is important to allow for the vector nature and the nontrivial polarization structure of electromagnetic radiation solitons since their scalar description cannot be quantitative. This approach can not only demonstrate the stability but can also be used to determine the corresponding quantitative characteristic (the eigenvalue γ). This value characterizes the internal modes of these solitons, i.e., the natural modes of small perturbations in an effective light guide induced by the "strong field" of a soliton in a nonlinear medium.

The results obtained here for weakly nonparaxial optical solitons serve as an additional argument to support the "needles of light" identified in [15], i.e., strongly nonparaxial solitons of width less than the wavelength of the radiation in a linear medium. We also note that since the nonlinear Schrödinger equation describes an extremely wide range of phenomena of various physical nature, this approach to analyze the perturbed Schrödinger equation may not be confined merely to optical problems.

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APPENDIX

We shall calculate the matrix elements appearing in (5.22) which can be reduced to single integrals (over the radial coordinate) of the functions $F_{0,1}(r)$ and their derivatives. Graphs of the functions are plotted in Fig. 1 and their integrals were calculated numerically:

$$m_{\varphi,\Gamma} = \frac{1}{2\Gamma}\int_0^\infty A_{s0}\frac{\partial\delta A_{s0}}{\partial\Gamma}\rho d\rho = \frac{1}{2k^4}\frac{\varepsilon_0}{\varepsilon_2}p_2,$$

$$p_2 = \int_0^\infty\left[3F_0(r)F_1(r) + F_0(r)\frac{dF_1(r)}{dr}\right]rdr = 5.821,$$

$$m_{\varphi,a} = -\frac{1}{32\Gamma(\Gamma^2-k^2)}\int_0^\infty\frac{\partial A_{s0}^2}{\partial\Gamma}\rho^3 d\rho = -\frac{\varepsilon_0}{\varepsilon_2}\frac{1}{k^2}$$

$$\times\frac{1}{32(\Gamma^2-k^2)}\frac{\partial}{\partial\Gamma}\left[(\Gamma^2-k^2)\int_0^\infty F_0^2(\sqrt{\Gamma^2-k^2}\rho)\rho^3 d\rho\right]$$

$$= \frac{\varepsilon_0}{\varepsilon_2}\frac{1}{k^2}\frac{1}{16(\Gamma^2-k^2)^2}p_3,$$

$$p_3 = \int_0^\infty F_0^2(r)r^3 dr = 2.211,$$

$$m_2 = m_{2s} + m_{2k} + m_{2z},$$

$$m_{2s} = \langle \delta\mathbf{A}^{+}_{\varphi 0}, \mathbf{M}_{2s}\delta\mathbf{A}_f \rangle = \frac{1}{16}\frac{\varepsilon_2}{\varepsilon_0}\frac{1}{\Gamma^2 - k^2}$$

$$\times \int_0^\infty A_{s0}\left[\left(\frac{dA_{s0}}{d\rho}\right)^2 A_{s0}\rho^2 + \left(\frac{d}{d\rho} + \frac{1}{\rho}\right)(A_{s0}\rho^2)\right]\rho d\rho$$

$$= \frac{1}{16k^4}\frac{\varepsilon_0}{\varepsilon_2}p_s,$$

$$p_s = \int_0^\infty \left[\left(\frac{dF_0(r)}{dr}F_0(r)r\right)^2\right.$$

$$\left. + F_0(r)\left(\frac{d}{dr} + \frac{1}{r}\right)\left(\frac{dF_0^2(r)}{dr}F_0(r)r^2\right)\right]rdr = -1.608,$$

$$m_{2k} = \langle \delta\mathbf{A}^{+}_{\varphi 0}, \mathbf{M}_{2k}\delta\mathbf{A}_f \rangle = \frac{k^2}{4(\Gamma^2 - k^2)}$$

$$\times \frac{\varepsilon_2}{\varepsilon_0}\int_0^\infty A^3_{s0}\delta A_{sx0}\rho^3 d\rho = \frac{1}{4k^4}\frac{\varepsilon_0}{\varepsilon_2}p_k,$$

$$p_k = \int_0^\infty F_0^{(3)}(r)F_1(r)r^3 dr = 1.926,$$

$$m_{2z} = \langle \delta\mathbf{A}^{+}_{\varphi 0}, \mathbf{M}_{2z}\delta\mathbf{A}_f \rangle = -\frac{1}{32(\Gamma^2 - k^2)}$$

$$\times \int_0^\infty d\rho\rho A_{s0}\left\{\frac{1}{k^2}[\Delta_\perp - (\Gamma^2 - k^2)]\right.$$

$$\times \left[\Delta_\perp - (\Gamma^2 - k^2) + k^2\frac{\varepsilon_2}{\varepsilon_0}A^2_{s0}\right](A_{s0}\rho^2)$$

$$+ \frac{\varepsilon_2}{\varepsilon_0}\left[3A^2_{s0}\Delta_\perp(A_{s0}\rho^2) - 2A^2_{s0}\rho^2\Delta_\perp A_{s0} - (\Gamma^2 - k^2)A^3_{s0}\rho^2\right.$$

$$\left.\left. + k^2\frac{\varepsilon_2}{\varepsilon_0}A^5_{s0}\rho^2\right]\right\} = -\frac{1}{32k^4}\frac{\varepsilon_0}{\varepsilon_2}p_z.$$

$$p_z = \int_0^\infty drrF_0(r)[\Delta_r - 1][\Delta_r - 1 + F_0^2(r)](F_0(r)r^2)$$

$$+ 3\int_0^\infty drrF_0^3(r)\Delta_r(F_0(r)r^2) - 2\int_0^\infty drr^3F_0^3(r)\Delta_r(F_0(r))$$

$$- \int_0^\infty drr^3F_0^4(r) + \int_0^\infty drr^3F_0^6(r) = 14.899,$$

$$\Delta_r = \frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr}.$$

After substituting these values into (5.22) we obtain (5.23).

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SPELL: striction, prolate, Poynting, adjoint