Global Coupling with Time Delay in an Array of Semiconductor Lasers

G. Kozyreff,[1] A. G. Vladimirov,[2] and Paul Mandel[1]

[1]*Optique Nonlinéaire Théorique, Université Libre de Bruxelles, Campus Plaine CP 231, B-1050 Bruxelles, Belgium*

[2]*Physics Faculty, St. Petersburg State University, 198904 St. Petersburg, Russia*

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Synchronization due to a weak global coupling with time delay in a semiconductor laser array is investigated both in the cw and self-pulsing regimes. A generalized form of the Kuramoto phase equations is derived and discussed analytically. The time delay is shown to induce in-phase synchronization in all dynamical regimes. Another form of synchronization is found which leads to local extinction of self-pulsing in the array.

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Many models developed in physics, chemistry, and biology are formulated in terms of weakly coupled identical oscillators. Since signals propagate with a finite velocity, the coupling among the oscillators is, in principle, time delayed. The influence of this delay was studied recently for various models such as pulse-coupled oscillators [1] and phase models [2]. Multistability [3,4] and oscillator death [5] were found to be possible consequences of the delay.

The focus of this Letter is on the nonlinear dynamics of globally coupled oscillators with time delay, illustrated by an array of semiconductor lasers (SCL). Aside from fundamental interest, this subject may also have technological applications. In arrays of semiconductor lasers, synchronizing the lasing elements in phase is of importance in order to obtain a large output power $\propto |\sum_j E_j|^2$ concentrated in a single-lobed far field pattern [6]. To this end, local coupling between the neighboring laser elements via overlapping evanescent fields was considered in [7]. More recently, local and global couplings were compared in the absence of delay [8] with the conclusion that global coupling is more efficient to achieve stationary in-phase operation. A similar result concerning nonstationary regimes has just been found numerically [9] for delayed global coupling via a feedback mirror. This is the situation we analyze in this Letter, for identical SCL.

SCL are known to be extremely sensitive to optical feedback and to undergo a series of instabilities from self-pulsing to coherence collapse as the feedback strength is increased [10]. We show that in the cw regime, characterized by time independent laser intensities, the time delay induces bistability between in-phase and antiphase states by increasing the stability domain of in-phase operation. For larger coupling strength, in-phase cw operation can be destabilized in two ways. First, there is a degenerate Hopf bifurcation to antiphase self-pulsing that exists independently of the delay. Second, if the delay is at least comparable to the relaxation oscillation period of the solitary SCL, another, nondegenerate, Hopf bifurcation exists which leads to in-phase self-pulsing. To study this nonstationary regime, a set of dynamical equations that generalize the Kuramoto model [11] is derived. In these equations, each SCL is a two-frequency oscillator with one phase variable describing oscillations at the optical frequency and a second phase variable related to oscillations in the laser intensities at the relaxation frequency. Our main result, obtained analytically, is that by properly tuning the position of the feedback mirror such an in-phased self-pulsing can be made stable for a broad range of coupling strength, hence preserving the in-phase synchronization in a nonstationary regime. In addition, we show that, if the time delay is large enough, the primary instability of the in-phase cw regime is always the Hopf bifurcation to an in-phase periodic regime. Numerically, in-phase synchronization is seen to persist in quasiperiodic and even in the chaotic regimes in a wide domain of parameter values. Additionally, if the number of elements in the array is odd, we found another form of synchronization, featuring localized extinction of self-pulsing in the array. Similar behavior was reported experimentally and theoretically for two mutually coupled SCL [12].

An array of SCL can be modeled by N coupled Lang-Kobayashi equations [13] written in dimensionless form:

$$\frac{dE_j}{dt} = (1 - i\alpha)Z_jE_j + i\,\frac{\eta}{N}\sum_{n=1}^{N} e^{-i\omega t_D - i\delta_{nj}} \times E_n(t - t_D), \qquad (1)$$

$$\gamma^{-1}\frac{dZ_j}{dt} = P - Z_j - (1 + 2Z_j)|E_j|^2. \qquad (2)$$

In these equations, E_j is the slowly varying envelope of the electric field and Z_j is the carrier excess density of the jth laser. The time unit is the photon cavity lifetime $\tau_p \simeq 2 \times 10^{-12}$ s, $\gamma = \tau_p/\tau_c \simeq 10^{-3}$ is the ratio of the photon to carrier lifetimes, $\alpha \simeq 5$ is the linewidth enhancement factor, and P is the pump parameter which is proportional to the injection current above threshold. The coupling strength between the lasers is characterized by the parameter η and is assumed to be small: $\eta \ll 1$. The phase of the coupling is $-\omega t_D$, where $\omega = \omega_0\tau_p$ is the normalized optical frequency and $t_D = 2L/(c\tau_p)$ is the external cavity round-trip time normalized by τ_p. We consider the most symmetric situation where all of the coupling terms in (1) have identical optical phases: $\delta_{nj} = 0$. This can be approximated experimentally, e.g., by placing a spherical feedback mirror at the focus of a converging lens (for example, mirror radius ~ 1 mm, array size $\sim 500\ \mu\text{m}, L \gtrsim 5$ cm).

The cw solutions of Eqs. (1) and (2) are of the form $E_j(t) = \mathcal{E} \exp[i\phi_j(t)]$ and $Z_j(t) = \mathcal{Z}$. In the in-phase state, $\phi_j = \phi$, whereas antiphase cw states are characterized by $\sum_n \exp(i\phi_n) = 0$. In what follows we will deal only with the antiphase states such that $\phi_n - \phi_j = 2(n - j)k\pi/N$.

We analyze Eqs. (1) and (2) in the domain of parameters $\gamma, \eta \ll 1$ and in the limit $\alpha \gg 1$, as in [14]. It turns out that the qualitative results obtained in that limit are remarkably robust when α is reduced to a more realistic value ($\alpha \simeq 5$). For $\alpha, \alpha\sqrt{\gamma}, \sqrt{\gamma}/\eta \gg 1$, Eqs. (1) and (2) admit solutions of the form

$$E_j = \mathcal{E}\left[1 + \frac{\rho_j}{\alpha}\sin(\Omega t + \theta_j)\right] \times \exp i[\phi_j - \rho_j \sin(\Omega t + \theta_j)],$$

$$Z_j = \mathcal{Z} + \Omega\,\frac{\rho_j}{\alpha}\cos(\Omega t + \theta_j).$$

Here $\Omega = (2\gamma P)^{1/2}$ is the relaxation oscillation frequency of the isolated SCL, which is in the GHz range. Hereafter, we will call ϕ_j and θ_j the optical and relaxation oscillation phases, respectively, of the laser j. With the complex amplitude $z_j = \rho_j \exp(i\theta_j)$, the normalized coupling parameter $K \equiv \alpha\eta\gamma^{-1}(P + 1/2)^{-1}$, the slow time $\tau \equiv \gamma(P + 1/2)t$, and $\tau_D \equiv \gamma(P + 1/2)t_D$, a two-time-scale analysis yields

$$\frac{d\phi_j}{d\tau} = -\frac{K}{N}\sum_{n=1}^{N} \sin(\phi_{nj})J_0(|z_{nj}|), \tag{3}$$

$$\frac{dz_j}{d\tau} = -z_j - \frac{K}{N}\sum_{n=1}^{N} z_{nj}\frac{J_1(|z_{nj}|)}{|z_{nj}|}\cos(\phi_{nj}), \tag{4}$$

where $z_{nj} = z_j - z_n(\tau - \tau_D)\exp(-i\Omega t_D)$, $\phi_{nj} \equiv \phi_n(\tau - \tau_D) - \phi_j - \omega t_D$, and $J_\nu(x)$ are Bessel functions of the first kind. Equations (3) and (4) allow a description of the array dynamics both in the cw and self-pulsing regimes.

Substituting $\rho_j = z_{nj} = 0$ into Eq. (3) produces coupled delayed Kuramoto phase equations [4,11] for the optical phases ϕ_j. These equations, which were studied in [4], can be used to describe the SCL array dynamics below the self-pulsing threshold. We note that the so-called "frustration parameter" in [4] is now ωt_D and is therefore delay dependent. In the cw in-phase state, $\phi_j = \phi = \Delta\omega\tau$, where $\Delta\omega = K\sin(\omega t_D + \Delta\omega\tau_D)$. The multiple solutions of this equation correspond to the external cavity modes [10]. From a linear stability analysis, it follows that the cw in-phased state is stable if $\pi/2 - K\tau_D < \omega t_D - 2\pi n < 3\pi/2 + K\tau_D$ with integer n. For the antiphase cw operation, stability requires $-\pi/2 + K\tau_D/2 < \omega t_D - 2n\pi < \pi/2 - K\tau_D/2$. This is in agreement with the results obtained in [4]. In these two stability conditions, the time delay appears in two well-separated time scales: τ_D and ωt_D. Moving the external mirror by one wavelength changes ωt_D by 4π while τ_D changes negligibly. Hence, these states are bistable over a range of the order of $K\tau_D/2$ for constant coupling strength and variable ωt_D. Furthermore, if $2K\tau_D > \pi$, stable in-phase operation is possible for any value of ωt_D. This is due to the overlap of the stability domains of two successive external cavity modes $\Delta\omega$. In this way, the increase of the time delay favors stable in-phase cw operation of the laser array. However, in the presence of noise, hopping may occur between bistable in-phase and antiphase regimes.

As the coupling strength η is increased, the in-phase cw state of (1) and (2) can undergo two Hopf bifurcations leading to undamped oscillations at a frequency close to Ω. One bifurcation point is at $K = K_{H1} = -2/\cos(\omega t_D + \Delta\omega\tau_D)$. It is $(N - 1)$-fold degenerate and gives rise to antiphase self-pulsing. The emerging solutions correspond to steady state solutions of (3) and (4) and can be written near the bifurcation point as $\phi_j = \phi = \Delta\omega\tau$, $z_j = \rho\exp(2ijk\pi/N)$. The other Hopf bifurcation, which is nondegenerate, leads to in-phase self-pulsing. It is characterized by $\phi_j = \phi = \Delta\omega\tau$, $z_j = \rho\exp(i\Delta\Omega\tau)$ and takes place at $K = K_{H2}$ with

$$K_{H2} = \frac{K_{H1}}{1 - \cos(\Omega t_D + \Delta\Omega\tau_D)}, \tag{5}$$

$$\Delta\Omega = \cot[(\Omega t_D + \Delta\Omega\tau_D)/2]. \tag{6}$$

Equation (6) possesses multiple solutions, each producing a distinct relaxation frequency. The expression for K_{H2} given by (5) coincides with the known result for a solitary SCL with optical feedback [15] in the limit $\alpha \gg 1$. As a function of the feedback delay, the critical normalized coupling strengths K_{H2} and K_{H1} have minima. This occurs for $\cos(\nu t_D + \Delta\nu\tau_D) = -1$, where ν can be either ω or Ω. However, if $t_D \ll (2\gamma P)^{-1/2}$, then $\cos(\Omega t_D + \Delta\Omega\tau_D) \simeq 1$ and K_{H2} diverges. Such a restriction on the magnitude of the delay does not exist for K_{H1}, provided that t_D is larger than a few optical periods. Thus, while the antiphase instability exists for very small delays, the in-phase instability requires a relatively large delay. In practice, this requirement is met if the external mirror is placed at a distance equal to or greater than $L = 0.5c(\tau_p\tau_c)^{1/2} \sim 1$ cm, where c is the speed of light. If this condition is fulfilled, the relative position of the two Hopf bifurcations K_{H1} and K_{H2} can be controlled by the position of the feedback mirror: $K_{H2} < K_{H1}$ if $\cos(\Omega t_D + \Delta\Omega\tau_D) < 0$. In this case the in-phase cw regime is destabilized via the nondegenerate Hopf bifurcation at $K = K_{H2}$ leading to in-phase self-pulsing solution. If the emerging periodic solution is stable the in-phase synchronization is maintained.

In the framework of the generalized Kuramoto equations (3) and (4) the stability of the in-phase periodic solution bifurcating at K_{H2} can be studied analytically. Because this solution lies in the synchronization manifold $\{\rho_j = \rho, \theta_j = \theta, \phi_j = \phi\}$, Eqs. (3) and (4) reduce to the single laser problem. The amplitude of the in-phase solution bifurcating at K_{H2} is thus determined by [14,16]: $K = -\tilde{\rho}/[(1 - \cos\varphi)\cos(\psi)J_1(\tilde{\rho})]$, where $\varphi \equiv \Omega t_D + \Delta\Omega\tau_D$, $\tilde{\rho} \equiv 2\rho\sin(\varphi/2)$, $\psi \equiv \omega t_D + \Delta\omega_{H2} \times \tau_D$, and the transcendental equation for the optical frequency shift $\Delta\omega_{H2}$ is now $\tilde{\rho}$ dependent. Linearization of Eqs. (3) and (4) around this solution yields stability conditions for perturbations transverse to the synchronization manifold. If $K_{H2} < K_{H1}$, the in-phase branch of periodic solutions is stable in the vicinity of the self-pulsing threshold. It is, however, destabilized at a second threshold K_ϕ. The condition $K = K_\phi$ defines a $(N - 1)$-fold degenerate steady state bifurcation of (3) and (4) which is a secondary bifurcation of the in-phase periodic solutions for (1) and (2). This bifurcation leads to a gradual desynchronization of the optical phases ϕ_j. On the contrary, if $K_{H2} > K_{H1}$, the cw regime is already unstable at the Hopf bifurcation $K = K_{H2}$ and the in-phase periodic solution emerging from this point is also unstable. However, the laser array can be stabilized in the in-phase state through a $(N - 1)$-fold degenerate Hopf bifurcation of (3) and (4) at $K = K_\theta$. It corresponds to a degenerate secondary Hopf bifurcation in the original laser equations by which the relaxation phases θ_j desynchronize. By further increasing K, the laser array again loses in-phase synchronization at $K = K_\phi > K_\theta$ (see Fig. 1). It is seen in Fig. 1 that in this case in-phase and antiphase self-pulsing can coexist.

Unlike the Hopf bifurcation at $K = K_{H2}$ which gives rise to a single periodic solution, the degenerate bifurcation at $K = K_{H1}$ produces multiple antiphase periodic solutions [17]. The stability properties of these solutions require separate consideration. In this Letter, we present only numerical results. If $K_{H1} < K_{H2}$, antiphase states of the type $\theta_n - \theta_j = 2(n - j)k\pi/N$ are observed near the self-pulsing threshold K_{H1}. Most often the array splits up into two equally populated self-pulsing clusters. In each cluster, the SCL are in phase while the intensities of SCL in different clusters oscillate with a relaxation phase shift equal to π. If the number of lasers is odd, the laser that does not belong to either clusters is cw. Such a behavior is described by Eqs. (3) and (4), with $z_1 = 0$, $z_{j>1} = (-1)^j z$, $\phi_{j>1} = \Delta\omega_{H1} \times \tau$, and $\phi_1 = \Delta\omega_{H1} \times \tau - \delta\phi$, where $\Delta\omega_{H1}$ and $\delta\phi$ are determined self-consistently. This solution emerges at $K = K_{H1}$ together with the other antiphase self-pulsing solutions (see Fig. 1). In the absence of local coupling, the initial conditions determine which laser is cw. This results from the symmetry of (1) and (2) with respect to permutations of the laser indices. If this permutation symmetry is broken, the position of the cw laser can be determined by the boundary conditions independently of the initial state. In particular, we perturbed Eq. (1) with a small nearest-neighbor coupling term $i\chi(E_{j-1} + E_{j+1})$ with $j = 1, \ldots, N$ and $E_0 = E_{N+1} = 0$. We found that the central laser emits cw and that each cluster is entirely located on each side of the central laser. In this case the cw laser can be regarded as a discrete domain wall in which a sudden relaxation phase jump of π occurs.

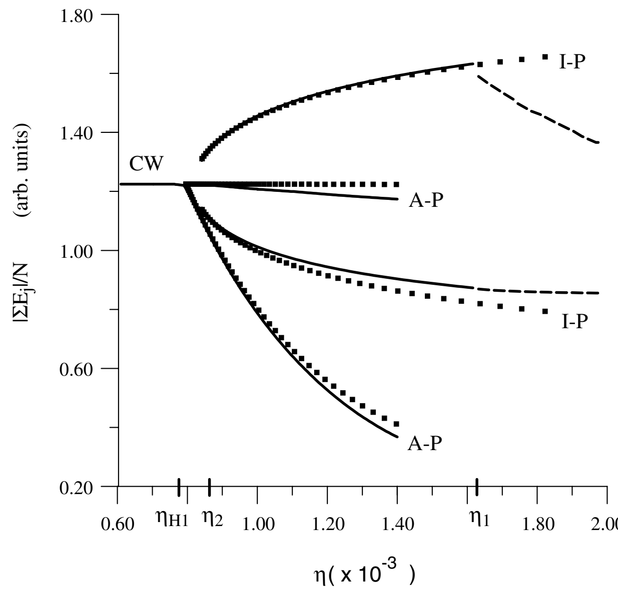

FIG. 1. Numerical bifurcation diagram computed for Eqs. (1) and (2) with $N = 5$, $\alpha = 5$, $\gamma = 0.001$, $P = 1.5$, $\omega t_D - 2n\pi = 3.0$, and $t_D = 91.7$. This corresponds to $\Omega t_D \simeq 5.02$ and $\tau_D \simeq 0.18$. Solid lines: numerical results. Dotted lines: analytical results computed from Eqs. (3) and (4). CW, stable cw in-phase state; I-P: in-phase self-pulsing; A-P, antiphase self-pulsing where one laser is cw. The cw solution is destabilized by a degenerate Hopf bifurcation at η_{H1} resulting in a decrease in the average total field. The threshold η_1 (η_2) corresponds to K_ϕ (K_θ).

If L is increased to $0.5c\tau_c \sim 30$ cm, τ_D becomes $\mathcal{O}(1)$. This changes qualitatively the array dynamics. The solutions to (6) become more and more closely spaced. For $\tau_D > \pi/2$, there is at least one relaxation frequency such that $K_{H2}(\Delta\Omega) < K_{H1}$. Hence, in this case the instability of the in-phase cw regime always leads to in-phase self-pulsing. Furthermore, a new secondary Hopf bifurcation appears within the synchronization manifold, above which the SCL intensities become in-phase quasiperiodic. According to the numerical simulations [18] of Eqs. (1) and (2) with delays $\tau_D = \mathcal{O}(1)$, this in-phase secondary bifurcation always precedes the desynchronizing one at $K = K_\phi$. The in-phase synchronization is thus maintained by the new instability within the synchronization manifold. Further increase of the coupling parameter gives rise to synchronized chaotic intensities (see Fig. 2).

In conclusion, we have derived a generalized set of Kuramoto phase equations (3) and (4) to study an array

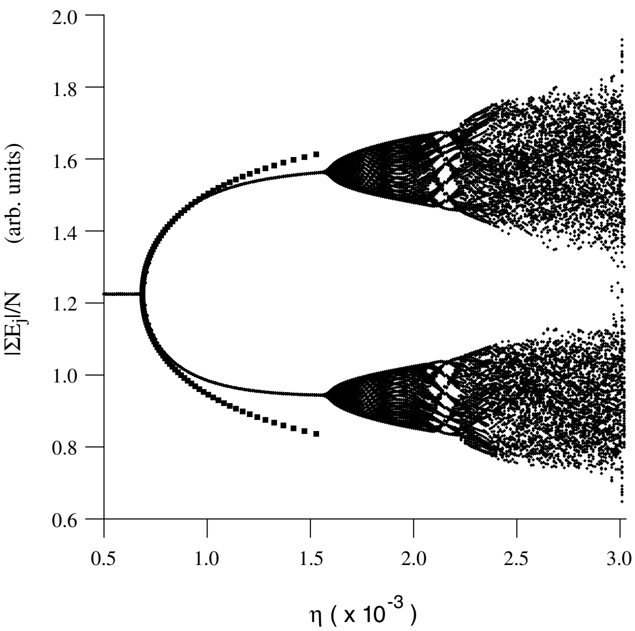

FIG. 2. Same parameter values as Fig. 1 but with $N = 4$ and $t_D = 917.06$, corresponding to $\Omega t_D \simeq 50.23$ and $\tau_D \simeq 1.83$. In cw, self-pulsing, quasiperiodic, and chaotic regimes, only in-phase synchronization is observed.

of globally coupled semiconductor lasers. This allows one to discuss analytically the dynamics below and above the self-pulsing threshold. Above this threshold, the self-pulsing elements of the array can be synchronized in phase by properly tuning the time delay on three different time scales τ_D, Ωt_D, and ωt_D simultaneously. To increase τ_D favors in-phase synchronization by increasing the stability domain of the existing solutions as well as by producing new branches of nonstationary in-phase solutions via in-phase instabilities. This process leads eventually to synchronized chaos. If the external cavity length L is such that $\tau_D \ll 1$, Ωt_D and ωt_D must be close to a resonance for the in-phase Hopf bifurcation to exist and to be the first instability ($K_{H2} < K_{H1}$). If, however, $K_{H1} < K_{H2}$, in-phase self-pulsing may still appear but for larger coupling strength. In this case this in-phase self-pulsing regime is bistable with antiphase regimes emerging at K_{H1}. It loses stability again for larger coupling strength. Among the types of antiphase synchronization that can be observed if $K_{H1} < K_{H2}$, we have described a particular state characterized by the extinction of sustained relaxation oscillations of a single laser in the array. Taking into account additional nearest-neighbor coupling that may arise from the proximity of the SCL, such a state would be a discrete domain wall in the laser array. Finally, we checked numerically that the synchronization properties reported in this Letter persist in an array of slightly nonidentical SCL with a weak nearest-neighbor coupling and a slightly asymmetric global coupling in (1): $0 < |\delta_{nj}| < 0.1$. For example, with $\eta = 10^{-3}$, $\chi = 10^{-4}$, variations $\delta\alpha = 10^{-1}$ around α, and $\delta P = 10^{-3}$ around P, both in-phase synchronization and local extinction of self-pulsing are still observed. The essential dynamics is, therefore, captured by the consideration of identical elements as with the Kuramoto model with delay [4].

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