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Stable autosolitons in dispersive media with saturable gain and absorption

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Abstract

We introduce the simplest one-dimensional model of a dispersive optical medium with saturable dissipative nonlinearity and filtering (dispersive loss) which gives rise to stable solitary pulses (autosolitons). In the particular case when the dispersive loss is absent, the same model may also be interpreted as describing a stationary field in a planar optical waveguide with uniformly distributed saturable gain and absorption. In a certain region of the model's parameter space, two coexisting solitary-pulse solutions are found numerically, one of which may be stable. Solving the corresponding linearized eigenvalue problem, we identify stability borders for the solitary pulses in their parametric plane. Beyond one of the borders, the symmetric pulse is destroyed by asymmetric perturbations, and at the other border it undergoes a Hopf bifurcation, which may turn it into a breather. © 2000 Elsevier Science B.V. All rights reserved.

1. Introduction

'Autosolitons' are robust localized pulses in models combining conservative and dissipative nonlinear and dispersive terms [1]. They occur and find important applications in nonequilibrium plasmas [1,2] and semiconductors [1], in hydrodynamics (Poiseille flow [3] and traveling-wave convection [4–6]), and in nonlinear optics [7–14].

The simplest model which gives rise to solitary pulses that may be interpreted as autosolitons is the cubic complex Ginzburg–Landau (GL) equation,

$$E_z = E + (1 - ic_1)E_{\tau\tau} - (1 + ic_2)|E|^2 E, \qquad (1)$$

which is written in the standard 'optical' notation [19], i.e., E is the local amplitude of the electromagnetic wave, z (to be treated as the evolutional variable) is the propagation distance, and $\tau = t - z/V_{gr}$ is the 'local time', t and V_{gr} being the physical time and the mean group velocity of the carrier wave. Further, c_1 and c_2 are coefficients of the (chromatic) dispersion and (Kerr) nonlinearity, while the terms the coefficients in front of which are normalized to be 1 account for, respectively, linear gain, dispersive

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losses (spectral filtering), and nonlinear losses (two-photon absorption). Eq. (1) always has a single exact solitary-pulse solution [2,3]; however, this solution is always unstable, as its background, the trivial solution $E=0$, is obviously unstable because of the presence of the linear gain.

The simplest possibility to modify the model so that to let it generate *stable* solitary pulses is to convert it into the quintic GL equation [20,21],

$$E_z = -E + (1-ic_1)E_{\tau\tau} + (1+ic_2)|E|^2E - (\Gamma + ic_3)|E|^5E, \tag{2}$$

where the linear gain and cubic loss are replaced, respectively, by linear loss and cubic gain, $\Gamma > 0$ is the coefficient of quintic loss, and c_3 accounts for a possible quintic correction to the Kerr effect. In this case, the trivial solution is stable, and two solitary-pulse solutions, one unstable and one stable, may coexist at fixed values of all the parameters in Eq. (2).

However, the quintic GL equation is a phenomenological model, and it would be very desirable to find more realistic models allowing for the existence of stable solitary pulses. One possibility is to consider a model of a *dual-core* nonlinear optical fiber, in which the linear gain, dispersion, filtering, and Kerr (purely cubic) nonlinearity are present in one (active) core, while the other (passive) one, linearly coupled to the active core, has only linear loss [9,10]. Thus, the model consists of a cubic GL equation linearly coupled to the second, purely linear, equation. It is easy to select parameters of the model so that to provide for the stability of the trivial solution. Then, two *exact* solitary-pulse solutions (autosolitons) can be found, following the pattern of the exact solution to the cubic GL equation, and direct numerical simulations clearly show that one of the two exact solitary pulses may be stable [9,10]. The simulations have revealed nontrivial autosoliton's stability borders in the parameter space of the model ('nontrivial' implies that the borders are different from obvious stability conditions for the zero background). Moreover, interactions between the stable autosolitons have also been simulated in detail in this model [11].

2. The model

In this work, we aim to demonstrate that stable optical autosolitons can also be easily found in a model similar to that introduced in Ref. [12], which combines *saturable* gain and *saturable* absorption in a lasing medium, while the Kerr nonlinearity may be completely neglected. Thus, the nonlinearity in the model is purely dissipative. The gain and loss coefficients in the model are to be chosen so that its linearized version has no gain, in order to provide for the stability of the zero background. The presence of the linear loss suggests that linear dispersive loss, i.e., the filtering (diffusion-like) term should also be included.

All the ingredients of the saturable model discussed above are dissipative (considering gain as negative dissipation). However, from the experience accumulated in the studies of the quintic GL equation in the strong-dissipation limit [15–17], it is known that, while solitary-pulse solutions may exist in a purely dissipative model, they can never be stable [18]. The presence of one, at least, conservative term (which may enter with a small coefficient [15–17]) is necessary to provide for the stability of the corresponding autosoliton. In optical media, this role may be very naturally played by the dispersion.

Thus, the model combining saturable gain, saturable absorption, dispersion and (not necessarily) filtering takes the form (cf. similar models for the lasers with saturable absorbers [12–14,22]):

$$E_z - ie^{-i\theta}E_{\tau\tau} = \left(-1 + \frac{g_0}{1+I/I_g} - \frac{a_0}{1+I/I_a}\right)E,$$
$$I \equiv |E|^2, \tag{3}$$

where g_0 and a_0 are positive gain and absorption coefficients in the linear approximation, and I_g and I_a are saturation intensities for the gain and absorption (the linear loss coefficient is normalized to be 1, cf. Eq. (2)). The real coefficient θ characterizes a relation between the dispersion and filtering (diffusion) coefficients, which are proportional, respectively, to $\cos\theta$ and $\sin\theta$. By means of an obvious rescaling, we may set $I_a = 1$, then all the remaining

parameters are fully independent, and their number cannot be further reduced.

The underlying condition of the stability of the zero background ($E=0$) takes the form

$$g_0 < 1 + a_0 , \tag{4}$$

which must be supplemented by $g_0 > 1$, as otherwise the model can never provide for an effective gain. The positiveness of the effective filtering coefficient imposes another necessary condition, $0 \leqslant \theta \leqslant \pi$. In fact, the range of the parameter θ can be restricted to

$$0 \leqslant \theta < \pi/2 , \tag{5}$$

as the region $\pi/2 < \theta \leqslant \pi$ can be mapped into $0 \leqslant \theta < \pi/2$ by means of complex conjugation, $E \to E^*$. The value $\theta = \pi/2$ is excluded, as it corresponds to the purely dissipative model that cannot give rise to stable pulses.

In the particular case $\theta = 0$ (no filtering), the same model with the temporal variable τ replaced by the transverse coordinate x has an alternative interpretation in nonlinear optics: it may describe a stationary field in a planar waveguide [19] with the saturable gain and absorption uniformly distributed in it. In that case, the term iE_{xx} accounts for the diffraction in the waveguide.

3. Autosolitons and their stability in the saturable model

An autosoliton (solitary-pulse) solution to Eq. (3) is sought for in an obvious form, $E(z,\tau) = \exp(i\alpha z) \cdot \mathscr{E}_0(\tau)$, where α is the propagation constant, and $\mathscr{E}_0(\tau)$ is a complex *even* function vanishing at $|\tau| \to \infty$ and determined by an equation

$$\frac{\mathrm{d}_0^2 \mathscr{E}}{\mathrm{d}\tau^2} = ie^{i\theta}\left(-1 + i\alpha + \frac{g_0}{1+|\mathscr{E}_0|^2/I_\mathrm{g}} - \frac{a_0}{1+|\mathscr{E}_0|^2/I_\mathrm{a}} \right)\mathscr{E}_0 . \tag{6}$$

As is well known [9,10,15–17], an autosoliton will have no chance to be stable if only *one* solitary-pulse solution exists at given values of the parameters. Indeed, due to the condition (4), the system has a trivial attractor, $E=0$. If there is also a nontrivial attractor in the form of an autosoliton, there must simultaneously exist an unstable solitary-pulse solution that plays the role of a *separatrix* between the attraction basins of the two attractors. It is well known too that the two solitary-pulse solutions may undergo a bifurcation at some critical point, where they merge and disappear, so that no autosoliton exists past this point.

In exact accordance with these expectations, numerical integration of Eq. (6) reveals that solitary pulses exist only in a pair in some parametric region, and do not exist at all in other regions. As a typical illustration, in Fig. 1 we display the autosoliton's propagation constant as a function of θ at fixed values of the other parameters. In the particular case shown in this figure, the autosolitons actually exist at $0 \leqslant \theta < 0.34$, i.e., inside a relatively small part of the formally available region (5), which implies that the saturable gain cannot compensate filtering losses when they are too strong (see also below). Note that, despite the large ratio $I_\mathrm{g}/I_\mathrm{a} = 10$ in the case shown in Fig. 1, we cannot use an approximation with $I_\mathrm{g} = \infty$: as it is follows from a simple consideration of Eq. (3), in this limit the model either can provide no effective gain at all, if $g_0 < 1$, or it will blow up (be unstable at $|E|^2 \to \infty$) in the opposite case. Above a certain threshold value of the linear gain parameter

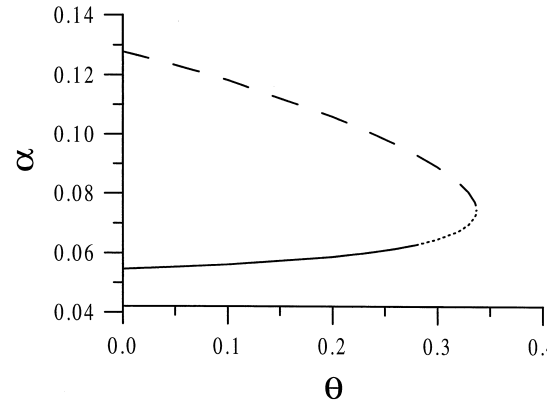

Fig. 1. The autosoliton's propagation constant versus the parameter θ controlling the ratio between the dispersion and filtering (diffusion) coefficients, that may vary in the interval (5). In this figure and in Fig. 2 below, the other parameters are $g_0 = 2.06$, $a_0 = 2$, and $I_\mathrm{g}/I_\mathrm{a} = 10$. Two branches of the autosoliton solutions correspond to the upper and lower curves in this plot. The whole upper branch, and the dotted part of the lower one correspond to unstable autosolitons; see below.

g_0 the upper and lower autosoliton branches do not merge any longer with the increase of θ and, hence, the saddle-node bifurcation shown in Fig. 1 disappears (see Fig. 2). In Fig. 2, which corresponds to $g_0 = 2.11$, T is the limit point for the stable autosoliton branch. When approaching this point from the left the width of the autosoliton solution tends to infinity.

For the study of the stability of the stationary autosolitons, we linearized the full Eq. (3) near the stationary solution, assuming $E(z,\tau) = \exp(i\alpha z) \cdot [\mathscr{E}_0(\tau) + \exp(\gamma z)\mathscr{E}_1(\tau)]$, where $\mathscr{E}_1(\tau)$ is an eigenmode of the infinitesimal perturbation, and γ is the corresponding instability growth rate. Because the unperturbed solution $\mathscr{E}_0(\tau)$ is even, the resultant linear eigenvalue problem for γ can be solved separately for even (symmetric) and odd (antisymmetric) eigenmodes $\mathscr{E}_1$ [23].

Numerical results for the corresponding eigenvalues are presented in Fig. 3 (for the same values of the parameters as in Fig. 1). As is it obvious from the figure, all the upper branch of the stationary solutions from Fig. 1 is unstable against symmetric perturbations, while all the lower branch is stable against them. However, a part of the stationary solutions belonging to the lower branch is destabilized by antisymmetric perturbations (this part of the lower branch is dotted in Fig. 1). It is noteworthy that the antisymmetric eigenmodes related to the upper branch in Fig. 1 undergo a bifurcation at $\theta \approx 0.22$, therefore the corresponding curve $\gamma_{as}(\theta)$ in Fig. 3b has several parts, marked by the numbers 2, 3, and 4 (mark 1 is reserved for the smooth curve $\gamma_{as}(\theta)$ corresponding to the lower-branch autosolitons).

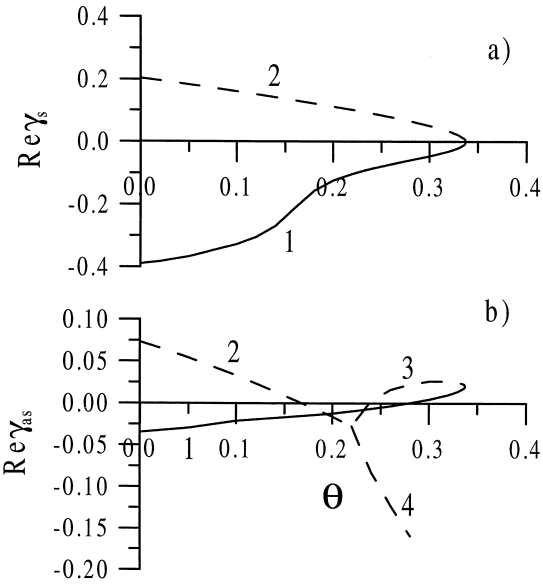

Fig. 3. The instability growth rates γ_s and γ_{as} for the symmetric (a) and antisymmetric (b) infinitesimal perturbations of the stationary autosolitons versus θ. The dashed and continuous curves $\gamma(\theta)$ pertain, respectively, to the upper and lower branches in Fig. 1.

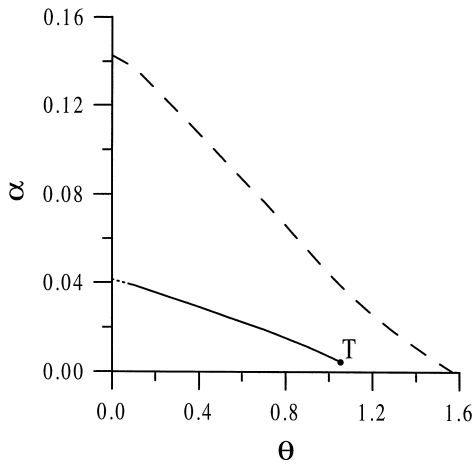

Fig. 2. Same as in Fig. 1, but for $g_0 = 2.11$. The upper autosoliton branch is always unstable and terminates at the point $\alpha = 0$, $\theta = \pi/2$, which corresponds to the unstable autosoliton of the purely dissipative model. The lower branch becomes unstable via a Hopf bifurcation for small values of the parameter θ (dotted line). This branch terminates at the point T.

To present the stability results in a possibly most general and compact form, we continued the numerical analysis, varying θ and, additionally, the gain parameter g_0, while for the absorption coefficient and the ratio of the saturation intensities the same fixed values were kept as in Figs. 1, 2, and 3, i.e., $a_0 = 2$ and $I_g/I_0 = 10$. This way to vary the parameters has a clear physical sense, as in the experiment the loss factor and saturation intensity ratio are both fixed for a given setup, while the gain can be readily adjusted changing the pump power.

The region in the parametric plane (g_0, θ) where the autosoliton is stable according to the numerical solution of the linearized eigenvalue problem is shown in Fig. 4. Above the upper border AS, the

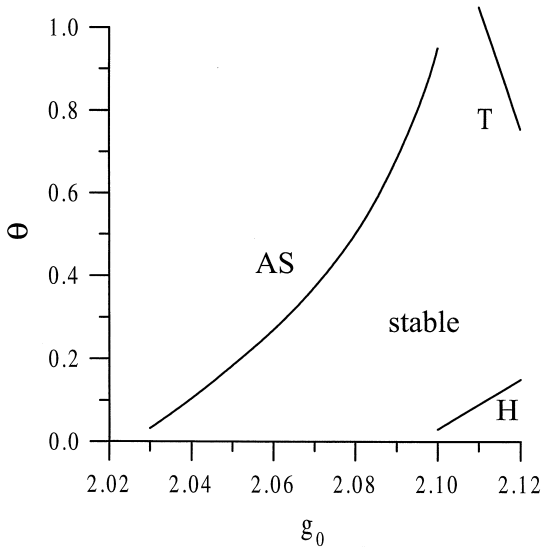

Fig. 4. Stability borders for the autosoliton in the (g_0, θ) plane. The other parameters are $a_0 = 2$ and $I_g / I_a = 10$.

autosoliton loses its stability to antisymmetric perturbations, similarly to what was discussed in detail above for the particular case $g_0 = 2.06$. Another upper border, T, corresponds to a bifurcation set where the stable autosoliton branch terminates as it is shown in Fig. 2. The lower border, H, starting at $g_0 \approx 2.094$, is a new one: at this border, the autosoliton loses stability against a perturbation eigenmode with a complex instability growth rate, corresponding to a *Hopf bifurcation* that is going to transform the stationary autosoliton into a breather (vibrating autosoliton). A detailed study of the breather is beyond the scope of the present work.

Note that the autosolitons do not exist at all at $g_0 < 2.026$, which can be easily explained by the fact that, when the gain is too weak, it cannot provide for the balance with loss, necessary for the existence of a stationary pulse. It is also noteworthy that the upper stability border is going up very steeply with the increase of g_0, which may be qualitatively realized too: an excessive gain makes it possible to compensate extra filtering losses proportional to $\sin\theta$. Besides that, it was argued above that the region $\pi/2 < \theta \leqslant \pi$ is fully symmetric to the region (5); from here it follows that, in order to comply with the symmetry, the upper border must turn back at $\theta = \pi/2$, which explains its steep ascent.

4. Conclusion

We have described the simplest model of a dispersive optical medium (which, in the general case, includes dispersive loss too) with saturable dissipative nonlinearity, which gives rise to stable solitary pulses. If the dispersive loss is absent, the same model may also be interpreted as describing a stationary field in a planar optical waveguide with uniformly distributed saturable gain and absorption. In a certain parametric region, two coexisting solitary-pulse solutions were found numerically, one of which may be stable. Numerical solution of the corresponding linearized eigenvalue problem has determined actual stability borders for the autosoliton. Beyond one of the borders, the symmetric autosoliton is destroyed by asymmetric perturbations, while at the other border it undergoes a Hopf bifurcation, which may turn it into a breather.

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