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THE COMPLEX LORENZ MODEL: GEOMETRIC STRUCTURE, HOMOCLINIC BIFURCATION AND ONE-DIMENSIONAL MAP

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It is shown that the phase space of the complex Lorenz model has the geometric structure associated with a fiber bundle. Using the equations of motion in the base space of the fiber bundle the surfaces bounding the attractors in this space are found. The homoclinic "butterfly" responsible for the Lorenz-like attractor appearance is shown to correspond to a codimension-two bifurcation. One-dimensional map describing bifurcation phenomena in the complex Lorenz model is constructed.

The Complex Lorenz Model (CLM)

$$\begin{aligned} \dot{x} &= -\sigma(x-y)\,, \\ \dot{y} &= -(1-i\delta)y + (r-z)x\,, \\ \dot{z} &= -bz + \frac{1}{2}(x^*y + xy^*)\,, \end{aligned} \tag{1}$$

was introduced by Gibbon & McGuinness [1982] as a generalization of the Lorenz model (LM), originally derived from the partial differential equations describing thermal convection of a liquid flow [Lorenz, 1963; Sparrow, 1982]. In Eqs. (1) x and y are complex and z is a real variable. Formally the complexity of the variables x and y in (1) (in LM these variables are real) is associated with the parameters δ and $r = r_1 + ir_2$, which is not so in LM. The generalization of LM by Gibbon & McGuinness is, however, much more meaningful and covers a variety of dynamical systems described by partial differential equations and possessing a certain type of instability. As a specific example of the system, where CLM results from the multiple scale analysis of the supercritical behavior, Gibbon & McGuinness [1982] treated the two-layer model of the baroclinic instability in the atmosphere, introduced by Phillips [1963]. CLM is also a basic model in the semiclassical theory of lasers and masers (see e.g. [Ning & Haken, 1990]), for which x and y are the slow complex amplitudes of the electric field and the atomic polarization, respectively, and z is the population difference of the energy levels for the resonance transition. Recent measurements of unstable far-infrared laser output has shown remarkable similarity to the results of the numerical integration of CLM [Tang & Weiss, 1994; Tang *et al.*, 1991; Weiss *et al.*, 1988].

The issue of the present work is to reveal the geometric structure of CLM and some general properties of the model associated with this structure. Following this we introduce a specific projective space, in which the states differing only by the common phase of the variables x and y are considered to be equivalent. The phase space and the projective space occur as parts of a principle fiber bundle [Kobayashi & Nomizu, 1969]. We show that all the physical information about the system can be extracted from the equations of motion in this projective space. Using these equations, we show the existence of a surface bounding the limit sets of trajectories in the projective space. This leads to the conclusion that for CLM the homoclinic bifurcation is with codimension-two. Based on these results, we derive a one-dimensional map, associated with CLM dynamics near the homoclinic bifurcation. Analyzing this map we reveal the hierarchy of bifurcations inherent to the "complex" behavior of CLM.

Alongside of Eqs. (1), we shall use another equivalent representations of CLM. Let

$$\sigma(r_1 - 1) - \frac{\delta^2}{4} \equiv \eta > 0, \quad b < 2\sigma\,.$$

Applying the change of variables

$$\begin{aligned} x' &= \eta^{-3/4} a x\,, \\ y' &= \eta^{-5/4}\sigma a\left(y - \left(1 + \frac{i\delta}{2\sigma}\right)x\right)\,, \\ z' &= \eta^{-1}\sigma\left(z - \frac{|x|^2}{2}\right)\,, \\ t' &= t\sqrt{\eta}\,, \end{aligned} \tag{2}$$

where

$$a = e^{-i\delta t/2}\sqrt{\frac{2\sigma - b}{2}}\,,$$

to Eqs. (1) one can obtain the following representation for CLM

$$\begin{aligned} \frac{dx'}{dt'} &= y'\,, \\ \frac{dy'}{dt'} &= (1 + i\nu)x' - \mu y' - x'z' - \rho x'|x'|^2\,, \\ \frac{dz'}{dt'} &= -\beta z' + |x'|^2\,. \end{aligned} \tag{3}$$

Here

$$\begin{aligned} \nu &= \frac{2r_2\sigma + \delta(\sigma - 1)}{2\eta}, \quad \mu = \frac{1 + \sigma}{\sqrt{\eta}}\,, \\ \rho &= \frac{\sqrt{\eta}}{2\sigma - b}\,, \quad \beta = \frac{b}{\sqrt{\eta}}\,. \end{aligned} \tag{4}$$

The Jacobian for the transformation (2) is equal to $|a|^4\sigma^3/\eta^5$. Therefore, for $\eta > 0$ and $b < 2\sigma$ the transformation (2) is a diffeomorphism, i.e. it is continuous and invertible, and the vector fields defined by (1) and (3) are topologically equivalent.

Neglecting the term $-\rho x|x|^2$ in (3) one can obtain the complex generalization of the Shimizu–Morioka equations [Shimizu, 1980], which were investigated in detail [Shilnikov, 1991] and were shown in [Vladimirov, 1993; Rucklidge,1993] to be the truncated normal form equations describing the chaotic phenomena near a bifurcation point at triply degenerate zero eigenvalue with geometrical multiplicity two. Note that all the coefficients in (3) are real when $\nu = 0$, i.e. when the condition (13) is satisfied.

Let us now observe that both Eqs. (1) and Eqs. (3) possess a symmetry group $U(1)$, which acts in the subspace $\mathcal{C}^2$ of the CLM phase space $\mathcal{H}$, related to the variables x and y for Eqs. (1), or to the variables x' and y' for Eqs. (3). These group transformations change the phases of x and y (or x' and y') to the same value. The invariance of the vector field defined by CLM under the $U(1)$ group action reflects the fact that the states differing only by the common phase in x and y, belong to the same physical state. Consider the real functions u, v and w of CLM phase variables

$$u = (|x'|^2 - |y'|^2)/2 \tag{5}$$

and

$$v + iw = x'^* y'\,. \tag{6}$$

Note that for $R = (u^2 + v^2 + w^2)^{1/2} = (|x'|^2 + |y'|^2)/2$, one can write $|x'|^2 = R + u$ and $|y'|^2 = R - u$. Being considered as the Cartesian coordinates in the Euclidean space $\mathcal{P}$, the functions u, v, w and z' provide the projection map $\Pi : \mathcal{H} \to \mathcal{P}$. This map projects all the elements of the CLM phase space $\mathcal{H}$, differing only by a common phase factor in x' and y', into the same point in $\mathcal{P}$. Differentiating (5) and (6) with respect to time and using Eqs. (1) one gets the following equations of motion

$$\begin{aligned} \dot{u} &= v + \mu(R - u) - \nu w - v(1 - z' - \rho(R + u))\,, \\ \dot{v} &= -\mu v + R - u + (R + u)(1 - z' - \rho(R + u))\,, \\ \dot{w} &= -\mu w + \nu(R + u)\,, \\ \dot{z}' &= -\beta z' + (R + u)\,. \end{aligned} \tag{7}$$

One can observe that the spaces $\mathcal{H}$ and $\mathcal{P}$ and the map Π form a fiber bundle [Kobayashi &

Nomizu, 1969] so that $\mathcal{H}$ is the bundle space, $\mathcal{P}$ is the base space and the structure group is $U(1)$. This fiber bundle is similar to the one known from quantum mechanics [Samuel & Bhandari, 1988] and formed by the Hilbert space of state vectors, the density matrix space and the corresponding projection map. This similarity becomes clear by identifying the "state vector" $|X\rangle$ for CLM with the pair of complex numbers x' and y' and noting that $|x'|^2 = R+u$ and $|y'|^2 = R-u$ are just the diagonal elements and $v+iw$ is the off-diagonal one for the corresponding "density matrix". The remarkable property of the fiber bundle ($\mathcal{H}$, $\mathcal{P}$, Π) is that the evolution in $\mathcal{H}$ may be extracted from the trajectory in $\mathcal{P}$. Indeed, it was proved [Samuel & Bhandari, 1988] that if the evolution of the state vector obeys the relation

$$\mathrm{Im}(\langle X|\dot{X}\rangle) = 0\,, \tag{8}$$

then the phase $\gamma_g(t) = \arg(\langle X(0)|X(t)\rangle)$ may be calculated as

$$\gamma_g = -\oint_{\Gamma T} A_s ds, \tag{9}$$

where A_s is given by

$$A_s = \mathrm{Im}\frac{\langle X(s)|d/ds|X(s)\rangle}{\langle X(s)|X(s)\rangle}\,. \tag{10}$$

Here $\langle|\rangle$ denotes the Hermitian scalar product on $\mathcal{C}_2$ and ΓT is the closed contour in $\mathcal{H}$ composed by the segment T of the trajectory between two states and any curve Γ, which projects onto the geodesic in $\mathcal{P}$. Since this phase is completely determined by the geometry of the contour, it was called geometric phase [Samuel & Bhandari, 1988]. For CLM the state vector may be made to obey Eq. (8) by means of the gauge transformation $|X\rangle \to |X\rangle \exp(i\gamma_d)$, where the "dynamic phase" γ_d is given by the equation [Toronov & Derbov, 1994]:

$$\gamma_d = \int_0^t \mathrm{Im}\left[\frac{\langle X|F\rangle}{\langle X|X\rangle}\right]_{t'} dt'\,, \tag{11}$$

$|F\rangle$ is the right-hand side vector for the first two equations in (3).

One can see from Eq. (11), that the dynamic phase is the function of the point in $\mathcal{P}$. To show that γ_g may be also extracted from $\mathcal{P}$, we introduce the spherical coordinates ρ, θ and ϕ

$$u = \rho\cos\theta,\quad v = \rho\sin\theta\cos\phi,\quad w = \rho\sin\theta\sin\phi\,.$$

Expressing A_s in (9) in terms of ρ, θ and ϕ, one gets:

$$\gamma_g = \oint_{\Gamma T} \sin^2\left(\frac{\theta}{2}\right) d\phi\,, \tag{12}$$

where the integral is taken in $\mathcal{P}$ along the contour composed by the trajectory and the geodesic. One can see that the right-hand side of (12) is nothing but half the solid angle subtended by the contour. Thus the evolution of the complete phase of $|X\rangle$, that is $\gamma_d + \gamma_g$, may be reconstructed from the trajectory in $\mathcal{P}$, determined by Eqs. (7). So, one may use these equations instead of (1) for studying CLM.

Consider now the plane in $\mathcal{P}$ given by the equation $w = 0$. It follows from Eqs. (7) that at $w = 0$ we have $\dot{w} = \nu(R+u) = \nu|x'|^2$. Therefore, the value of dw/dt is non-negative when $\nu > 0$ and non-positive in the opposite case. Thus for $\nu > (<)0$ the trajectories on the plane are tangent to it, or directed towards the region of $\mathcal{P}$ $w > (<)0$. It follows from Eqs. (7) that the surface $w = 0$ is globally stable and flow invariant for $\nu = 0$. The condition $\nu = 0$ may be rewritten in the form

$$r_2 = r_{2c} = \delta\frac{1-\sigma}{2\sigma}\,. \tag{13}$$

Let us show that for $r_2 > (<)r_{2c}$ every trajectory starting in the region $w < (>)0$ is attracted to other trajectories where $w \geq (\leq)0$. Let r_2 be greater than r_{2c}, which is the same as $\nu > 0$. Consider the family of hyperplanes $w = C < 0$ in $\mathcal{P}$. It may be seen from Eqs. (7) that for $\nu > 0$ the value of $\dot{w}$ is positive on each of these hyperplanes. Therefore, a trajectory, being started somewhere in the half-space $w < 0$, intersects all these hyperplanes reaching the hyperplane $w = 0$. There exists a zero-measured set of trajectories that tend to the origin in the limit $t \to \infty$ (see below). All other trajectories once and forever come to the half-space $w > 0$. To show that for $r_2 < r_{2c}$ every trajectory tends to the set of points in $\mathcal{P}$ for which $w \leq 0$ one should consider the family of hyperplanes given by $w = C > 0$. We omit the detailed consideration here since it is completely analogous to the case $r_2 > r_{2c}$.

An important outcome of the existence of the bounding surface $w = 0$ is the restriction for the homoclinic bifurcation in the parameter space. It is known that in LM the homoclinic orbits that exist for certain parameter values [Sparrow, 1982] are very important structures responsible for the formation of a chaotic set of trajectories. These structures are formed by the trajectories which are bi-asymptotic to the saddle $x = y = z = 0$. The necessary condition for the existence of these structures is the intersection of stable and unstable invariant manifolds of a saddle [Shil'nikov, 1981]. Since CLM include LM as a particular case at $\delta = 0$ and $r_2 = 0$, the corresponding homoclinic structures are also present in the CLM.

The local structure of invariant manifolds of a saddle may be realized from the linear stability analysis of the steady-state solution of Eqs. (3) $x' = y' = z' = 0$. This analysis gives the eigenvalues

$$\begin{aligned} \lambda_1 &= -\frac{\mu}{2} + \sqrt{1 + \frac{\mu^2}{4} + i\nu}\,, \\ \lambda_2 &= -\frac{\mu}{2} - \sqrt{1 + \frac{\mu^2}{4} + i\nu}\,, \\ \lambda_3 &= -\beta\,. \end{aligned} \tag{14}$$

If $\mathrm{Re}[\lambda_1] > 0$, the origin is a saddle. The eigenvectors in $\mathcal{H}$ corresponding to the eigenvalues $\lambda_{1,2,3}$ are:

$$\mathbf{V}_1 = N_1^{-1}\begin{pmatrix} 1 \\ \lambda_1 \\ 0 \end{pmatrix}, \quad \mathbf{V}_2 = N_2^{-1}\begin{pmatrix} 1 \\ \lambda_2 \\ 0 \end{pmatrix}, \quad \mathbf{V}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

where $N_{1,2} = \sqrt{1 + |\lambda_{1,2}|^2}$. It follows from (14) that for sufficiently small ν we have $\mathrm{Re}\,\lambda_1 > 0$, $\mathrm{Re}\,\lambda_2 < 0$ and $\lambda_3 < 0$. In addition, for $\nu > 0$

$$\mathrm{Im}\,\lambda_1 = -\mathrm{Im}\,\lambda_2 > 0\,. \tag{15}$$

We consider only the case when $|\lambda_3| < |\mathrm{Re}\,\lambda_2|$ since it corresponds to the appearance of a Lorenz attractor.

The linear subspaces $E^u = \mathrm{span}\{\mathbf{V}_1\}$ and $E^s = \mathrm{span}\{\mathbf{V}_2, \mathbf{V}_3\}$ are are given by

$$\begin{aligned} E^u &: \; y' = x'\lambda_1,\; z' = 0\,, \\ E^s &: \; y' = x'\lambda_2\,. \end{aligned}$$

The unstable manifold W^u (stable manifold W^s) of the origin, is tangent to E^u (E^s) at $x' = y' = z' = 0$. Since the z'-axis is flow-invariant and belongs to W^s, in the small sphere defined by $|x'|^2 + |y'|^2 + z'^2 \le \varepsilon^2$ these manifolds can be written in the form

$$W^u_{\mathrm{loc}} : y' = x'\{\lambda_1 + \mathrm{O}(\varepsilon)\}, \quad z' = \mathrm{O}(\varepsilon^2)\,, \tag{16}$$

$$W^s_{\mathrm{loc}} : y' = x'\{\lambda_2 + \mathrm{O}(\varepsilon)\}\,. \tag{17}$$

Consider the projections $\Pi(W^u)$ and $\Pi(W^s)$. Being flow invariant, they should also be invariant under the U(1)-group action. Their dimensions are also smaller by one than the dimensions of W^u and W^s. Hence, $\Pi(W^u)$ is a one-dimensional manifold and $\Pi(W^s)$ is a three-dimensional one. From (6), (16) and (17) we obtain

$$\begin{aligned} w|_{\Pi(W^u_{\mathrm{loc}})} &= |x'|^2\{\mathrm{Im}\,\lambda_1 + \mathrm{O}(\varepsilon)\}, \\ w|_{\Pi(W^s_{\mathrm{loc}})} &= |x'|^2\{\mathrm{Im}\,\lambda_2 + \mathrm{O}(\varepsilon)\}\,. \end{aligned} \tag{18}$$

Taking (15) into account one can conclude that for $\nu > 0$ all points of the projection $\Pi(W^u_{\mathrm{loc}})$ are in the half-space $w \ge 0$. Then, using the formal solution of the third equation in (11)

$$w(t') = w(0) + \nu e^{-\mu t'}\int_0^{t'}(R(s) + u(s))ds\,. \tag{19}$$

we conclude that all points of $\Pi(W^u)$ are in this half-space. It follows from (18) that all points of the projection $\Pi(W^s_{\mathrm{loc}})$, except for the points lying on the z'-axis, are in the half-space $w < 0$. Hence, $\Pi(W^u)$ and $\Pi(W^u_{\mathrm{loc}})$ can intersect only along the z'-axis. However, it is impossible since this axis does not belong to W^u. Therefore, for $\nu > 0$ Eqs. (1) and (5) cannot possess orbits homoclinic to the origin. Using the invariance of the sets of Eqs. (10) and (11) with respect to the transformation

$$\nu \to -\nu,\; w \to -w \tag{20}$$

one can easily show that such orbits cannot exist for $\nu < 0$ also. Thus, $\nu = 0$ or

$$r_2 = \frac{\delta(1-\sigma)}{2\sigma}$$

is the necessary condition for the existence of the Lorenz homoclinic "butterfly".

Let us assume that

$$\alpha = -\frac{\lambda_3}{\lambda_1} < 1\,,$$

where λ_1 and λ_3 are given by Eqs. (14) with $\nu = 0$. This inequality corresponds to the case when a strange invariant set appears after the destruction of the "butterfly" (for $\delta = r_2 = 0$). In LM the associated bifurcation phenomena has been described by means of the one-dimensional map [Shil'nikov, 1981b]

$$\begin{aligned} &\xi \to \mathrm{sign}\,\xi(-\varepsilon_1 + \mathrm{sign}\,A|\xi|^\alpha), \quad 0 < |\xi| \ll 1\,, \\ &0 \le \varepsilon_1 \ll 1\,, \end{aligned} \tag{21}$$

where ξ is a real variable, A is the separatrix value, and ε_1 describes the small deviation of the parameter values from the homoclinic bifurcation point. Below A is assumed to be positive.

Now we shall construct a similar map for CLM. Let x', y' be complex and $\nu = 0$ in (3). Then, as shown above, the limit sets of trajectories of Eqs. (3) belong to the globally stable hypersurface $x'^*y' - x'y'^* = w = 0$. Every trajectory lying on this hypersurface has the form $(x'(t')e^{i\psi}, y'(t')e^{i\psi}, z'(t'))$, where $(x'(t'), y'(t'), z'(t'))$ is the solution of Eqs. (3) with real x' and y', and ψ is a constant depending on initial conditions. Based on this fact one can rewrite (21) in the form that is valid for complex x' and y' in Eqs. (3)

$$\xi \to e^{i\arg\xi}(-\varepsilon_1 + |\xi|^\alpha)\,, \tag{22}$$

Here, unlike (21), ξ is complex. Taking into account possible rescaling of ξ, the coefficient A have been set to unit magnitude.

The coordinate change (5, 6) transforms the homoclinic "butterfly" into a single homoclinic orbit in the projective space $\mathcal{P}$. The one-dimensional map describing bifurcation phenomena in $\mathcal{P}$ associated with this orbit can be easily obtained from (22)

$$\Xi \to (-\varepsilon_1 + \Xi^{\alpha/2})^2, \quad 0 < \Xi \ll 1, \quad 0 \le \varepsilon_1 \ll 1\,, \tag{23}$$

Here $\Xi = |\xi|^2$. Together with (22) the map (23) is valid only for $\nu = 0$. For $0 < \nu \ll 1$ we have

$$\Xi \to G(\Xi,\, \varepsilon_1,\, \nu), \quad 0 < \Xi \ll 1, \quad 0 \le \varepsilon_1 \ll 1\,, \tag{24}$$

where $G(\Xi,\, \varepsilon_1,\, 0) = (-\varepsilon_1 + \Xi^{\alpha/2})^2$. Assuming that for small Ξ and ε_1 there exists the derivative

$$G_{\nu\nu}(\Xi, \varepsilon_1,\, 0) = \left(\frac{\partial^2 G(\Xi,\, \varepsilon_1,\, \nu)}{\partial \nu^2}\right)_{\nu=0}\,,$$

for small ν from (24) we obtain

$$\Xi \to G(\Xi,\, \varepsilon_1,\, 0) + \frac{\nu^2}{2} G_{\nu\nu}(\Xi,\, \varepsilon_1,\, 0) + \mathrm{O}(\nu^4)\,. \tag{25}$$

Here $G(\Xi,\, \varepsilon_1,\, 0)$ is defined by (23). Due to the symmetry property (20) of Eqs. (3) linear and cubic terms in ν are absent in (25). Since $\Xi, \varepsilon_1 \ll 1$ we neglect the dependence of $G_{\nu\nu}(\Xi,\, \varepsilon_1, 0)$ on Ξ and ε_1. Then, omitting the small terms $\mathrm{O}(\nu^4)$ in (25), we get

$$\begin{gathered}\Xi \to (-\varepsilon_1 + \Xi^{\alpha/2})^2 + \varepsilon_2^2, \quad 0 < \Xi \ll 1\,, \\ 0 \le \varepsilon_1, \varepsilon_2 \ll 1\,,\end{gathered} \tag{26}$$

where $\varepsilon_2^2 = (\nu^2/2)G_{\nu\nu}(0,\, 0,\, 0)$. Since $G(\Xi,\, \varepsilon_1,\, \nu) \ge 0$ and $G(0,\, 0,\, 0) = 0$ we have $G_{\nu\nu}(0,\, 0,\, 0) \ge 0$. We will assume that $G_{\nu\nu}(0,\, 0,\, 0) > 0$. Note, that the point $\varepsilon_1 = \varepsilon_2 = 0$ corresponds to the codimension-two homoclinic bifurcation. The parameter ε_2 is proportional to the small quantity ν.

Substituting

$$\Xi = \varepsilon_1^{2/\alpha}\left\{1 + \frac{2\lambda}{\alpha^2}(1 - 2\zeta)\varepsilon_1^{2(1-\alpha)/\alpha}\right\}\,,$$

and

$$\varepsilon_2 = \varepsilon_1^{1/\alpha}\left\{1 + \frac{\lambda(2-\lambda)}{2\alpha^2}\varepsilon_1^{2(1-\alpha)/\alpha}\right\}\,, \tag{27}$$

into (26) we obtain the logistic map

$$\zeta \to \lambda\zeta(1-\zeta) + \mathrm{O}(\varepsilon_1^{2(1-\alpha)/\alpha})\,.$$

Thus, the bifurcations exhibited by the map (26) are similar to those exhibited by the logistic map. Moreover, in the small vicinity of the codimension-two point ($\varepsilon_1 = 0$, $\varepsilon_2 = 0$) one can obtain asymptotic expressions for the bifurcation sets of the map (26) by substituting the bifurcation values of the parameter λ calculated for the logistic map into Eq. (27). In particular, the first two bifurcations of the logistic map are the saddle-node ($\lambda = 1$) and the period-doubling one ($\lambda = 3$). Hence, asymptotic expressions for the bifurcation curves on the parameter plane (ε_1, ε_2) corresponding to these bifurcations are

$$\varepsilon_2 = \varepsilon_1^{1/\alpha}\left\{1 + \frac{1}{2\alpha^2}\varepsilon_1^{2(1-\alpha)/\alpha} + \mathrm{O}(\varepsilon_1^{4(1-\alpha)/\alpha})\right\}\,, \tag{28}$$

and

$$\varepsilon_2 = \varepsilon_1^{1/\alpha}\left\{1 - \frac{3}{2\alpha^2}\varepsilon_1^{2(1-\alpha)/\alpha} + \mathrm{O}(\varepsilon_1^{4(1-\alpha)/\alpha})\right\}\,, \tag{29}$$

respectively. Since $0 < \alpha < 1$, one can see from Eqs. (28) and (29) that both curves match each other in the vicinity of the codimension-two point $(\varepsilon_1 = 0, \varepsilon_2 = 0)$.

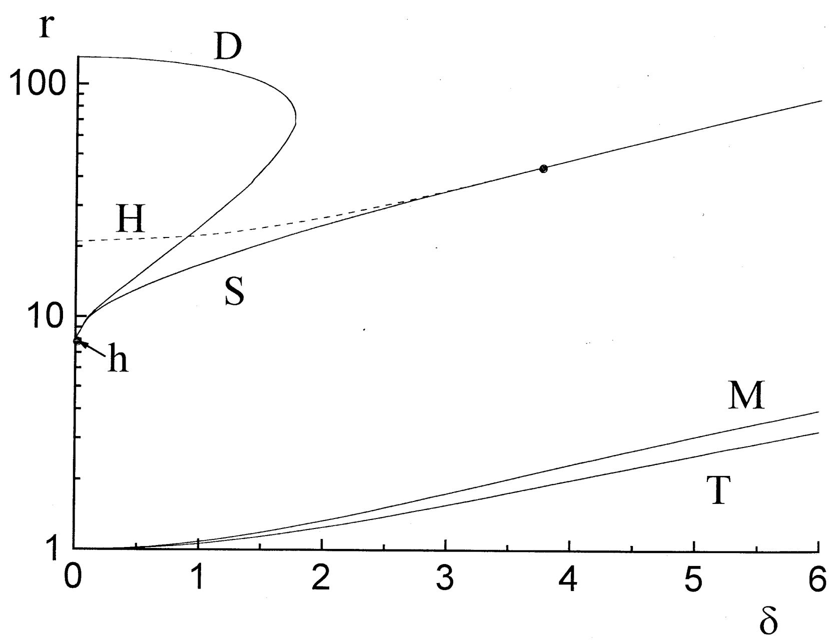

Fig. 1. Bifurcation set for Eqs. (7). The parameters are defined by (4). $\sigma = 3$, $r_2 = 0$, $b = (1/9)$. T — linear threshold. H — Hopf bifurcation. S — saddle-node bifurcation of the period-one solution. D — period doubling bifurcation.

To check the results obtained with the help of the one-dimensional map (26) we have calculated numerically several bifurcation curves of Eqs. (1) with real r. These curves are shown in Fig. 1 in the $(\delta,\, r)$-plane. The curve T indicates the stability boundary of the trivial steady-state solution $x' = y' = z' = 0$ that is unstable above this curve. The reduction of Eqs. (1) to Eqs. (3) is valid above the curve M defined by the equality $\sigma(r_1-1)-\delta^2/4 = 0$. The point h is the codimension-two homoclinic bifurcation point. According to our considerations it should be a limit point for infinite number of bifurcation curves each corresponding to a certain bifurcation of the logistic map. Two of these curves are shown in Fig. 1. The curve S corresponds to the saddle-node bifurcation (28). When crossing this curve from the right two limit cycles appear in the projective space $\mathcal{P}$. One of them is stable and the other one is unstable. When the parameter δ is further decreased the stable limit cycle undergoes a period-doubling bifurcation (curve D) which is the first bifurcation of the infinite period-doubling cascade leading to a chaotic attractor. This result of our numerical calculations is in good agreement with the predictions based on the one-dimensional map (26). Namely, the behavior of the curves D and S in the vicinity of the point h is similar to that of the curves described by Eqs. (28) and (29) up to the diffeomorphysm of the plane $(\varepsilon_1,\, \varepsilon_2)$ onto the plane $(\delta,\, r)$. The curve H in Fig. 1 corresponds to the Hopf bifurcation of the nontrivial steady-state solution in the projective space $\mathcal{P}$. The dashed (solid) line indicates the subcritical (supercritical) branch of this curve. Note that the intersection of the curve H with the r-axis corresponds to the subcritical Hopf bifurcation that is known to take place in the parameter space of the real Lorenz model.

Our results may be summarized as follows. We have shown that all the dynamical properties of CLM including the peculiarities of the phase evolution can be revealed using the representation of the model equations in the projective space. This representation provides an efficient and clear method for studying the properties of CLM. The surface bounding the attractors in the projective space is found. The codimension-two origin of the homoclinic bifurcation in CLM is shown. For the parameter values close to the codimension-one bifurcation point a one-dimensional map equivalent to the logistic map is obtained. The correspondence between the bifurcation hierarchy in CLM and that obtained from the asymptotic expressions in the vicinity of codimension-two bifurcation is shown numerically.

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