

Optics Communications

Optics Communications 149 (1998) 67-72

# Bifurcation analysis of a bidirectional class B ring laser

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#### Abstract

A model of a ring class B laser operating in two counterpropagating modes is considered. A set of reduced equations are derived which provide the simplest mathematical description of this model. Homoclinic bifurcations leading to the appearance of low-frequency regular and chaotic antiphase oscillations of the counterpropagating wave intensities are described. The effect of symmetry breaking caused by weak backscattering on the laser dynamics is discussed. © 1998 Published by Elsevier Science B.V.

## 1. Introduction

Ring lasers have attracted considerable attention of researchers due to their applications as laser gyro and their complicated dynamics. In particular, bidirectional solidstate and CO<sub>2</sub> lasers can exhibit undamped regular and chaotic antiphase oscillations of counterpropagating wave (CPW) intensities [1-15]. In the course of these oscillations CPW intensities are time-periodic with period T and phase shift T/2 between the waves. The total intensities of the CPWs have a much smoother behavior than the intensities of individual waves. Therefore, they can be considered as a particular case of antiphase oscillations described in multimode lasers [16]. Antiphase CPW oscillations in class B lasers were a subject of numerous experimental [1-3.6.7.9.14.15 and theoretical studies [1.3-6.8-13.15]. It was found that linear coupling between the CPWs resulting from backscattering can lead to regimes with antiphase near sinusoidal pulsations of the CPW intensities (selfmodulation of the first kind [6,15]). On the other hand, theoretical studies demonstrated that even in the absence of linear coupling between the CPWs frequency detuning can destabilize unidirectional operation of a class B laser [17] and give rise to undamped low-frequency alternation of the CPWs [6,12,15]. Such pulsations (self-modulation of the second kind [6,15]) were observed experimentally (see, for example, Refs. [1-3,6,7,14,15]). They arise due to

nonlinear coupling of CPWs caused by an induced grating of the inversion in the active medium. It is well known that the self-modulation of the first kind can appear after a supercritical Hopf bifurcation of the "travelling-wave" solution (see, for example, Ref. [13]). However, bifurcation mechanisms leading to low-frequency self-modulation of the second kind are not entirely known yet. In particular, in Ref. [12] a model of a bidirectional class B laser without backscattering was studied theoretically. The steady-state solution corresponding to unidirectional lasing was found to lose stability via a Hopf bifurcation. However, the frequency corresponding to the imaginary parts of the eigenvalues responsible for this instability is missing in the spectrum of the intensity pulsations which appear after the Hopf bifurcation. Therefore, the main goal of the present paper is to investigate bifurcation mechanisms leading to regular and chaotic self-modulation of the second kind.

### 2. Laser model and basic equations

The model of a single-mode bidirectional class B laser considered here is similar to that described in Ref. [6]. It is given by the following set of four ordinary differential equations,

$$\frac{\mathrm{d}f_+}{\mathrm{d}t'} = -(1-\delta-i\Delta_{\rm c})f_+ + i\rho_+ \exp(i\theta_+)f_- \\ + \mathscr{L}(1+i\Delta)(f_+n_0+f_-n_2),$$

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$$\begin{aligned} \frac{\mathrm{d}f_{-}}{\mathrm{d}t'} &= -\left(1 + \delta + i\Delta_{\mathrm{c}}\right)f_{-} + i\rho_{-}\exp(i\theta_{-})f_{+} \\ &+ \mathscr{L}(1 + i\Delta)\left(f_{-}n_{0} + f_{+}n_{2}^{*}\right), \\ \frac{\mathrm{d}n_{0}}{\mathrm{d}t'} &= -\gamma \Big[-A + n_{0} + \mathscr{L}\left(|f_{+}|^{2}n_{0} + |f_{-}|^{2}n_{0} \\ &+ f_{+}^{*}f_{-}n_{2} + f_{+}f_{-}n_{2}^{*}\right)\Big], \\ \frac{\mathrm{d}n_{2}}{\mathrm{d}t'} &= -\gamma \Big[n_{2} + \mathscr{L}\left(|f_{+}|^{2}n_{2} + |f_{-}|^{2}n_{2} + f_{+}f_{-}^{*}n_{0}\right)\Big], \end{aligned}$$
(1)

for dimensionless variables, which are the complex electric field amplitudes of the CPWs  $(f_+)$  and the amplitudes of the first two spatial harmonics of the inversion  $(n_0 \text{ and }$  $n_2$ ). The real variable  $n_0$  corresponds to the homogeneous component of the inversion. The complex variable  $n_2$ describes the induced grating of the inversion. The normalized time is  $t' = t\kappa$ ,  $\kappa = (\kappa_+ + \kappa_-)/2$ , where  $\kappa_+$  are the dumping rates of the CPWs. The parameters  $\delta = (\kappa_{-} - \kappa_{-})^{-1}$  $(\kappa_{+})/2\kappa$  and  $\Delta_{\rm c} = (\omega_{\rm c}^{-} - \omega_{\rm c}^{+})/2\kappa$  are the amplitude and the phase nonreciprocity respectively ( $\omega_c^{\pm}$  are the cavity eigenfrequencies of the CPWs). The parameters  $\rho_{+}\exp(i\theta_{+}) = \zeta_{+}/\kappa$  describe the linear coupling between the waves due to backscattering.  $\gamma = \gamma_{\parallel} / \kappa$  is the normalized longitudinal relaxation constant,  $\Delta$  is the normalized frequency detuning from the line center. A is the pump parameter and  $\mathscr{L} = (1 + \Delta^2)^{-1}$ .

If the pump parameter A slightly exceeds its threshold value  $(1 > A \mathcal{L} - 1 \equiv \varepsilon > 0)$  then, introducing the new set of variables

$$\begin{split} z_{\pm} &= f_{\pm} \sqrt{\mathscr{D}/\varepsilon} \exp\left(-i\Delta t' + \frac{i\theta}{2}\right), \quad N_0 = \frac{n_0 \mathscr{D} - 1}{\sqrt{\gamma\varepsilon}}, \\ N_2 &= \frac{n_2 \mathscr{D}}{\sqrt{\gamma\varepsilon}}, \quad \tau = t' \sqrt{\gamma\varepsilon}, \end{split}$$

we can rewrite (1) in the form

$$\frac{dz_{+}}{d\tau} = (B + iC)z_{+} + iR_{+}\exp(i\theta)z_{-} 
+ (1 + i\Delta)(z_{+}N_{0} + z_{-}N_{2}), 
\frac{dz_{-}}{d\tau} = -(B + iC)z_{-} + iR_{-}\exp(i\theta)z_{+} 
+ (1 + i\Delta)(z_{-}N_{0} + z_{+}N_{2}^{*}), 
\frac{dN_{0}}{d\tau} = -\Gamma N_{0} + 1 - |z_{+}|^{2} - |z_{-}|^{2}, 
\frac{dN_{2}}{d\tau} = -\Gamma N_{2} - z_{+}z_{-}^{*}.$$
(2)

Here  $\Gamma = \sqrt{\gamma/\varepsilon}$ ,  $B = \delta/\sqrt{\gamma\varepsilon}$ ,  $C = \Delta_c/\sqrt{\gamma\varepsilon}$ ,  $R_{\pm} = \rho_{\pm}/\sqrt{\gamma\varepsilon}$ , and  $\theta = (\theta_+ + \theta_-)/2$ . We have neglected in (2) the third order terms proportional to  $\varepsilon\Gamma = \sqrt{\gamma\varepsilon}$ . Since the values of the parameter  $\gamma$  typical of solid-state and CO<sub>2</sub> lasers are very small, we obtain  $\sqrt{\gamma\varepsilon} \ll 1$ .

In Refs. [18–20] the equations for a class A laser with two and three transverse modes were derived. For a laser operating in two Gauss-Laguerre modes with angular indices  $\pm n$  these equations are similar to those describing CPW interaction in a bidirectional ring class A laser. Starting from this result and comparing (2) with the equations obtained in Ref. [21] for a multi-transverse-mode class B laser one can easily conclude that Eqs. (2) can be also applied to study the dynamics of a class B laser operating in two Gauss-Laguerre modes with opposite angular indices  $\pm n$ . In particular, if these two modes have equal cavity eigenfrequencies and losses then B = C = 0in Eqs. (2).

The vector field defined by Eqs. (2) is invariant under the transformation  $z_{\pm} \rightarrow z_{\pm} \exp(i\psi)$ . For  $R_{\pm} = 0$ , Eqs. (2) have an additional symmetry group generated by

$$z_{\pm} \rightarrow z_{\pm} \exp(\pm i\phi), \quad N_2 \rightarrow N_2 \exp(2i\phi).$$
 (3)

Here  $\phi$  and  $\psi$  are arbitrary constants. Moreover, for  $R_+=0$  Eqs. (2) are invariant under the transformations

$$(z_{\pm}, N_2, B, C) \rightarrow (z_{\mp}, N_2^*, -B, -C),$$
  
$$(z_{\pm}, B, \Delta) \rightarrow (z_{\mp}^*, -B, -\Delta).$$
(4)

It follows from the symmetry properties (4) that for  $R_{\pm} = B = C = 0$  the vector field, defined by (2) is invariant under  $Z_2$  symmetry group action

$$z_{\pm} \to z_{\mp}, \quad N_2 \to N_2^*. \tag{5}$$

The symmetry property (5) reflects the equal status of the CPWs in the absence of nonreciprocity. Due to the symmetry properties (4) we restrict our consideration to the case when  $\Delta \ge 0$  and  $B \ge 0$ .

Let us consider the case when there is no backscattering in the laser ( $R_+ = R_- = 0$  in Eqs. (2)). The two solutions of Eqs. (2) corresponding to cw unidirectional laser operations are  $S_{\pm}$ :  $|z_{\pm}|^2 = 1 \pm \Gamma B$ ,  $z_{\pm} = 0$ ,  $N_0 = -B$ ,  $N_2 = 0$ . The linear stability analysis of the solutions  $S_{\pm}$ yields the relaxation frequencies  $\omega_{\rm R}^{\pm}$  (see Ref. [6])

$$\frac{\omega_{\rm R}^{\pm}}{\kappa\sqrt{\gamma\varepsilon}} = \sqrt{2(1\pm\Gamma B) - \Gamma^2/4} \,. \tag{6}$$

For B = 0 these frequencies coincide with the well-known relaxation frequency of a single-mode class B laser. The unidirectional solution  $S_+$  is stable when  $\Delta_- < \Delta < \Delta_+$ ; otherwise it is unstable and the laser starts to operate in two CPWs. Here

$$\Delta_{\pm} = (1 + \Gamma B)^{-1} \Big[ \pm (\Gamma + 2B) \sqrt{1 + 3\Gamma B + C^2} \\ -C(\Gamma - 2B) \Big].$$
(7)

It follows from the invariance of (2) under the transformations (4) that the instability threshold for the solution  $S_{-}$ can be obtained from (7) by means of the substitution  $(B,C) \rightarrow (-B, -C)$ .



Fig. 1. Branches of the solutions  $S_+$ ,  $S_-$ ,  $S_1$ ,  $S_2$ , and  $S_3$ .  $R_+ = R_- = 0$ ,  $\Delta = 0.5$ . (a) B = C = 0; (b) B = 0.1, C = 0; (c) B = 0, C = 0.1. Thick (thin) lines correspond to stable (unstable) solutions.

Apart from the solutions  $S_+$  and  $S_-$ , Eqs. (2) possess three branches of solutions with time-independent CPW intensities. We label these solutions  $S_{1,2,3}$ . Consider the case without cavity nonreciprocity (B = C = 0). Then the explicit expressions for the solutions  $S_{1,2,3}$  are given by

$$S_2: |z_+|^2 = |z_-|^2 = \frac{1}{3}, \quad N_0 = \frac{1}{3\Gamma}, \quad N_2 = \frac{2}{9\Gamma^2},$$
 (8)

$$S_{1,3}: |z_{\pm}|^{2} = \frac{d}{2} \left( 1 \pm \frac{\Delta}{2\Gamma} \sqrt{3d-2} \right), \quad N_{0} = \frac{1}{\Gamma} (1-d),$$

$$N_2 = \frac{a}{\Gamma^2} (1 - d),$$
 (9)

where  $d = (2 - \Gamma^2)/(2 - \Delta^2)$ . Eqs. (8) correspond to the standing-wave solution with equal CPW intensities, while (9) corresponds to the solutions with unequal nonzero CPW intensities. These solutions were not described earlier for a bidirectional class B laser. However, as it will be shown in the following section, they play an important role in the appearance of the self-modulation regime of the

second kind. The standing-wave solution (8) is stable for  $\Gamma > \sqrt{3/4}$  and  $\Delta^2 > 3\Gamma^2/2 - 1$ . The conditions  $\Gamma = \sqrt{3/4}$  and  $\Delta > 1$  correspond to a Hopf bifurcation of this solution. The condition  $\Delta^2 = 3\Gamma^2/2 - 1$  corresponds to a pitchfork bifurcation point where the solutions (9) bifurcate from (8) (see Fig. 1a). The codimension-two point ( $\Gamma = \sqrt{3/4}$ ,  $\Delta = 1$ ) corresponds to  $Z_2$  symmetric Bogdanov-Takens bifurcation where the characteristic equation determining the stability of (8) has a double zero root. The solutions (9) are always unstable.  $S_3$  ( $S_2$ ) bifurcates from  $S_+$  ( $S_-$ ) at the points  $\Delta = \pm \Gamma$ , which are obtained by substituting B = C = 0 into (7) (see Fig. 1a).

In the presence of nonreciprocity ( $B \neq 0$  and/or  $C \neq 0$ )  $Z_2$  symmetry (5) of Eqs. (2) is broken and the pitchfork bifurcation is destroyed. As it is shown in Figs. 1b, 1c, this bifurcation is replaced by a saddle-node one (see Ref. [22] for the description of imperfect bifurcations).

#### 3. Bifurcation diagrams

Bifurcations of Eqs. (2) with  $R_+ = R_- = 0$  have been studied numerically using the programs LINLBF and LIN-BAS which were designed to calculate bifurcation curves of steady-state and limit cycle solutions respectively [23]. The program INTSEP [24] has been used to calculate one-dimensional unstable manifolds of steady-state solutions.

Several bifurcation curves of Eqs. (2) in the  $(\Delta, \Gamma)$ plane for the case without nonreciprocity (B = C = 0) are shown in Fig. 2. Here the curve p  $(\Delta^2 = 3\Gamma^2/2 - 1)$ corresponds to a pitchfork bifurcation of the solution  $S_2$ , and the line H corresponds to a supercritical Hopf bifurcation of this solution. The curve t  $(\Delta = \Gamma)$  corresponds to



Fig. 2. Bifurcation curves for Eqs. (2) with  $R_+ = R_- = B = C = 0$ in ( $\Gamma$ ,  $\Delta$ )-plane. The lines H, t, p, and h correspond to Hopf, steady-state, pitchfork, and heteroclinic bifurcation, respectively. The point TB is the codimension-two Bogdanov–Takens bifurcation point. The point SF divides the curve h into two parts. The upper (lower) part of the curve h corresponds to a pair of heteroclinic orbits connecting two saddles (saddle-foci).

the points where the unstable steady-state solution  $S_3(S_1)$ bifurcates from the solution  $S_{+}$  ( $S_{-}$ ) (see Fig. 1a). For  $\Delta^2 < 2$  the solutions  $S_+$  and  $S_-$  are unstable below this line. The point TB in Fig. 2 is a codimension-two bifurcation point of Bogdanov-Takens type [25,26]. Due to the symmetry property (5) this is  $Z_2$ -symmetric bifurcation. A point of Bogdanov-Takens bifurcation with Z<sub>2</sub> symmetry is known to be a limit point for either the heteroclinic or homoclinic bifurcation line [25,26]. In Fig. 2, curve h terminating at the point TB corresponds to a heteroclinic loop connecting  $S_3$  and  $S_1$ . The point SF divides h into two parts. On the upper part the solutions  $S_3$  and  $S_1$  are saddles whereas on the lower part they are saddle-foci. A stable limit cycle emerges from the homoclinic loop when crossing the line h from above. This limit cycle corresponds to the low-frequency antiphase pulsations of the CPW intensities (self-modulation regime of the second kind). When approaching the heteroclinic bifurcation line h from below the period of the limit cycle tends to infinity. The phase portrait of this cycle is shown in Fig. 3a for the parameter values close to heteroclinicity. In the region between the lines h and t stable antiphase pulsations coexist with the stable solutions  $S_{\pm}$ . Note, however, that for  $\Gamma, \Delta \ll 1$  the bistability region is extremely narrow. Below the line t the solutions  $S_{\pm}$  are unstable.

As it was mentioned in Section 2,  $Z_2$  symmetry (5) of Eqs. (2) is broken in the presence of nonreciprocity.



Fig. 3. Limit cycles of Eqs. (2) with  $R_+ = R_- = 0$  corresponding to antiphase pulsations of the CPWs. Parameter values are close to heteroclinic (homoclinic) bifurcation. (a)  $\Gamma = 0.741$ ,  $\Delta = 0.7$ , B = C = 0; (b)  $\Gamma = 0.315$ ,  $\Delta = 0.8$ , B = 0.25, C = 0.



Fig. 4. Bifurcation curves for Eqs. (2) with  $R_{+} = R_{-} = B = 0$  and C = 0.2. The lines  $H_{1,2}$ ,  $s_{\pm}$ , and  $t_{\pm}$ , correspond to Hopf, saddlenode, and steady-state bifurcations, respectively. The point CT is a codimension-two bifurcation point with a single zero and a pair of pure imaginary eigenvalues.

Therefore, the pitchfork bifurcation of  $S_2$  that exists for B = C = 0 is replaced by a saddle-node bifurcation. For small nonzero B and/or C heteroclinic bifurcation leading to the appearance of the antiphase pulsations is replaced by a homoclinic one (see Fig. 3b). Bifurcation curves of Eqs. (2) in the  $(\Delta, \Gamma)$ -plane for the case with cavity phase nonreciprocity (B = 0, C = 0.2) are presented in Fig. 4. For  $\Delta^2 < 2$ , the solutions  $S_1$  and  $S_2$  exist below the saddle-node bifurcation curve s. They collide with each other and disappear on this curve. The solution  $S_1$  is always unstable, while  $S_2$  is stable between the curves s and H<sub>2</sub>. Here H<sub>2</sub> corresponds to the supercritical Hopf bifurcation of  $S_2$ . A stable limit cycle bifurcates from  $S_2$ when crossing the line H<sub>2</sub> from above. The line H<sub>1</sub> corresponds to the subcritical Hopf bifurcation of the unstable solution  $S_1$ . This bifurcation leads to the emergence of an unstable limit cycle below the line H<sub>1</sub>. The solution  $S_+$  ( $S_-$ ) is stable above the line  $t_+$  ( $t_-$ ) corresponding to steady-state bifurcation points. At these points the unstable solution  $S_3(S_1)$  bifurcates from the solution  $S_+$  ( $S_-$ ). The curves  $H_1$  and  $H_2$  terminate at the codimension-two point CT which is characterized by a single zero and a pair of pure imaginary eigenvalues. It was shown that this kind of bifurcation can lead to the appearance of two-dimensional tori and, in generic case, to their break-up resulting in chaotic phenomena that includes Shil'nikov's homoclinic behavior [26,27].

Fig. 5 displays the enlarged vicinity of the point CT in the  $(\Gamma, \Delta)$ -plane. The line T<sub>1</sub> corresponds to a bifurcation



Fig. 5. Enlarged vicinity of the codimension-two point CT from Fig. 4. The lines  $T_1$  and  $T_2$  correspond to bifurcations of limit cycles into two-tori. The lines  $D_1$  and  $D_2$  correspond to period-doubling bifurcations. The solid (dashed) parts of  $D_1$  and  $D_2$  indicate period-doubling bifurcations of a stable (unstable) limit cycle.

into a two-torus of a limit cycle emerging from the Hopf bifurcation on the line  $H_2$ . The lines  $D_1$  and  $D_2$  correspond to the first two bifurcations of the period-doubling sequence starting from this limit cycle (the left dashed parts of these lines correspond to a period-doubling of unstable cycles). The line  $T_2$  corresponds to the bifurcation of a period-two cycle into a period-two torus. Truncated normal form equations for the bifurcation point with a single zero and a pair of pure imaginary eigenvalues together with different types of unfolding diagrams produced by these equations can be found in Ref. [26].

#### 4. The effect of small backscattering

For  $R_{\pm} = 0$  and B = C = 0 the self-modulation regime of the second kind that emerges from the homoclinic loop connecting the saddle-foci  $S_3$  and  $S_1$  is expected to exhibit oscillations with two basic frequencies. The first of them corresponds to a slow alternation of the CPWs. When approaching the homoclinic bifurcation point this frequency tends to zero. The second frequency corresponds to the revolutions of the phase trajectory around the solutions  $S_3$  and  $S_1$ . The complex eigenvalue of  $S_3$  ( $S_1$ ) with smallest absolute value of its real part is determined. For small  $\Gamma$  this frequency is very close to the relaxation frequency defined by Eq. (6). For small  $\Gamma$  and  $\Delta$  the homoclinic bifurcation curve is in a close proximity to the transcritical bifurcation curve (see curves h and t in Fig. 2). Hence, when the homoclinic bifurcation takes place, the solution  $S_3$  ( $S_1$ ) is very close to  $S_+$  ( $S_-$ ). A pair of complex eigenvalues of  $S_3$  ( $S_1$ ) with smallest absolute value of the real part differs slightly from the eigenvalues of the solution  $S_+$  ( $S_-$ ) which have the imaginary parts defined by Eq. (6).

Linear coupling between the CPWs  $(R_{+} \neq 0)$  breaks the symmetry (3) of Eqs. (2). As a consequence of this, the unstable solutions  $S_1$  and  $S_3$  that exist for  $R_{+}=0$  and correspond to stationary CPW intensities are transformed into saddle limit cycles,  $LC_1$  and  $LC_3$ , corresponding to time-periodic CPW intensities. Steady-state bifurcations of the solutions  $S_+$  and  $S_-$  are transformed into Hopf bifurcations. Since there exists a bifurcation set where  $S_1$  and  $S_3$  possess heteroclinic orbits, for sufficiently small nonzero values of  $R_+$  the stable (unstable) manifold of the cycle  $LC_2$  and the unstable (stable) manifold of the cycle  $LC_3$ are expected to intersect in a certain domain of parameter space. It is known that this kind of intersection leads to a very complicated behavior. Recently, a laser with a saturable absorber was shown to exhibit similar behavior [31]. Thus, symmetry breaking associated with weak linear coupling between CPWs may lead to the transition from regular antiphase pulsations to irregular ones.

### 5. Conclusion

Global bifurcations leading to the emergence of lowfrequency antiphase pulsations of the CPW intensities (self-modulation regime of the second kind) in a model of a single-mode bidirectional ring class B laser without backscattering have been described. The new branches of unstable solutions corresponding to stationary CPW intensities,  $S_1$  and  $S_3$  (see Section 3), have been found. These are precisely the solutions that play a crucial role in the origin of the self-modulation regime of the second kind. It has been shown that the limit cycle corresponding to periodic antiphase oscillations emerges either from the homoclinic loop connecting  $S_1$  and  $S_3$  (in the case without nonreciprocity) or from the orbit homoclinic to one of these solutions (in the case with small nonreciprocity). A bistability region in parameter space where undamped antiphase pulsations coexist with unidirectional cw operation has been found. A codimension-two bifurcation point leading to the appearance of two-tori and chaotic phenomena associated with their break-up has been shown to exist in the laser parameter space. In the presence of weak backscattering mechanisms of the antiphase pulsations appearance become more complicated and they include chaotic phenomena associated with the intersection of stable and unstable manifolds of the saddle cycles. It is likely that this is why self-modulation regime of the second kind may not be strictly periodic [10,13,15].

As it was already mentioned in Section 2, the results obtained can be applied to study the dynamics of a class B laser operating in two transverse modes. It was shown in experiment that a bimode CO<sub>2</sub> laser with a saturable absorber can exhibit "mode-hopping", that is periodic jumps from one transverse mode to another one [28]. The phase trajectory obtained by the phase-space reconstruction method was found to make revolutions around "pure-mode" solutions corresponding to laser operation in a single transverse mode. The regimes periodic alternating between transverse modes were also observed experimentally [29-31] and found numerically (see Fig. 10 in Ref. [32]). These regimes are very similar to those emerging from the heteroclinic bifurcation described in Section 3. According to the results obtained, homoclinic loop connects the unstable "mixed-mode" solutions  $(S_1 \text{ and } S_3)$ rather than the "pure-mode" ones  $(S_{+} \text{ and } S_{-})$ . Therefore, the phase trajectory must revolve around these "mixed-mode" solutions. Note, however, that for small values of the parameter  $\Gamma$  typical of class B lasers the "mixed-mode" solutions  $S_1$  ( $S_3$ ) are in close proximity to the "pure-mode" one  $S_{\perp}$  (S\_) (see Fig. 2, where the heteroclinic bifurcation curve h tends to the steady-state bifurcation curve t as  $\Gamma \rightarrow 0$ ). It seems likely that for this reason "mixed-mode" solutions were not detected in the experiment of Ref. [29].

#### Acknowledgements

The author is grateful to Professor Ya.I. Khanin for suggesting the problem, his constant attention to this work, useful discussions and remarks.

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