Optics Communications 100 (1993) 351-360
North-Holland

OPTICS
COMMUNICATIONS

Full length article

Low-intensity chaotic operations of a laser with a saturable absorber

A.G. Vladimirov
Research Institute of Physics, St. Petersburg State University,
1, Ulianovskaya st., St. Petersburg-Petrodvorets 198904, Russian Federation

and

D.Yu. Volkov
Research Institute of Mathematics and Mechanics, St. Petersburg State University,
2, Bibliotechnaya pl., St. Petersburg-Petrodvorets 198904, Russian Federation

Received 22 June 1992; revised manuscript received 27 January 1993

We study the dynamics of a laser with a saturable absorber for parameter values close to a multiple bifurcation point. We show that different chaotic attractors exist in a small vicinity of the zero-intensity steady state near the laser threshold.

1. Introduction

It is well known that the Maxwell–Bloch equations for a single-mode ring laser exhibit the Lorenz instability leading to chaotic dynamics [1]. However, the threshold for this instability (the second laser threshold) requires a set of parameter values that are not very accessible in experiments. In several recent studies it has been shown by means of numerical calculations that adding a saturable absorber into the laser resonator may produce a substantial reduction of the second threshold (see, for example, refs. [2–6]). We demonstrate here that for suitable parameter values the Maxwell–Bloch equations for a laser with a saturable absorber (LSA) can exhibit small-intensity chaotic solutions near the lasing threshold (first laser threshold). In ref. [7] this kind of chaotic solutions was called an asymptotic chaos. Our investigation also provides a qualitative understanding of the bifurcation sequences leading to chaos in LSA equations.

We study the simplest theoretical model of a ring single-mode LSA in which each atomic species is modeled by a set of homogeneously broadened two-level atoms. For this model the semiclassical theory yields five mean-field ordinary differential LSA equations for three complex and two real variables [6,8,9]. Neglecting the phases of the electric field and atomic polarizations in the case of perfect tuning we have five real LSA equations [9].

We consider the bifurcation points in the parameter space, for which the jacobian matrix of the perturbations of the real LSA equations evaluated at the zero-intensity steady state has a triply degenerate zero eigenvalue with geometrical multiplicity two. In the vicinity of these points the application of normal form theory yields the reduced equations which describe the local dynamics of LSA equations. We investigate the bifurcations leading to chaotic oscillations in the reduced equations. We show that there exists a case in which the reduced equations may be transformed into the Shimizu–Morioka equations [10]. It is known that the Shimizu-Morioka equations which were proposed in ref. [11] exhibit various kinds of chaotic behavior. The bifurcation phenomena in these equations have been studied in great detail as reported in ref. [12].

We discuss the influence of the phases of the electric field and the atomic polarizations in the LSA equations on the stability of the solutions under consideration. We show that the LSA equations exhibit two kinds of chaotic solutions. One of them is stable with respect to phase variables and the other is unstable.

2. Laser model and reduction to normal form

The LSA model studied is similar to that considered in refs. [8,9]. In the case of perfect tuning it is given by the following set of five ordinary differential equations:

$$dE/dt=-E+P_1+P_2\,,\quad dP_1/dt=-P_1\delta_1-E(M_{01}+M_1)\,,\quad dP_2/dt=-P_2\delta_2-E(M_{02}+M_2)\,,$$
$$dM_1/dt=-\rho_1M_1+(E^*P_1+P_1^*E)/2\,,\quad dM_2/dt=-\rho_2M_2+\beta(E^*P_2+P_2^*E)/2\,, \tag{1}$$

for dimensionless variables, which are the electric field amplitude E, the atomic polarization amplitudes P_k, and the deviations of the population differences M_k from their values M_{0k} without the laser field. Here the subscript $k=1$ pertains to the amplifying medium and $k=2$ pertains to the absorbing medium. $M_{01}<0$, $M_{02}>0$. The variables E, P_1, and P_2 are complex, the variables M_1 and M_2 are real. The parameters δ_k and ρ_k are transversal and longitudinal relaxation constants divided by the resonator halfwidth which has been used to renormalize the time t. The parameter β is the ratio of the saturation intensity in the amplifying medium to the saturation intensity in the absorbing medium. Note that the LSA equations (1) are equivariant under the O(2) symmetry group action generated by $(E, P_1, P_2)\to\exp(i\phi)\,(E, P_1, P_2)$ and $(E, P_1, P_2)\to(E^*, P_1^*, P_2^*)$.

A linear stability analysis of the zero-intensity stationary solution $E=P_k=M_k=0$ ($k=1, 2$) of eqs. (1) leads to the characteristic equation of the form

$$(\lambda^3+A_2\lambda^2+A_1\lambda+A_0)(\lambda+\rho_1)(\lambda+\rho_2)=0\,, \tag{2}$$

where

$$A_0=\delta_1\delta_2+\delta_2M_{01}+\delta_1M_{02}\,,\quad A_1=\delta_1+\delta_2+\delta_1\delta_2+M_{01}+M_{02}\,,\quad A_2=1+\delta_1+\delta_2\,. \tag{3}$$

Note that $A_0=0$ corresponds to the linear laser threshold whereas $A_1A_2-A_0=0$ and $A_0<0$ correspond to the Hopf bifurcation of the zero-intensity steady state. If two conditions $A_0=0$ and $A_1=0$ are satisfied simultaneously then we have bifurcation of the Takens–Bogdanov type corresponding to double zero λ in eq. (2). This bifurcation in LSA equations was studied in refs. [13,14]. Here we consider two multiple bifurcation points with the codimension greater than that of the Takens–Bogdanov bifurcation. They are determined by three conditions

$$A_0=0\,,\quad A_1=0\,,\quad \rho_1=0 \text{ or } \rho_2=0\,. \tag{4}$$

At these points the jacobian matrix J of eqs. (1) (with real E and P_k) evaluated at the zero-intensity stationary solution has a triply degenerate zero eigenvalue with geometrical multiplicity two.

Let us select M_{01}, M_{02}, and ρ_1 (M_{01}, M_{02}, and ρ_2) as our bifurcation parameters. Then, taking into account eqs. (3), we conclude that two bifurcation points (4) exist when $\delta_1>\delta_2$ and they are given by

$$M_{01}=\bar{M}_{01}\,,\quad M_{02}=\bar{M}_{02}\,,\quad \rho_1=0\,, \tag{5a}$$

and

$$M_{01}=\bar{M}_{01}\,,\quad M_{02}=\bar{M}_{02}\,,\quad \rho_2=0\,, \tag{5b}$$

where

$$\bar{M}_{01}=-\delta_1^2(1+\delta_2)/(\delta_1-\delta_2)\,,\quad \bar{M}_{02}=\delta_2^2(1+\delta_1)/(\delta_1-\delta_2)\,.$$

Consider the LSA equations (1) with the bifurcation parameters fixed by eqs. (5a) (eqs. (5b)). In order to obtain the reduced form of eq. (1) we first introduce a linear change of variables $(E, P_1, P_2, M_1, M_2) \to (u', v', w', \zeta_1', \zeta_2')$,

$$E=u'+v'+w' , \quad P_1=[u'\delta_1(\delta_2+1)+v'(\delta_1-1)(\delta_2+1)-w'\delta_1^2]/\delta_1-\delta_2) ,$$

$$P_2=-[u'\delta_2(\delta_1+1)+v'(\delta_2-1)(\delta_1+1)-w'\delta_2^2]/(\delta_1-\delta_2) , \quad M_k=\zeta_k' \ (k=1,2) ,$$

which transforms the jacobian matrix J of eqs. (1) at the bifurcation points (5a, b) into the Jordan normal form

$$\begin{matrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\Lambda_1 & 0 \\ 0 & 0 & 0 & 0 & -\Lambda_2 \end{matrix} ,$$

where $\Lambda_1=1+\delta_1+\delta_2>0$, $\Lambda_2=\rho_2>0$ ($\Lambda_2=\rho_1>0$) for the bifurcation point (5a) (bifurcation point (5b)).

Next, using the center manifold theory, we eliminate the variables w', ζ_2' (ζ_1') corresponding to the negative eigenvalues $-\Lambda_1$ and $-\Lambda_2$ and we apply the normal form technique [15,16] to eliminate certain nonlinear terms by the transformation preserving O(2) symmetry

$$u=u'+u[R_{1k}|u|^2+R_{2k}(v^*u+u^*v)] ,$$

$$v=v'+u[R_{3k}|u|^2+R_{4k}(v^*u+u^*v)+R_{5k}\zeta_k+R_{6k}\zeta_k^2]+v[R_{7k}|u|^2+R_{8k}(v^*u+u^*v)+R_{9k}\zeta_k+R_{10k}\zeta_k^2] ,$$

$$\zeta_k=\zeta_k'+|u|^2(R_{11k}+R_{12k}\zeta_k)+(v^*u+u^*v)(R_{13k}+R_{14k}\zeta_k) .$$

Here $k=1$ ($k=2$) corresponds to the bifurcation point (5a) (bifurcation point (5b)). Real coefficients R_{jk}, ($j=1$, 14; $k=1$, 2) depend on the parameters of the LSA equations. We do not present explicit expressions for these coefficients since they are very cumbersome and irrelevant in further analysis. The resulting normal form equations calculated up to the third order terms are given by

$$du/dt=v+O(4) , \quad dv/dt=F_k(u, u^*, v, v^*, \zeta_k)+O(4) , \quad d\zeta_k/dt=G_k(|u|^2, \zeta_k)+O(4) , \tag{6}$$

where O(4) denotes quartic and higher-order terms in $(u, u^*, v, v^*, \zeta_k)$ and

$$F_k(u, u^*, v, v^*, \zeta_k)=-a_k\zeta_k u-b_k\zeta_k v+d_k u|u|^2+e_k v|u|^2+f_k u|v|^2+g_k u^2v^*+h_k\zeta_k^2u+r_k\zeta_k^2v . \tag{7}$$

$$G_k(|u|^2, \zeta_k)=c_k|u|^2+s_k|u|^2\zeta_k \quad (k=1,2) .$$

Explicit expressions for the coefficients a_k, b_k, c_k, d_k, e_k, f_k, g_k, h_k, r_k, and s_k are given in the Appendix.

Now let us take into account small deviations of the parameters M_{01}, M_{02}, and ρ_1 (ρ_2) from the bifurcation points given by eqs. (5a, b). To do this we introduce linear unfolding terms into eqs. (6) according to standard approach [15,16]. Omitting O(4) terms we have

$$du/dt=v , \quad dv/dt=\epsilon_1 u+\epsilon_2 v+F_k(u, u^*, v, v^*, \zeta_k) , \quad d\zeta_k/dt=-\gamma_k\zeta_k+G_k(|u|^2, \zeta_k) . \tag{8}$$

In the case of perfect tuning the unfolding parameters ϵ_1, ϵ_2, and γ_k are real. They describe small perturbations of the coefficients in the characteristic equation determining the linear stability of the zero-intensity stationary solution $u=v=w=0$ of eqs. (8). This equation is given by

$$(\lambda^2+\epsilon_2\lambda+\epsilon_1)(\lambda+\gamma_k)=0 ,$$

and its solutions are equal to those solutions of eq. (2) which vanish at the bifurcation points (5a, b). Using this fact and neglecting the second order terms with respect to small deviations of the parameters M_{01} and M_{02} from their bifurcation values we obtain

$$\epsilon_1 = [\delta_2(\bar{M}_{01} - M_{01}) + \delta_1(\bar{M}_{02} - M_{02})]/(1+\delta_1+\delta_2) ,$$

$$\epsilon_2 = [(1+\delta_1)(\bar{M}_{01} - M_{01}) + (1+\delta_2)(\bar{M}_{02} - M_{02})]/(1+\delta_1+\delta_2)^2 , \quad \gamma_k = \rho_k .$$

In the next section we shall restrict our consideration to the specially chosen ϵ-neighborhood of the zero-intensity steady state where the term $-b_k\zeta_k v$ and the qubic terms in the normal form equations can be neglected since they are much smaller then the other terms of eqs. (8), (7). However, for certain values of the parameters, small amplitude solutions of eqs. (1) which are not contained in this neighborbood also can bifurcate from the trivial steady state at the points (5). We do not consider such solutions but we note that some of the terms neglected here can be necessary to analyze them.

3. Small amplitude chaotic solutions

The equations (8) describe the small amplitude dynamics of the original LSA equations in the small ϵ-neighborhood of the bifurcation points (5a, b). We define this neighborhood as

$$\epsilon_1 = \mathrm{sign}(\epsilon_1)\epsilon^2 , \quad \epsilon_2 = \mu\epsilon , \quad \gamma_k = \alpha_k \epsilon > 0 , \quad \epsilon > 0 , \tag{9}$$

where μ and α_k ($k=1, 2$) do not depend on ϵ. Let $a_k \neq 0$ and $c_k \neq 0$ in eqs. (7). Balancing the low-order terms by rescaling the variables in eqs. (8)

$$u = x\epsilon^{3/2}/\sqrt{|a_k c_k|} , \quad v = y\epsilon^{5/2}/\sqrt{|a_k c_k|} , \quad \zeta_k = z\epsilon^2 \mathrm{sign}(c_k)/a_k , \quad t = \tau/\epsilon , \tag{10}$$

and neglecting $O(\epsilon)$ terms we have the reduced equations

$$\mathrm{d}x/\mathrm{d}\tau = y , \quad \mathrm{d}y/\mathrm{d}\tau = \mathrm{sign}(\epsilon_1)x + \mu y - \mathrm{sign}(c_k)xz , \quad \mathrm{d}z/\mathrm{d}\tau = -\alpha_k z + |x|^2 . \tag{11}$$

Let $\mathrm{sign}(\epsilon_1)=1$ and x, y be real. These conditions correspond to laser operation above the linear threshold and to the neglect of the phases of the electric field and the atomic polarizations in the LSA equations (1). Consider eqs. (11) for the bifurcation point (5a) ($k=1$). It follows from the formulas given in the Appendix that $\mathrm{sign}(c_1)=1$. Hence, instead of eqs. (11), we have

$$\mathrm{d}x/\mathrm{d}\tau = y , \quad \mathrm{d}y/\mathrm{d}\tau = x + \mu y - xz , \quad \mathrm{d}z/\mathrm{d}\tau = -\alpha_1 z + x^2 . \tag{12}$$

These equations have three stationary solutions which we denote as S_0 and $S_\pm$. The stationary solution S_0 ($x=y=z=0$) corresponds to laser being off. The solutions $S_\pm$ correspond to stationary laser operation and are given by $x = \pm\sqrt{\alpha_1}$, $y=0$, $z=1$. For $\mu<0$, eqs. (12) are equivalent to the Shimizu–Morioka equations which were obtained from the well known Lorenz equations in the limit of large Rayleigh numbers (Ra→∞) [11]. The dynamics of the Shimizu–Morioka equations has been extensively studied. It has been shown that eqs. (12) demonstrate period-doubling cascades, an infinite number of various homoclinic and heteroclinic connections, Lorenz-like attractors, and chaotic quasi-attractors [12].

Figure 1 represents the x–z projections of the phase trajectories of the Shimizu–Morioka equations (12) for several values of the parameter μ and $\alpha_1 = 0.3$. Figs. 1a–e illustrate the period-doubling transition from a symmetric (invariant under the transformation $x \to -x$, $y \to -y$) limit cycle to a pair of asymmetric chaotic quasi-attractors with the increase of the parameter $-\mu$. At still higher values of $-\mu$ the asymmetric chaotic quasi-attractors undergo a symmetry-restoring crisis leading to a single symmetric quasi-attractor, which is shown in fig. 1f. The projections of the phase trajectories corresponding to the spiral chaotic quasi-attractor and the Lorenz attractor are shown in fig. 1g and fig. 1h, respectively.

A detailed investigation of the bifurcation set for the Shimizu–Morioka equations has been made by A.L. Shil'nikov [12]. We have calculated only a few of the bifurcation curves for eqs. (12) (see fig. 2). To calculate the bifurcations of the periodic solutions a specially designed program [17] has been used. In fig. 2 the curve

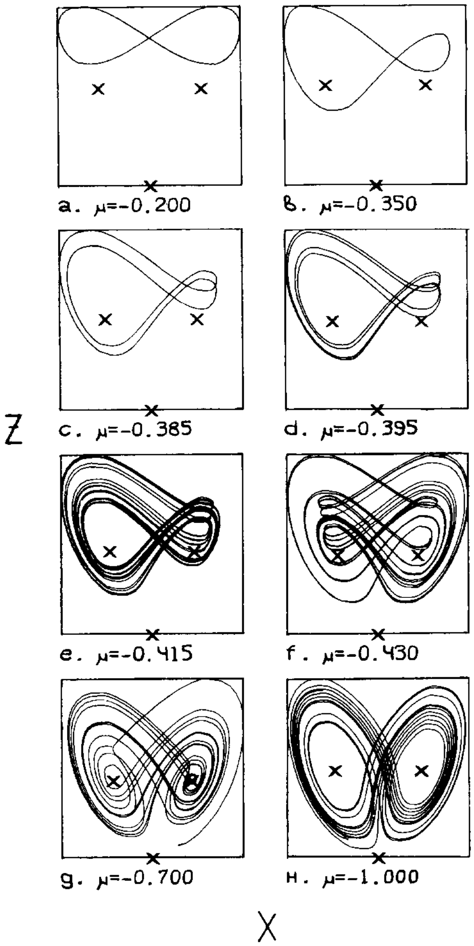

Fig. 1. Numerical solutions of eqs. (8) with $\alpha_1=0.3$. (a) Symmetric limit cycle, (b) period-one asymmetric limit cycle, (c) period-two limit cycle, (d) period-four limit cycle, (e), (f), (g), (h) chaotic attractors. Crosses indicate the positions of the stationary solutions S_0 and $S_\pm$.

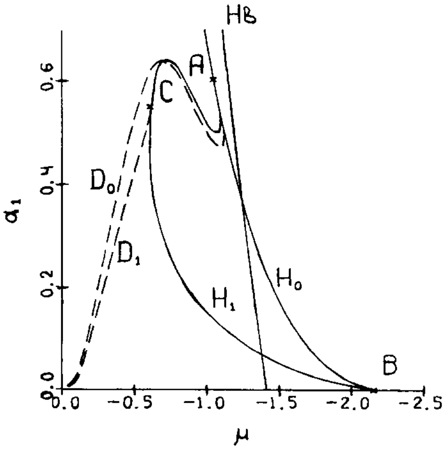

Fig. 2. Bifurcation curves for eqs. (8). The curve H_b corresponds to the Hopf bifurcation of the stationary solutions $S_\pm$. Dashed line D_0 (D_1) corresponds to pitchfork (period-doubling) bifurcations of periodic solution. Solid lines H_0 and H_1 correspond to homoclinic bifurcations.

of principal homoclinic orbits is denoted by the symbol H_0 and corresponds to a homoclinic "butterfly" which is shown in fig. 3a. The "butterfly" consists of two orbits homoclinic to the trivial stationary solution S_0. For sufficiently large values of α_1 this "butterfly" transforms into the homoclinic "figure-eight" which has been shown to exist near the Takens–Bogdanov bifurcation point [15,16]. The truncated normal form equations for this bifurcation can be obtained by the elimination of the variable ζ_k from the eqs. (8) using the center manifold theorem.

It can be shown that the periodic solutions emerging from the homoclinic orbits forming the "butterfly" are unstable if the "saddle quantity", $\sigma=\lambda_1+\lambda_2$, is positive. Here $\lambda_1>\lambda_2>\lambda_3$ are the eigenvalues of S_0. Using eqs. (12) we obtain $\sigma=-\alpha_1+(\mu+\sqrt{\mu^2+4})/2$. The condition $\sigma>0$ is fulfilled for the lower part of the curve H_0 lying between the points A and B. When crossing this part of H_0 from the right, the invariant hyperbolic chaotic set arises [18]. The codimension-two points A and B are the limit points for an infinite number of bifurcation curves. The point B ($\alpha_1=0$, $\mu\approx-2.153$) is the limit point for the curves corresponding to multi-circuited ho-

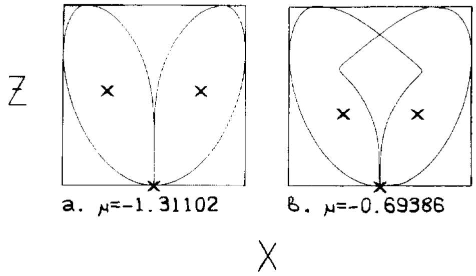

Fig. 3. Orbits homoclinic to the stationary solution S_0 of eqs. (8) with $\alpha_1=0.3$. (a) Homoclinic "butterfly", (b) double-circuited homoclinic orbits. Crosses indicate the positions of the stationary solutions.

moclinic orbits. The bifurcation phenomena near point A ($\sigma=0$) in the parameter space has been considered in ref. [12]. It has been shown that Lorenz-like attractors exist in the vicinity of the point in the α_1–μ plane. Two of the curves starting from the point A are shown in fig. 2. The curve D_0 corresponds to a pitchfork bifurcation of the symmetric periodic solution into a pair of asymmetric periodic solutions (see figs. 1a, b). The curve H_1 corresponds to the existence of two double-circuited orbitals homoclinic to the stationary state S_0. These orbits are shown in fig. 3b. The curve D_1 indicates the first bifurcation of the period-doubling sequence starting from the periodic solution which is shown in fig. 1b. The point C on the curve H_1 is the limit point for the curve D_1. Together with the point A this point belongs to the boundary of the existence of the Lorenz attractor [12]. The curve H_b corresponds to the Hopf bifurcations of the solutions $S_\pm$. This bifurcation is supercritical (subcritical) for $\alpha_1>\bar{\alpha}$ ($\alpha_1<\bar{\alpha}$), where $\bar{\alpha}\simeq 0.11248$.

Now let us consider the eqs. (11) for the bifurcation point (5b) ($k=2$, $\mathrm{sign}(c_2)=-1$). Let x and y be real and $\mathrm{sign}(\epsilon_1)=-1$ (laser operation below the linear threshold). Then instead of eq. (11) we have

$$\mathrm{d}x/\mathrm{d}\tau=y\,,\quad \mathrm{d}y/\mathrm{d}\tau=-x+\mu y+xz\,,\quad \mathrm{d}z/\mathrm{d}\tau=-\alpha_2 z+x^2\,. \tag{13}$$

Note that equations equivalent to eq. (13) were obtained in ref. [19] from the equations describing two-dimensional thermosolutal convection in a Boussinesq fluid by a different method. The relationship between the LSA equations and the equations describing convective instabilities in a two-component fluid was established earlier [2]. As do the Shimizu-Morioka equations, the eqs. (13) have three steady state solutions. One of them (S_0) is located at the origin $x=y=z=0$. The other two solutions ($S_\pm$) are given by $x=\pm\sqrt{\alpha_2}$, $y=0$, $z=1$. The origin $x=y=z=0$ is stable for $\mu<0$ and looses its stability at $\mu=0$ via a Hopf bifurcation leading to the emergence of a symmetric periodic solution which is shown in fig. 4a. It appears from the information of fig. 4 that the increase of the parameter μ in eqs. (13) leads to the chaotic behavior associated with period doubling cascades (see figs. 4b–f). Thus the small amplitude chaotic solutions exist near the bifurcation point (5b) as well as near the bifurcation point (5a). For sufficiently large positive values of the parameter μ, solutions of eqs. (13) become unbounded. This means that no attractor of real LSA equations lies in the neighborhood defined by eqs. (9) and (10).

Figure 5 represents several bifurcation curves for the solutions of eqs. (13) in the μ–α_2 plane. The curves D_0 and D_1 are similar to those in fig. 2 and indicate the pitchfork bifurcation and the first bifurcation of period doubling sequences. The parts of these curves lying below the line $\alpha_2=\mu$ correspond to the bifurcations of unstable periodic solutions. The curve of the principal heteroclinic loop H_0 corresponds to the existence of a pair of orbits connecting two saddle-foci $S_\pm$. This orbits are shown in fig. 6a. For large values of α_2 the principal loop transforms into the orbit connecting two saddles with real eigenvalues. The chaotic set arises when crossing the lower part of the curve H_0 for which Shil'nikov's criterion [18] is fulfilled. This part of H_0 is limited by the codimension-two point A corresponding to the zero "saddle quantity" of $S_\pm$ ($\sigma=\lambda_1+\mathrm{Re}\,\lambda_{2,3}=0$). The curve H_1 corresponds to the existence of homoclinic orbits connecting two saddle-foci $S_\pm$. One of these orbits is shown in fig. 6b.

Volume 100, number 1,2,3,4 OPTICS COMMUNICATIONS 1 July 1993

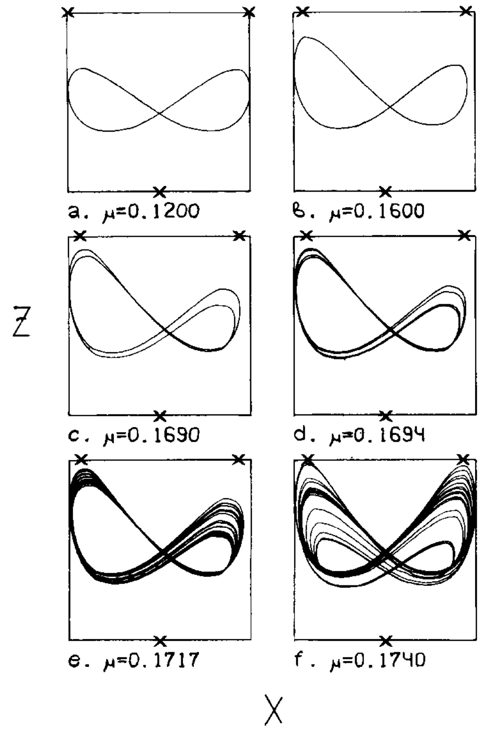

Fig. 4. Numerical solutions of eqs. (9) with $\alpha_2=0.5$. (a) Symmetric limit cycle, (b) period-one asymmetric limit cycle, (c) period-two limit cycle, (d) period-four limit cycle, (e), (f) chaotic attractors. Crosses indicate the positions of stationary solutions.

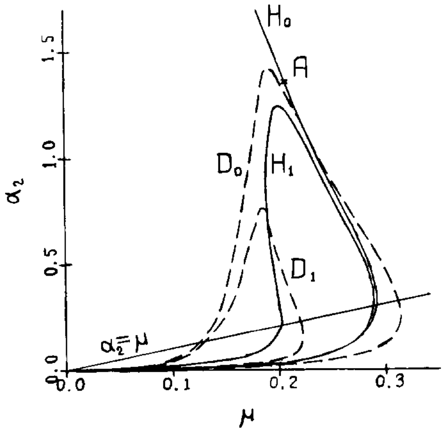

Fig. 5. Bifurcation curves for eqs. (9). Dashed line D_0 (D_1) corresponds to pitchfork (period-doubling) bifurcations of periodic solutions. Below the line $\alpha_2=\mu$ the vector field given by the eqs. (8) has negative divergence. In this region D_0 and D_1 indicate bifurcations of unstable solutions. Solid line H_0 (H_1) corresponds to heteroclinic (homoclinic) bifurcations.

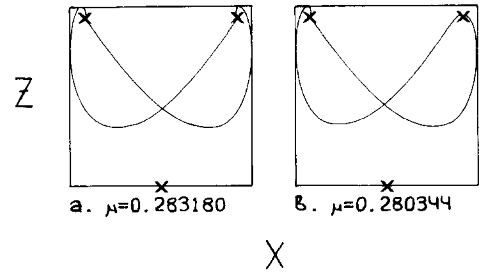

Fig. 6. Heteroclinic and homoclinic orbits of eqs. (8) with $\alpha_2=0.5$. (a) Orbits heteroclinic to stationary solutions $S_\pm$, (b) orbital homoclinic to stationary solution S_-. Crosses indicate the positions of stationary solutions.

4. The case with phases

Now consider the eqs. (11) with complex x and y. The phase space of these equations contains the flow invariant set given by $\Theta=xy^*-yx^*=0$, in which x and y may be considered as real variables ($\arg(x)-\arg(y)=\pm\pi N$, $N=0, 1, 2, \ldots$). Differentiating Θ with respect to time τ we obtain from eqs. (11)

$$\mathrm{d}\Theta/\mathrm{d}\tau=\mu\Theta\,. \tag{14}$$

It follows from eq. (14) that the invariant set $\Theta=0$ in the phase space of eqs. (11) is always stable for $\mu<0$.

This condition is fulfilled for the Shimizu–Morioka equations. Thus, all solutions of the complex Shimizu–Morioka equations (eqs. (11) with $\mathrm{sign}(\epsilon_1)=\mathrm{sign}(c_k)=1$ and $\mu<0$) tend to the asymptotically stable invariant set $\Theta=0$. This property of the complex Shimizu–Morioka equations is identical to that of the complex Lorenz equations [20]. Since eqs. (11) are equivariant under the O(2) action generated by $(x, y)\to\exp(\mathrm{i}\phi)(x, y)$ and $(x, y)\to(x^*, y^*)$, every solution of the complex Shimizu–Morioka equations on this set has the form $(x(\tau)\exp(\mathrm{i}\phi), y(\tau)\exp(\mathrm{i}\phi), z(\tau))$ with ϕ constant. Here $(x(\tau), y(\tau), z(\tau))$ is the solution of the real Shimizu–Morioka equations. Thus the small amplitude chaotic solutions of the LSA equations (1) near the bifurcation point (5a) are stable with respect to phase variables, but they are not asymptotically stable because the perturbations in the phase ϕ correspond to a zero eigenvalue. Here we do not consider the case of nonzero values of the detuning parameters. For the complex Lorenz equations this case was studied in refs. [21–24].

Another situation appears in the vicinity of the bifurcation point (5b) where the small amplitude chaotic solutions correspond to laser operation below the linear threshold ($\mathrm{sign}(\epsilon_1)=-1$ and $\mu>0$ in eqs. (11). It follows from eq. (14) that for $\mu>0$ the invariant set $\Theta=0$ of eqs. (11) is unstable. Thus the chaotic solutions emerging from the bifurcation point (5b) are unstable with respect to phase variables. This kind of chaotic solutions was calculated numerically in ref. [25].

5. Conclusion

We have shown that the small intensity chaotic solutions of the LSA equations (1) exist in the vicinity of the bifurcation point (5a). Thus solutions emerging above the linear laser threshold are similar to those in the Lorenz equations, which describe the single-mode laser, and they are stable with respect to the phase variables. In contradistinction to the Lorenz equations, the bifurcation point (5a) leading to the emergence of chaotic solutions in the LSA equations corresponds to finite LSA parameter values. This gives a qualitative understanding of the phenomenon of the second threshold reduction in a laser with a saturable absorber. We have shown that as distinct from the complex Lorenz equations the LSA equations (1) possess chaotic solutions which emerge below the linear laser threshold from the bifurcation point (5b) and that these solutions are unstable with respect to phase variables. Note that the multiple bifurcation points (5a, b) persist when polarization in the amplifier is adiabatically eliminated.

Though our bifurcation analysis is local in the parameter space the results of it in many cases remain qualitatively correct for parameter values that are sufficiently removed from the original bifurcation points (5a, b). Therefore the reduced equations (11) provide the simplest mathematical description of the qualitative aspects of low-dimensional chaotic dynamics in the LSA equations as well as in several other laser equations which possess the symmetries of the Lorenz equations (see refs. [26–31] for the examples of laser equations for which the analysis described here could be applied).

Acknowledgements

It is a pleasure to thank Prof. N.B. Abraham, Prof. L.P. Shil'nikov, and especially Prof. Ya.I. Khanin for their attention to this work and useful remarks. The authors also express their gratitude to Dr. A.L. Shil'nikov for fruitful discussions.

Appendix

Coefficients a_k, b_k, c_k, d_k, e_k, f_k, g_k, h_k, r_k, and s_k in eqs. (7) are given below. Here the subscript $k=1$ pertains to the bifurcation point (5a) and $k=2$ pertains to the bifurcation point (5b).

$$a_1=\delta_2/\Lambda_1>0\,,\quad b_1=(\delta_1+1)/\Lambda^2>0\,,\quad c_1=\delta_1(1+\delta_2)/(\delta_1-\delta_2)>0\,,$$

$$d_1=-\delta_2[\rho_2(2\delta_1-1)(1+\delta_2)-2\beta\delta_1(1+\delta_1)/[2\Lambda_1\rho_2(\delta_1-\delta_2)]\,,$$

$$e_1=\{\rho_2^2(\delta_2+1)[\delta_1\delta_2\Lambda_1-2(\delta_1+1)(2\delta_1-1)]$$
$$+\beta(\delta_1+1)[\rho_2(4\delta_2(1+\delta_2)-\delta_1\Lambda_1)-2\delta_1\delta_2\Lambda_1]\}/[2\Lambda_1^2\rho_2^2(\delta_1-\delta_2)]\,,$$

$$f_1=\{\rho_2^3(\delta_2+1)(\delta_1-1)[\delta_2\Lambda_1-2(\delta_1+1)]+\beta(\delta_1+1)[\rho_2^2(2\delta_2(2\delta_2+1)-\delta_1\Lambda_1-2)$$
$$-2\rho_2(\delta_2\delta_1(\Lambda_1-1)-\delta_1(\delta_1+1)+2\delta_2(\delta_2+1))-4\delta_1\delta_2\Lambda_1]\}/[2\Lambda_1^2\rho_2^3(\delta_1-\delta_2)]\,,$$

$$g_1=\{\beta\delta_1(1+\delta_1)[\rho_2(2\delta_2-1)-2\delta_2]-\rho_2^2\delta_2(1+\delta_2)(\delta_1-1)\}/[2\Lambda_1\rho_2^2(\delta_1-\delta_2)]\,,$$

$$h_1=\delta_2[(\delta_2-1)(\Lambda_1+1)\Lambda_1+\delta_2]/\Lambda_1^4\,,\quad r_1=(1+\delta_1)[(\delta_2-1)(\Lambda_1+1)\Lambda_1+2\delta_2]/\Lambda_1^5\,,$$

$$s_1=-[\Lambda_1(2\delta_1-1)(1+\delta_2)(\Lambda_1\delta_2-\delta_1-1)-\delta_1(1+\delta_1)(\Lambda_1-2\delta_1)]/[\Lambda_1^3(\delta_1-\delta_2)]\,,$$

$$a_2=\delta_1/\Lambda>0\,,\quad b_2=(\delta_2+1)/\Lambda^2>0\,,\quad c_2=-\beta\delta_2(1+\delta_1)/(\delta_1-\delta_2)<0\,,$$

$$d_2=\delta_1[\beta\rho_1(2\delta_2-1)(1+\delta_1)-2\delta_2(1+\delta_2)/[2\Lambda_1\rho_1(\delta_1-\delta_2)]\,,$$

$$e_2=-\{\beta\rho_1^2(\delta_1+1)[\delta_1\delta_2\Lambda_1-2(\delta_2+1)(2\delta_2-1)]$$
$$+(\delta_2+1)[\rho_1(4\delta_1(1+\delta_1)-\delta_2\Lambda_1)-2\delta_1\delta_2\Lambda_1]\}/[2\Lambda_1^2\rho_2^2(\delta_1-\delta_2)]\,,$$

$$f_2=-\{\beta\rho_1^3(\delta_1+1)(\delta_2-1)[\delta_1\Lambda_1-2(\delta_2+1)]+(\delta_2+1)[\rho_1^2(2\delta_1(2\delta_1+1)-\delta_2\Lambda_1-2)$$
$$-2\rho_1(\delta_1\delta_2(\Lambda_1-1)-\delta_2(\delta_2+1)+2\delta_1(\delta_1+1))-4\delta_2\delta_1\Lambda_1]\}/[2\Lambda_1^2\rho_1^3(\delta_1-\delta_2)]\,,$$

$$g_2=-\{\delta_2(1+\delta_2)[\rho_1(2\delta_1-1)-2\delta_1]-\beta\rho_1^2\delta_1(1+\delta_1)(\delta_2-1)\}/[2\Lambda_1\rho_1^2(\delta_1-\delta_2)]\,,$$

$$h_2=\delta_1[(\delta_1-1)(\Lambda_1+1)\Lambda_1+\delta_1]/\Lambda_1^4\,,\quad r_2=(1+\delta_2)[(\delta_1-1)(\Lambda_1+1)\Lambda_1+2\delta_1]/\Lambda_1^5\,,$$

$$s_2=\beta[\Lambda_1(2\delta_2-1)(1+\delta_1)(\Lambda_1\delta_1-\delta_2-1)-\delta_2(1+\delta_2)(\Lambda_1-2\delta_2)]/[\Lambda_1^3(\delta_1-\delta_2)]\,,$$

where

$$\Lambda_1=1+\delta_1+\delta_2\,.$$

References

[1] H. Haken, Phys. Lett. A 53 (1975) 77.
[2] M.G. Verlarde and J.C. Antoranz, Progr. Theor. Phys. 66 (1981) 717.
[3] L.A. Zenteno, K.E. Aldous and G.H.C. New, J. Mod. Optics 35 (1988) 1281.
[4] E.V. Grigoryeva, L.A. Kotomtseva and A.M. Samson, Phys. Lett. A 130 (1988) 57.
[5] V.S. Anischenko, M.A. Safonova, S.A. Tatarkova and V.V. Tuchin, Opt. Spektrosk. 70 (1991) 1092.
[6] N.B. Abraham, P. Mandel and L.M. Narducci, in: Progress in optics, vol. 25, ed. E. Wolf (North-Holland, Amsterdam, 1988) pp. 1–190.
[7] A. Arneodo, P.H. Coullet, E.A. Spiegel and C. Tresser, Physica D 14 (1985) 327.
[8] L.A. Lugiato, P. Mandel, S.T. Dembinski and A.A. Kossakowski, Phys. Rev. A 18 (1978) 1438.
[9] P. Mandel and T. Erneux, Phys. Rev. A 30 (1984) 1893.
[10] A.G. Vladimirov and D.Yu. Volkov, in: Quantum electronics laser science, Technical Digest Series, Vol. 11 (Optical Society of America, Washington, D.C., 1991) p. 98.
[11] T. Shimizu and N. Morioka, Phys. Lett. A 76 (1980) 201.
[12] A.L. Shil'nikov, Selecta Math. Sovietica 10 (1991) 105 (originally published in: Methods of qualitative theory of differential equations (in russian) (Gor'kov. Gos. Univ., Gor'ki, 1986) pp. 180–193); Physica D, to be published.
[13] P. Peplowski and H. Haken, Physica D 30 (1988) 135.

[14] G. Dangelmayr and M. Neveling, J. Phys. A: Math. Gen. 22 (1989) 1291.
[15] V.I. Arnold, Geometrical methods in the theory of ordinary differential equations (Springer, Berlin, 1983).
[16] J. Guckenheimer and P. Holmes, Nonlinear oscillations, dynamical systems and bifurcations of vector fields (Springer, Berlin, 1983).
[17] A.I. Hibnik, in: Algorithms and programs in FORTRAN: materials for the mathematical support of computers, Vol. 5. (Special National Information Center, Scientific Center for Biological Research, Pushchino, 1979).
[18] V.S. Afraimovich, V.V. Bykov and L.P. Shil'nikov, Trans. Mosc. Math. Soc. 2 (1983) 153.
[19] M.R.E. Proctor and N.O. Weiss, Nonlinearity 3 (1990) 619.
[20] S. Sarkar, J.S. Satchell and H.J. Carmichael, J. Phys. A: Math. Gen. 19 (1986) 2765.
[21] A.C. Fowler, J.D. Gibbon and M.J. McGuinness, Physica D 4 (1982) 139.
[22] H. Zeghlache and P. Mandel, J. Opt. Soc. Am. B 2 (1985) 18.
[23] H. Zeghlache, P. Mandel, N.B. Abraham and C.O. Weiss, Phys. Rev. A 38 (1988) 3128.
[24] C. Ning and H. Haken, Phys. Rev. A 41 (1990) 3826.
[25] A.G. Vladimirov, E.B. Pelyukhova and E.E. Fradkin, in: OSA Proc. on Nonlinear dynamics in optical systems, Vol. 7, eds. N.B. Abraham, E.M. Garmire and P. Mandel (Optical Society of America, Washington DC, 1991) p. 527.
[26] N.B. Abraham, D. Dangoisse, P. Glorieux and P. Mandel, J. Opt. Soc. Am. B 2 (1985) 23.
[27] R. Graham and Y. Cho, Optics Comm. 47 (1983) 52.
[28] P.A. Khandokhin, Ya.I. Khanin and I.V. Koryukin, Optics Comm. 65 (1988) 367.
[29] B. Meziane, Optics Comm. 75 (1990) 287.
[30] Li Ruo-Ding and P. Mandel, in: OSA Proc. on Nonlinear dynamics in optical systems, Vol. 7, eds. N.B. Abraham, E.M. Garmire and P. Mandel (Optical Society of America, Washington DC, 1991) p. 241.
[31] F. de Tomasi, B. Zambon and E. Arrimondo, in: OSA Proc. on Nonlinear dynamics in optical systems, Vol. 7, eds. N.B. Abraham, E.M. Garmire and P. Mandel (Optical Society of America, Washington DC, 1991) p. 519.