Low-intensity chaotic operations of a laser with a saturable absorber

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We study the dynamics of a laser with a saturable absorber for parameter values close to a multiple bifurcation point. We show that different chaotic attractors exist in a small vicinity of the zero-intensity steady state near the laser threshold.

1. Introduction

It is well known that the Maxwell–Bloch equations for a single-mode ring laser exhibit the Lorenz instability leading to chaotic dynamics [1]. However, the threshold for this instability (the second laser threshold) requires a set of parameter values that are not very accessible in experiments. In several recent studies it has been shown by means of numerical calculations that adding a saturable absorber into the laser resonator may produce a substantial reduction of the second threshold (see, for example, refs. [2–6]). We demonstrate here that for suitable parameter values the Maxwell–Bloch equations for a laser with a saturable absorber (LSA) can exhibit small-intensity chaotic solutions near the lasing threshold (first laser threshold). In ref. [7] this kind of chaotic solutions was called an asymptotic chaos. Our investigation also provides a qualitative understanding of the bifurcation sequences leading to chaos in LSA equations.

We study the simplest theoretical model of a ring single-mode LSA in which each atomic species is modeled by a set of homogeneously broadened two-level atoms. For this model the semiclassical theory yields five mean-field ordinary differential LSA equations for three complex and two real variables [6,8,9]. Neglecting the phases of the electric field and atomic polarizations in the case of perfect tuning we have five real LSA equations [9].

We consider the bifurcation points in the parameter space, for which the jacobian matrix of the perturbations of the real LSA equations evaluated at the zero-intensity steady state has a triply degenerate zero eigenvalue with geometrical multiplicity two. In the vicinity of these points the application of normal form theory yields the reduced equations which describe the local dynamics of LSA equations. We investigate the bifurcations leading to chaotic oscillations in the reduced equations. We show that there exists a case in which the reduced equations may be transformed into the Shimizu–Morioka equations [10]. It is known that the Shimizu–Morioka equations which were proposed in ref. [11] exhibit various kinds of chaotic behavior. The bifurcation phenomena in these equations have been studied in great detail as reported in ref. [12].

We discuss the influence of the phases of the electric field and the atomic polarizations in the LSA equations.
on the stability of the solutions under consideration. We show that the LSA equations exhibit two kinds of chaotic solutions. One of them is stable with respect to phase variables and the other is unstable.

2. Laser model and reduction to normal form

The LSA model studied is similar to that considered in refs. [8, 9]. In the case of perfect tuning it is given by the following set of five ordinary differential equations:

\[ \frac{dE}{dt} = -E + P_1 + P_2, \quad \frac{dP_1}{dt} = -P_1 \delta_1 - E(M_{01} + M_1), \quad \frac{dP_2}{dt} = -P_2 \delta_2 - E(M_{02} + M_2), \]

\[ \frac{dM_1}{dt} = -\rho_1 M_1 + \frac{(E^*P_1 + P_1^*E)}{2}, \quad \frac{dM_2}{dt} = -\rho_2 M_2 + \beta(E^*P_2 + P_2^*E)/2. \] (1)

for dimensionless variables, which are the electric field amplitude \( E \), the atomic polarization amplitudes \( P_k \), and the deviations of the population differences \( M_k \) from their values \( M_{0k} \) without the laser field. Here the subscript \( k = 1 \) pertains to the amplifying medium and \( k = 2 \) pertains to the absorbing medium. \( M_{01} < 0, M_{02} > 0 \).

The variables \( E, P_1, \) and \( P_2 \) are complex, the variables \( M_1 \) and \( M_2 \) are real. The parameters \( \rho_k \) are transversal and longitudinal relaxation constants divided by the resonator halfwidth which has been used to renormalize the time \( t \). The parameter \( \beta \) is the ratio of the saturation intensity in the amplifying medium to the saturation intensity in the absorbing medium. Note that the LSA equations (1) are equivariant under the \( O(2) \) symmetry group action generated by \((E, P_1, P_2) \to \exp(i\phi) (E, P_1, P_2) \) and \((E, P_1, P_2) \to (E^*, P_1^*, P_2^*) \).

A linear stability analysis of the zero-intensity stationary solution \( E = P_k = M_k = 0 \) \((k = 1, 2)\) of eqs. (1) leads to the characteristic equation of the form

\[ (\lambda^2 + A_2 \lambda + A_1 \lambda + A_0)(\lambda + \rho_1)(\lambda + \rho_2) = 0, \] (2)

where

\[ A_0 = \delta_1 \delta_2 + \delta_2 M_{01} + \delta_1 M_{02}, \quad A_1 = \delta_1 + \delta_2 + \delta_1 \delta_2 + M_{01} + M_{02}, \quad A_2 = 1 + \delta_1 + \delta_2. \] (3)

Note that \( A_0 = 0 \) corresponds to the linear laser threshold whereas \( A_1 A_2 - A_0 = 0 \) and \( A_0<0 \) correspond to the Hopf bifurcation of the zero-intensity steady state. If two conditions \( A_0 = 0 \) and \( A_1 = 0 \) are satisfied simultaneously then we have bifurcation of the Takens–Bogdanov type corresponding to double zero \( \lambda \) in eq. (2). This bifurcation in LSA equations was studied in refs. [13, 14]. Here we consider two multiple bifurcation points with the codimension greater than that of the Takens–Bogdanov bifurcation. They are determined by three conditions

\[ A_0 = 0, \quad A_1 = 0, \quad \rho_1 = 0 \text{ or } \rho_2 = 0. \] (4)

At these points the jacobian matrix \( J \) of eqs. (1) (with real \( E \) and \( P_k \)) evaluated at the zero-intensity stationary solution has a triply degenerate zero eigenvalue with geometrical multiplicity two.

Let us select \( M_{01}, M_{02}, \) and \( \rho_1 \) \((M_{01}, M_{02}, \) and \( \rho_2 \)) as our bifurcation parameters. Then, taking into account eqs. (3), we conclude that two bifurcation points (4) exist when \( \delta_1 > \delta_2 \), and they are given by

\[ M_{01} = M_{01}, \quad M_{02} = M_{02}, \quad \rho_1 = 0, \] \( \text{(5a)} \)

and

\[ M_{01} = \tilde{M}_{01}, \quad M_{02} = \tilde{M}_{02}, \quad \rho_2 = 0, \] \( \text{(5b)} \)

where

\[ \tilde{M}_{01} = -\delta_1^2(1 + \delta_2)/(\delta_1 - \delta_2), \quad \tilde{M}_{02} = \delta_2^2(1 + \delta_1)/(\delta_1 - \delta_2). \]

Consider the LSA equations (1) with the bifurcation parameters fixed by eqs. (5a) (eqs. (5b)). In order to
obtain the reduced form of eq. (1) we first introduce a linear change of variables \((E, P_1, P_2, M_1, M_2) \rightarrow (u', v', w', \xi_1, \xi_2)\),

\[
E = u' + v' + w', \quad P_1 = [u' \delta_1 (\delta_2 + 1) + v' (\delta_1 - 1) (\delta_2 + 1) - w' \delta_2^2] / (\delta_1 - \delta_2),
\]

\[
P_2 = -[u' \delta_2 (\delta_1 + 1) + v' (\delta_2 - 1) (\delta_1 + 1) - w' \delta_1^2] / (\delta_1 - \delta_2), \quad M_k = \xi_k (k = 1, 2),
\]

which transforms the jacobian matrix \(J\) of eqs. (1) at the bifurcation points (5a, b) into the Jordan normal form

\[
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -A_1 \\
0 & 0 & 0 & 0 & -A_2
\end{bmatrix}
\]

Next, using the center manifold theory, we eliminate the variables \(w', \xi_2^2 (\xi_1)\) corresponding to the negative eigenvalues \(-A_1\) and \(-A_2\) and we apply the normal form technique \([15, 16]\) to eliminate certain nonlinear terms by the transformation preserving \(O(2)\) symmetry

\[
u = u' + u [R_{1k} |u|^2 + R_{2k} (v^* u + u^* v)],
\]

\[
v' = v' + u [R_{3k} |u|^2 + R_{4k} (v^* u + u^* v) + R_{6k} \xi_k^2] + v [R_{7k} |u|^2 + R_{8k} (v^* u + u^* v) + R_{9k} \xi_k + R_{10k} \xi_k^2],
\]

\[
\xi_k = \xi_k^2 + |u|^2 (R_{11k} + R_{12k} \xi_k^2) + (v^* u + u^* v) (R_{3k} + R_{4k} \xi_k).
\]

Here \(k = 1 (k = 2)\) corresponds to the bifurcation point (5a) (bifurcation point (5b)). Real coefficients \(R_{jk}\) \((j = 1, 14; k = 1, 2)\) depend on the parameters of the LSA equations. We do not present explicit expressions for these coefficients since they are very cumbersome and irrelevant in further analysis. The resulting normal form equations calculated up to the third order terms are given by

\[
du/dt = v + O(4), \quad dv/dt = F_k (u, u^*, v, v^*, \xi_k) + O(4), \quad d\xi_k/dt = G_k (|u|^2, \xi_k) + O(4),
\]

where \(O(4)\) denotes quartic and higher-order terms in \((u, u^*, v, v^*, \xi_k)\) and

\[
F_k (u, u^*, v, v^*, \xi_k) = -a_k \xi_k u - b_k \xi_k v + d_k u |u|^2 + e_k v |u|^2 + f_k u |v|^2 + g_k u^2 v^* + h_k \xi_k^2 u + r_k \xi_k v.
\]

\[
G_k (|u|^2, \xi_k) = c_k |u|^2 + s_k |u|^2 \xi_k (k = 1, 2).
\]

Explicit expressions for the coefficients \(a_k, b_k, c_k, d_k, e_k, f_k, g_k, h_k, r_k, s_k\) are given in the Appendix.

Now let us take into account small deviations of the parameters \(M_{01}, M_{02}, \) and \(\rho_1 (\rho_2)\) from the bifurcation points given by eqs. (5a, b). To do this we introduce linear unfolding terms into eqs. (6) according to standard approach \([15, 16]\). Omitting \(O(4)\) terms we have

\[
u = v + \epsilon_1 u + \epsilon_2 v + F_k (u, u^*, v, v^*, \xi_k), \quad d\xi_k/dt = -y_k \xi_k + G_k (|u|^2, \xi_k).
\]

In the case of perfect tuning the unfolding parameters \(\epsilon_1, \epsilon_2, \) and \(y_k\) are real. They describe small perturbations of the coefficients in the characteristic equation determining the linear stability of the zero-intensity stationary solution \(u = v = w = 0\) of eqs. (8). This equation is given by

\[
(\lambda^2 + \epsilon_2 \lambda + \epsilon_1)(\lambda + y_k) = 0,
\]

and its solutions are equal to those solutions of eq. (2) which vanish at the bifurcation points (5a, b). Using this fact and neglecting the second order terms with respect to small deviations of the parameters \(M_{01}\) and \(M_{02}\) from their bifurcation values we obtain
In the next section we shall restrict our consideration to the specially chosen $\epsilon$-neighborhood of the zero-intensity steady state where the term $-b_k \zeta_k^3 v$ and the cubic terms in the normal form equations can be neglected since they are much smaller than the other terms of eqs. (8), (7). However, for certain values of the parameters, small amplitude solutions of eqs. (7) which are not contained in this neighborhood also can bifurcate from the trivial steady state at the points (5). We do not consider such solutions but we note that some of the terms neglected here can be necessary to analyze them.

3. Small amplitude chaotic solutions

The equations (8) describe the small amplitude dynamics of the original LSA equations in the small $\epsilon$-neighborhood of the bifurcation points (5a, b). We define this neighborhood as

$$\epsilon_1 = \text{sign}(\epsilon_1) \epsilon^2, \quad \epsilon_2 = -\mu \epsilon, \quad \zeta_k = \text{sign}(\epsilon_1) \epsilon^0, \quad \epsilon > 0,$$

where $\mu$ and $\alpha_k$ ($k = 1, 2$) do not depend on $\epsilon$. Let $a_k \neq 0$ and $c_k \neq 0$ in eqs. (7). Balancing the low-order terms by rescaling the variables in eqs. (8)

$$u = x^3/\sqrt{|a_k c_k|}, \quad v = y^{3/2}/\sqrt{|a_k c_k|}, \quad \zeta_k = x^2 \text{sign}(c_k) / a_k, \quad t = \tau / \epsilon,$$

and neglecting $O(\epsilon)$ terms we have the reduced equations

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = x + \mu y - \text{sign}(c_k) x z, \quad \frac{dz}{dt} = -\alpha_z z + x^2.$$

Let $\text{sign}(\epsilon_1) = 1$ and $x, y$ be real. These conditions correspond to laser operation above the linear threshold and to the neglect of the phases of the electric field and the atomic polarizations in the LSA equations (1). Consider eqs. (11) for the bifurcation point (5a) ($k = 1$). It follows from the formulas given in the Appendix that $\text{sign}(c_1) = 1$. Hence, instead of eqs. (11), we have

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = x + \mu y - x z, \quad \frac{dz}{dt} = -\alpha_z z + x^2.$$

These equations have three stationary solutions which we denote as $S_0$ and $S_\pm$. The stationary solution $S_0$ ($x = y = z = 0$) corresponds to laser being off. The solutions $S_\pm$ correspond to stationary laser operation and are given by $x = \pm \sqrt{\alpha_1}, y = 0, z = 0$. For $\mu < 0$, eqs. (12) are equivalent to the Shimizu-Morioka equations which were obtained from the well known Lorenz equations in the limit of large Rayleigh numbers ($Ra \rightarrow \infty$) [11]. The dynamics of the Shimizu-Morioka equations has been extensively studied. It has been shown that eqs. (12) demonstrate period-doubling cascades, an infinite number of various homoclinic and heteroclinic connections, Lorenz-like attractors, and chaotic quasi-attractors [12].

Figure 1 represents the $x$-$z$ projections of the phase trajectories of the Shimizu-Morioka equations (12) for several values of the parameter $\mu$ and $\alpha_1 = 0.3$. Figs. 1a-e illustrate the period-doubling transition from a symmetric (invariant under the transformation $x \rightarrow -x, y \rightarrow -y$) limit cycle to a pair of asymmetric chaotic quasi-attractors with the increase of the parameter $-\mu$. At still higher values of $-\mu$ the asymmetric chaotic quasi-attractors undergo a symmetry-restoring crisis leading to a single symmetric quasi-attractor, which is shown in fig. 1f. The projections of the phase trajectories corresponding to the spiral chaotic quasi-attractor and the Lorenz attractor are shown in fig. 1g and fig. 1h, respectively.

A detailed investigation of the bifurcation set for the Shimizu-Morioka equations has been made by A.L. Shilnikov [12]. We have calculated only a few of the bifurcation curves for eqs. (12) (see fig. 2). To calculate the bifurcations of the periodic solutions a specially designed program [17] has been used. In fig. 2 the curve
Fig. 1. Numerical solutions of eqs. (8) with $\alpha_1 = 0.3$. (a) Symmetric limit cycle, (b) period-one asymmetric limit cycle, (c) period-two limit cycle, (d) period-four limit cycle, (e), (f), (g), (h) chaotic attractors. Crosses indicate the positions of the stationary solutions $S_0$ and $S_2$.

Fig. 2. Bifurcation curves for eqs. (8). The curve $H_b$ corresponds to the Hopf bifurcation of the stationary solutions $S_2$. Dashed line $D_0$ ($D_1$) corresponds to pitchfork (period-doubling) bifurcations of periodic solution. Solid lines $H_0$ and $H_1$ correspond to homoclinic bifurcations.

of principal homoclinic orbits is denoted by the symbol $H_0$ and corresponds to a homoclinic “butterfly” which is shown in fig. 3a. The “butterfly” consists of two orbits homoclinic to the trivial stationary solution $S_0$. For sufficiently large values of $\alpha_1$, this “butterfly” transforms into the homoclinic “figure-eight” which has been shown to exist near the Takens–Bogdanov bifurcation point [15,16]. The truncated normal form equations for this bifurcation can be obtained by the elimination of the variable $\zeta_4$ from the eqs. (8) using the center manifold theorem.

It can be shown that the periodic solutions emerging from the homoclinic orbits forming the “butterfly” are unstable if the “saddle quantity”, $\sigma = \lambda_1 + \lambda_2$, is positive. Here $\lambda_1 > \lambda_2 > \lambda_3$ are the eigenvalues of $S_0$. Using eqs. (12) we obtain $\sigma = -\alpha_1 + (\mu + \sqrt{\mu^2 + 4})/2$. The condition $\sigma > 0$ is fulfilled for the lower part of the curve $H_0$ lying between the points A and B. When crossing this part of $H_0$ from the right, the invariant hyperbolic chaotic set arises [18]. The codimension-two points A and B are the limit points for an infinite number of bifurcation curves. The point B ($\alpha_1 = 0, \mu \approx -2.153$) is the limit point for the curves corresponding to multi-circuited ho-
Fig. 3. Orbits homoclinic to the stationary solution $S_0$ of eqs. (8) with $\alpha_1=0.3$. (a) Homoclinic “butterfly”, (b) double-circuited homoclinic orbits. Crosses indicate the positions of the stationary solutions.

homoclinic orbits. The bifurcation phenomena near point A ($\sigma=0$) in the parameter space has been considered in ref. [12]. It has been shown that Lorenz-like attractors exist in the vicinity of the point in the $\alpha_1-\mu$ plane. Two of the curves starting from the point A are shown in fig. 2. The curve $D_0$ corresponds to a pitchfork bifurcation of the symmetric periodic solution into a pair of asymmetric periodic solutions (see figs. 1a, b). The curve $H_1$ corresponds to the existence of two double-circuited orbital homoclinic to the stationary state $S_0$. These orbits are shown in fig. 3b. The curve $D_1$ indicates the first bifurcation of the period-doubling sequence starting from the periodic solution which is shown in fig. 1b. The point C on the curve $H_1$ is the limit point for the curve $D_1$. Together with the point A this point belongs to the boundary of the existence of the Lorenz attractor [12]. The curve $H_b$ corresponds to the Hopf bifurcations of the solutions $S_k$. This bifurcation is supercritical (subcritical) for $(\gamma_1 > \bar{\alpha}$, $(\gamma_1 < \bar{\alpha}$), where $\bar{\alpha} \approx 0.11248$.

Now let us consider the eqs. (11) for the bifurcation point (5b) ($k=2$, $\text{sign}(c_2) = -1$). Let $x$ and $y$ be real and $\text{sign}(E, ) = -1$ (laser operation below the linear threshold). Then instead of eq. (11) we have

$$
\begin{align*}
dx/\mathrm{d}\tau &= y, \\
y/\mathrm{d}\tau &= -x + \mu y + x z, \\
z/\mathrm{d}\tau &= -\alpha_2 z + x^2.
\end{align*}
$$

Note that equations equivalent to eq. (13) were obtained in ref. [19] from the equations describing two-dimensional thermodilutional convection in a Boussinesq fluid by a different method. The relationship between the LSA equations and the equations describing convective instabilities in a two-component fluid was established earlier [2]. As do the Shimizu-Morioka equations, the eqs. (13) have three steady state solutions. One of them ($S_0$) is located at the origin $x=y=z=0$. The other two solutions ($S_{\pm}$) are given by $x = \pm \sqrt{\alpha_2}$, $y=0$, $z=1$. The origin $x=y=z=0$ is stable for $\mu<0$ and loses its stability at $\mu=0$ via a Hopf bifurcation leading to the emergence of a symmetric periodic solution which is shown in fig. 4a. It appears from the information of fig. 4 that the increase of the parameter $\mu$ in eqs. (13) leads to the chaotic behavior associated with period doubling cascades (see figs. 4b–f). Thus the small amplitude chaotic solutions exist near the bifurcation point (5b) as well as near the bifurcation point (5a). For sufficiently large positive values of the parameter $\mu$, solutions of eqs. (13) become unbounded. This means that no attractor of real LSA equations lies in the neighborhood defined by eqs. (9) and (10).

Figure 5 represents several bifurcation curves for the solutions of eqs. (13) in the $\mu-\alpha_2$ plane. The curves $D_0$ and $D_1$ are similar to those in fig. 2 and indicate the pitchfork bifurcation and the first bifurcation of period doubling sequences. The parts of these curves lying below the line $\alpha_2=\mu$ correspond to the bifurcations of unstable periodic solutions. The curve of the principal heteroclinic loop $H_0$ corresponds to the existence of a pair of orbits connecting two saddle-foci $S_{\pm}$. This orbits are shown in fig. 6a. For large values of $\alpha_2$ the principal loop transforms into the orbit connecting two saddles with real eigenvalues. The chaotic set arises when crossing the lower part of the curve $H_0$ for which Shil'nikov's criterion [18] is fulfilled. This part of $H_0$ is limited by the codimension-two point A corresponding to the zero “saddle quantity” of $S_{\pm}$ ($\sigma=\lambda_1 + \text{Re} \lambda_2,\lambda_3=0$). The curve $H_1$ corresponds to the existence of homoclinic orbits connecting two saddle-foci $S_0$. One of these orbits is shown in fig. 6b.
4. The case with phases

Now consider the eqs. (11) with complex \( x \) and \( y \). The phase space of these equations contains the flow invariant set given by \( \Theta = xy^* - yx^* = 0 \), in which \( x \) and \( y \) may be considered as real variables \((\arg(x) - \arg(y) = \pm \pi N, N = 0, 1, 2, \ldots)\). Differentiating \( \Theta \) with respect to time \( \tau \) we obtain from eqs. (11)

\[
\frac{d\Theta}{d\tau} = \mu \Theta.
\]

It follows from eq. (14) that the invariant set \( \Theta = 0 \) in the phase space of eqs. (11) is always stable for \( \mu < 0 \).
This condition is fulfilled for the Shimizu–Morioka equations. Thus, all solutions of the complex Shimizu–Morioka equations (eqs. (11) with \( \text{sign}(\epsilon_1) = \text{sign}(c_k) = 1 \) and \( \mu < 0 \)) tend to the asymptotically stable invariant set \( \Theta = 0 \). This property of the complex Shimizu–Morioka equations is identical to that of the complex Lorenz equations [20]. Since eqs. (11) are equivariant under the \( O(2) \) action generated by \( (x, y) \to \exp(i\theta)(x, y) \) and \( (x, y) \to (x^*, y^*) \), every solution of the complex Shimizu–Morioka equations on this set has the form \( (x(\tau) \exp(i\phi), y(\tau) \exp(i\phi), \tau(\tau)) \) with \( \phi \) constant. Here \( (x(\tau), y(\tau), \tau(\tau)) \) is the solution of the real Shimizu–Morioka equations. Thus the small amplitude chaotic solutions of the LSA equations (1) near the bifurcation point (5a) are stable with respect to phase variables, but they are not asymptotically stable because the perturbations in the phase \( \phi \) correspond to a zero eigenvalue. Here we do not consider the case of nonzero values of the detuning parameters. For the complex Lorenz equations this case was studied in refs. [21–24].

Another situation appears in the vicinity of the bifurcation point (5b) where the small amplitude chaotic solutions correspond to laser operation below the linear threshold (\( \text{sign}(\epsilon_1) = -1 \) and \( \mu > 0 \)) in eqs. (11). It follows from eq. (14) that for \( \mu > 0 \) the invariant set \( \Theta = 0 \) of eqs. (11) is unstable. Thus the chaotic solutions emerging from the bifurcation point (5b) are unstable with respect to phase variables. This kind of chaotic solutions was calculated numerically in ref. [25].

5. Conclusion

We have shown that the small intensity chaotic solutions of the LSA equations (1) exist in the vicinity of the bifurcation point (5a). Thus solutions emerging above the linear laser threshold are similar to those in the Lorenz equations, which describe the single-mode laser, and they are stable with respect to the phase variables. In contradistinction to the Lorenz equations, the bifurcation point (5a) leading to the emergence of chaotic solutions in the LSA equations corresponds to finite LSA parameter values. This gives a qualitative understanding of the phenomenon of the second threshold reduction in a laser with a saturable absorber. We have shown that as distinct from the complex Lorenz equations the LSA equations (1) possess chaotic solutions which emerge below the linear laser threshold from the bifurcation point (5b) and that these solutions are unstable with respect to phase variables. Note that the multiple bifurcation points (5a, b) persist when polarization in the amplifier is adiabatically eliminated.

Though our bifurcation analysis is local in the parameter space the results of it in many cases remain qualitatively correct for parameter values that are sufficiently removed from the original bifurcation points (5a, b). Therefore the reduced equations (11) provide the simplest mathematical description of the qualitative aspects of low-dimensional chaotic dynamics in the LSA equations as well as in several other laser equations which possess the symmetries of the Lorenz equations (see refs. [26–31] for the examples of laser equations for which the analysis described here could be applied).

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Appendix

Coefficients \( a_k, b_k, c_k, d_k, e_k, f_k, g_k, h_k, r_k, \) and \( s_k \) in eqs. (7) are given below. Here the subscript \( k = 1 \) pertains to the bifurcation point (5a) and \( k = 2 \) pertains to the bifurcation point (5b).
\[ a_1 = \frac{\delta_2}{A_1} > 0, \quad b_1 = \frac{(\delta_1 + 1) / A^2 > 0, \quad c_1 = \frac{\delta_1}{(1 + \delta_2) / (\delta_1 - \delta_2) > 0,} \]

\[ d_1 = -\delta_2 [2 \beta (2 \delta_1 - 1) (1 + \delta_2) - 2 \beta (1 + \delta_1) / [2 A_1 \rho(2 \delta_1 - 2)]] \]

\[ e_1 = \rho_1^2 (\delta_1 + 1) [\delta_1, \delta_2, A_1 - 2(\delta_1 + 1)(2\delta_1 - 1)] \]

\[ + \beta (\delta_1 + 1) [\rho_1 (4\delta_2 (1 + \delta_1 - \delta_1 A_1) - 2 \delta_1, \delta_2) / [2 \beta A_1^2 \rho_1^2 (\delta_1 - \delta_2)], f_1 = \rho_1^2 (\delta_1 + 1)(\delta_1 - 1) [2 \delta_2, A_1 - 2(\delta_1 + 1)] + \beta (\delta_1 + 1) [\rho_1^2 (2 \delta_2 (2\delta_2 - 1) - \delta_1 A_1 - 2) \]

\[ - 2 \rho_1 (\delta_2, \delta_1 (\delta_1 + 1) + 2 \delta_2, (\delta_1 + 1) - 4 \delta_2, \delta_2 A_1)] / [2 \beta A_1^2 \rho_1^2 (\delta_1 - \delta_2)], g_1 = \delta_2 [\rho_1 (2 \delta_2 - 1) - \delta_1 A_1 + \delta_2] / [A_1^2], \]

\[ h_1 = \delta_2 ((\delta_2 - 1) (A_1 + 1) A_1 + \delta_2) / [A_1^2], \quad r_1 = (1 + \delta_1) [(\delta_2 - 1) (A_1 + 1) A_1 + \delta_2] / [A_1^2, \]

\[ s_1 = -(A_1 (2 \delta_1 - 1) (1 + \delta_2) (A_1, \delta_2 - 1) - \delta_1 (1 + \delta_1) (A_1, \delta_2 - 1)] / [A_1^2 (\delta_1 - \delta_2)], \]

\[ a_2 = \delta_1 / A > 0, \quad h_2 = (\delta_2 + 1) / A^2 > 0, \quad c_2 = -\beta (1 + \delta_1) / (\delta_1 - \delta_2) < 0, \]

\[ d_2 = \delta_2 \beta (2 \delta_2 - 1) (1 + \delta_1 - 2 \delta_2) / [2 A_1 \rho_1 (\delta_1 - \delta_2)], \]

\[ e_2 = -\beta (\delta_1 + 1) [\delta_1, \delta_2, A_1 - 2(\delta_2 + 1) (2 \delta_2 - 1)] \]

\[ + (\delta_1 + 1) [\rho_1 (4 \delta_1 (1 + \delta_1 - \delta_2, A_1) - 2 \delta_1, \delta_2) A_1)] / [2 \beta A_1^2 \rho_1^2 (\delta_1 - \delta_2)], \]

\[ f_2 = -\beta \delta_2 [(\delta_1 + 1) (\delta_2 - 1) [\rho_1 (2 \delta_2 - 1) 2(\delta_2 + 1) + (\delta_2 + 1) [\rho_1^2 (2 \delta_2 - 1) - \delta_2, A_1 - 2) \]

\[ - 2 \rho_1 (\delta_2, \delta_1 (\delta_1 + 1) + 2 \delta_2, (\delta_1 + 1) - 4 \delta_2, \delta_2 A_1)] / [2 \beta A_1^2 \rho_1^2 (\delta_1 - \delta_2)], \]

\[ g_2 = -(\delta_2 + 1) [\rho_1 (2 \delta_2 - 1) - \delta_2, A_1 + \delta_2] / [2 \beta A_1^2 \rho_1^2 (\delta_1 - \delta_2)], \]

\[ h_2 = \delta_2 [(\delta_1 - 1) (A_1 + 1) A_1 + \delta_2) / [A_1^2, \quad r_2 = (1 + \delta_2) [(\delta_1 - 1) (A_1 + 1) A_1 + \delta_2) / [A_1^2 (\delta_1 - \delta_2)], \]

where

\[ A_1 = 1 + \delta_1 + \delta_2. \]

References