Properties of the phase space and bifurcations in the complex Lorenz model

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The geometrical structure of the phase space and bifurcations in the complex Lorenz model are investigated. It is shown that the hierarchy of bifurcations in a single-mode laser with detuning of the resonator frequency from the frequency of a spectral line is similar to the hierarchy of bifurcations of the logistic map. *© 1998 American Institute of Physics.* [S1063-7842(98)00108-1]

INTRODUCTION

The complex Lorenz equations, or complex Lorenz model

$$\dot{x}=-\sigma(x-y),$$

$$\dot{y}=-(1-i\delta)y+(r-z)x,$$

$$\dot{z}=-bz+\frac{1}{2}(x^*y+xy^*), \tag{1}$$

was first introduced by Gibbon and McGuinness[1] as a generalization of the standard Lorenz model.[2,3] The complex Lorenz model differs from the latter in that *x* and *y* are complex. Formally, this complexity stems from the presence of the real parameter δ and the complex parameter $r=r_1+ir_2$, which are not a part of the original Lorenz model. The complex Lorenz model has important bearing on nonlinear dynamics because it is a universal finite-dimensional approximation for the class of distributed systems that exhibit so-called dispersion instability of a steady-state solution at a point of parameter space $\mu=\mu_c$ (Ref. 1). For such systems it has been shown[1] that the expansion of the vector representing the perturbation of the steady-state solution in powers of the small parameter $\varepsilon=(|\mu-\mu_c|)^{1/2}$ in a certain approximation leads to a system of equations, equivalent to (1), for the coefficients of the expansion. As an example, the model of a baroclinic instability in the atmosphere[4,5] is investigated in Ref. 1.

The variables in Eqs. (1) are the perturbation amplitudes relative to a spatially homogeneous solution of partial differential equations and, as such, do not admit a clear-cut physical interpretation in application to a baroclinic instability. However, there are systems for which the variables of the complex Lorenz model are observable quantities. These systems are lasers and masers, for which *x* and *y* represent the slowly varying complex amplitudes of the electric field and polarization of the medium, respectively, and *z* denotes the population difference between the energy levels of the working transition. In reality Eqs. (1) appeared in quantum electronics long before their ''discovery'' by Gibbon and McGuinness[1] (see, e.g., Ref. 6). The direct observability of the variables makes lasers the most suitable object for experimental implementation of the dynamics associated with the complex Lorenz model. Indeed, the results of experiments with single-mode, far-infrared lasers have been found to best match the results of numerical integration of the system (1) (Ref. 7). And even though Eqs. (1) provide the most realistic description of this type of laser exclusively, they are the simplest model reflecting such fundamental laser properties as the threshold character of lasing, frequency pulling, and the capability of generating complex wave modes.

However, despite a wealth of papers on the complex Lorenz model,[1,8–17] to this day it has not received the attention that it deserves. The investigations of this model are concerned primarily with the analysis of particular regimes, usually by numerical methods. In our opinion, the most incisive results have been obtained in Refs. 10 and 11 in regard to the stability analysis of simple periodic solutions corresponding to steady-state lasing. The objective of the present study is to investigate the geometrical structures of the complex Lorenz model and certain global properties of its solutions in connection with distinctive features of this structure.

An important geometrical property of the complex Lorenz model is its invariance under the transformation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} xe^{i\psi} \\ ye^{i\psi} \\ z \end{pmatrix}, \tag{2}$$

where ψ is an arbitrary phase constant.

This transformation corresponds to the group $U(1)$ acting in that subspace C^2 of the total phase space $\mathcal{H}$ which refers to the variables *x* and *y*. To picture the role of $U(1)$ — the symmetry for the structure of the limit sets in the phase space of the complex Lorenz model, it is sufficient to consider the case $\delta=r_2=0$. It can be shown[17] that in this case any trajectory in $\mathcal{H}$ is attracted to the invariant three-dimensional hyperspace

$$\frac{\mathrm{Re}(x)}{\mathrm{Im}(x)}=\frac{\mathrm{Re}(y)}{\mathrm{Im}(y)}=\mathrm{const}, \tag{3}$$

where the constant on the right side depends on the initial conditions.

On this hypersurface the real and imaginary parts of x and y vary synchronously with time, satisfying the equations of the original Lorenz model. Even under these conditions, however, the attractors of the original Lorenz model and its complex generalization are not identical. Indeed, if a given trajectory is attracted to a given limit set situated on the hypersurface defined by Eqs. (3), it follows from the symmetry property (2) that a trajectory that differs from the given trajectory by a certain common phase of x and y will be attracted to a set that is the image of the given set under the action (2) with the corresponding phase ψ. Consequently, an attractor of the Lorenz model in the given situation represents the direct sum of an infinite number of sets isomorphic to an attractor of the original model.

This example graphically demonstrates the more complex structure of the phase space of the complex Lorenz model in comparison with the original model and the role of symmetry in this greater complexity. However, it will be shown below that symmetry is not a tool that can be used to simplify the problem. Here we note an interesting analogy between the role of symmetry for physical systems described by the complex Lorenz model and for quantum systems described by a wave function. For neither system does the general phase of the state vector (which would be the vector with components x and y in the case of the complex Lorenz model) carry any information about the physical state; it merely characterizes the result of interference of a given state with certain other states. In quantum mechanics ''spurious'' information can be filtered out by application of the density matrix formalism. If a given state vector in a Hilbert space is described by N complex numbers, the corresponding density matrix is characterized by $2N-1$ real numbers.

The indicated analogy permits this approach to be applied to the complex Lorenz model. In Sec. 1. we introduce a special projective space, in which states differing by the common phase of x and y are treated as equivalent, and we derive equations of motion for the complex Lorenz model in this space. We also show how all information on the physical state of a system and phase evolution can be reproduced by means of these equations. In Sec. 2. we use these equations to analyze the boundedness properties of the limit sets of the complex Lorenz model in the projective space, and we show how these properties are related to prominent features of the phase dynamics and homoclinic bifurcations. In particular, the well-known bifurcation of generation of a homoclinic ''butterfly'' in the original Lorenz model is found to have codimensionality 2 in the complex model. On the basis of these results, in Sec. 3. we construct and analyze a one-dimensional map in the vicinity of a homoclinic bifurcation point for the domain of parameters of the complex Lorenz model. The results of our analysis of the map are compared with the results of a direct numerical investigation of bifurcations.

1. NORMAL FORM OF THE EQUATIONS AND THE PROJECTIVE SPACE

It will be advantageous below to use another system of variables in addition to x, y, and z. Let

$$\sigma(r_1-1)-\frac{\delta^2}{4}\equiv\eta>0,\quad b<2\sigma. \tag{4}$$

We introduce the change of variables

$$x'=\eta^{-3/4}ax,\quad y'=\eta^{-5/4}\sigma a\left(y-\left(1+\frac{i\delta}{2\sigma}\right)x\right),$$

$$z'=\eta^{-1}\sigma\left(z-\frac{xx^*}{2}\right),\quad t'=t\sqrt{\eta}, \tag{5}$$

where

$$a=e^{-i\delta t/2}\sqrt{\frac{2\sigma-b}{2}}.$$

After this substitution, which is similar to one proposed in Refs. 18 and 19 for the Lorenz model, Eqs. (1) assume the form

$$\frac{dx'}{dt'}=y',\quad \frac{dy'}{dt'}=(1+i\nu)x'-\mu y'-x'z'-\varrho x'|x'|^2,$$

$$\frac{dz'}{dt'}=-\beta z'+|x'|^2. \tag{6}$$

Here

$$\nu=\frac{2r_2\sigma+\delta(\sigma-1)}{2\eta},\quad \mu=\frac{1+\sigma}{\sqrt{\eta}},$$

$$\varrho=\frac{\sqrt{\eta}}{2\sigma-b},\quad \beta=\frac{b}{\sqrt{\eta}}. \tag{7}$$

The Jacobian of the substitution (5) is equal to $|a|^4\sigma^3/\eta^5$. Consequently, for $\eta>0$ and $b<2\sigma$ the change of variables (5) specifies a one-to-one continuous map (diffeomorphism) of the phase space of the system (1) $\mathcal{H}$ onto the phase space of the system (6) $\mathcal{H}'$ and vector fields, specified by the systems (1) and (6), which are topologically equivalent. In other words, when the conditions imposed on the values of the parameters are satisfied, the dynamics of the system (6) is equivalent to the dynamics of the complex Lorenz model.

If we ignore the term $\varrho x|x|^2$ in (6), we obtain a complex generalization of the Shimizu–Morioka equations, which have been investigated in detail in Ref. 20. It has been shown[21,22] that in a certain approximation the Shimizu–Morioka equations are also the normal form of the equations describing the chaotic dynamics near a bifurcation point with a triple zero eigenvalue having geometric multiplicity 2. We note that in (6) all the coefficients are real when $\nu=0$, i.e., when condition (33) holds (see below).

We consider the map Π: $\mathcal{H}(x,y,z)\to\mathcal{P}$, where $\mathcal{P}$ is a projective space with Cartesian coordinates u,v,w,z:

$$u=(|x|^2-|y|^2)/2,\quad v=\mathrm{Re}(x^*y),\quad w=\mathrm{Im}(x^*y). \tag{8}$$

The idea of using this map is based on the analogy of the complex Lorenz model with a two-level quantum-mechanical system. If we regard the variables x and y as components of a Schrödinger state vector, we can express the corresponding density matrix in terms of a linear combination of Pauli matrices with u, v, and w as the coefficients of this expansion. Note that

$$|x|^2=R+u, \quad |y|^2=R-u, \quad x^*y=v+iw, \tag{9}$$

where

$$R=(u^2+v^2+w^2)^{1/2}=(|x|^2+|y|^2)/2. \tag{10}$$

It is evident from Eqs. (9) that the variables u, v, w, and z contain the sum-total of information about the state of the system in the sense discussed in the Introduction. The map Π associates with each point of $\mathcal{H}$ that differ only by the common phase of x and y the map Π associates the same point in $\mathcal{P}$, whereas the images of states that differ by the amplitudes and/or the phase difference of x and y differ. We shall borrow the term *ray* from geometry[23] to designate the set of points of $\mathcal{H}$ corresponding to the same point of $\mathcal{P}$, and we shall refer to $\mathcal{P}$ as ''ray space.''

Differentiating Eqs. (8) with respect to the time and making use of Eqs. (1), we obtain the equations of motion in ray space

$$\dot{u}=-(\sigma+1)u+(\sigma-r_1+z)v-r_2w-(\sigma-1)R,$$

$$\dot{v}=-(\sigma+1)v-\delta w-(\sigma-r_1+z)u+(\sigma+r_1-z)R,$$

$$\dot{w}=-(\sigma+1)w+\delta v+r_2(R+u), \quad \dot{z}=-bz+v. \tag{11}$$

Accordingly, the image of system (6) under the map Π: $\mathcal{H}'(x',y',z')\to\mathcal{P}'$, where $\mathcal{P}'$ is a projective space equivalent to $\mathcal{P}$, is the system of equations

$$\dot{u}'=v'+\mu(R'-u')-\nu w'-v'(1-z'-\rho(R'+u')),$$

$$\dot{v}'=-\mu v'+R'-u'+(R'+u')(1-z'-\rho(R'+u')),$$

$$\dot{w}'=-\mu w'+\nu(R'+u'), \quad \dot{z}'=-\beta z'+(R'+u'). \tag{12}$$

We note that in phase spaces $\mathcal{H}$ and $\mathcal{H}'$ the sets $\mathcal{Z}$ and $\mathcal{Z}'$ of points on the z and z' axes are invariant under the flows specified by the systems of equations (1) and (6), respectively. The same is true of the corresponding point sets in spaces $\mathcal{P}$ and $\mathcal{P}'$. It follows, therefore, that the point sets $\mathcal{H}/Z$, $\mathcal{H}'/Z'$, $\mathcal{P}/Z$, and $\mathcal{P}'/Z'$ are also invariant under the flows specified by (1) and (6).

Before using Eqs. (11) and (12) in place of (1) and (6), we need to find a way to obtain information about the motion in $\mathcal{H}$ from the solutions of Eqs. (11). We consider the relation between such characteristics of the dynamical state of the system in $\mathcal{H}$ and in $\mathcal{P}$ as the Lyapunov characteristic exponents and fractal dimensionality of an attractor. For a given trajectory $X_0(t)$ of the dynamical system $dX/dt=F(X)$ the spectrum of Lyapunov exponents Λ_i is defined as

$$\Lambda_i=\lim_{t\to\infty} t^{-1}\ln\left(\frac{|e_i(t)|}{|e_i(0)|}\right), \tag{13}$$

where $e_i(t)$ denotes the fundamental solutions of the linear system of equations

$$\frac{dY}{dt}=\left.\frac{\partial F}{\partial X}\right|_{X=X_0(t)} Y. \tag{14}$$

We introduce local coordinates ξ_i in a neighborhood of a point $X_0\in\mathcal{H}/Z$:

$$\xi_1=u(x,y), \quad \xi_2=v(x,y), \quad \xi_3=w(x,y),$$

$$\xi_4=z, \quad \xi_5=\frac{\mathrm{Im}(\langle X_0,X\rangle)}{\langle X_0,X_0\rangle}. \tag{15}$$

From now on we use angle brackets to denote the Hermitian scalar product defined on C^2. The Jacobian of the transformation at the point $X=X_0$ is equal to $|x_0|^2+|y_0|^2>0$, so that (15) is a diffeomorphism in a certain neighborhood of X_0.

Writing Eqs. (14) for the system (1) in the local coordinates $\xi(\xi_1),\dots,\xi_5)$ defined by (15), we obtain

$$\frac{d\xi}{dt}=\begin{pmatrix} \begin{pmatrix} & & \\ & \hat{A} & \\ & & \end{pmatrix} & \begin{matrix}0\\0\\0\\0\end{matrix} \\ 0,0,0,0 & 0 \end{pmatrix}\xi, \tag{16}$$

where $\hat{A}$ is the Jacobian matrix of the system of equations (11), evaluated at the point $\Pi(X_0(t))$

It is evident from Eq. (16) that the matrix $\hat{A}$ specifies the evolution of perturbations orthogonal to ξ_5 in $X_0(t)$, whereas perturbations along ξ_5 remain neutral. Since (15) is a diffeomorphism, it follows from Eqs. (13) and (16) that the spectrum of Lyapunov exponents for a trajectory in $\mathcal{H}/Z$ differs from the spectrum for its projection in $\mathcal{P}$ only by the presence of a single additional null exponent. In particular, this implies that if a given set in $\mathcal{H}/Z$ is an attractor with Lyapunov dimensionality D_L, its image in $\mathcal{P}$ is an attractor with Lyapunov dimensionality D_L—1. This relation is also valid for fractal dimensionalities of a limit set in $\mathcal{H}/Z$ and its projection in $\mathcal{P}/Z$. The latter result follows from the fact that every limit set in $\mathcal{H}/Z$ can be represented locally (in a neighborhood of the given ray) by the direct product of a set in $\mathcal{P}/Z$ and the ray, i.e., the set $\mathcal{R}^1$.

Another piece of physical information associated with the trajectories in $\mathcal{H}$ and, in our opinion, missing from Eqs. (11) is the relative phase of two states. A rule for comparing the phases of two states of a physical system described by a complex state vector has been introduced by Pancharatnam for the states of classical polarized light[24] and has subsequently been generalized to the case of quantum systems.[25,26] It can be stated as follows from the complex Lorenz model: Two states X_1 and X_2 are said to be in phase if the norm $\|X_1+X_2\|$ is the maximum of all possible values of the total phases of X_1 and X_2. We note that this rule can be used to compare the phases of states associated with different rays. The value of the norm for two given rays is defined as the phase of the complex number $\langle X_1,X_2\rangle$. This phase is called

Pancharatnam's phase. Drawing on the analogy between the complex Lorenz model and the Schrödinger equations, Toronov *et al.*[15] have shown that Pancharatnam's phase occurs naturally in the laser dynamics problem. We now discuss this problem briefly from the standpoint of differential geometry.

The triplet $(\mathcal{H}/Z, P/Z, \Pi)$ forms a fiber bundle ''2'' (Refs. 23 and 27), for which $\mathcal{H}/Z$ is the fiber bundle space, $\mathcal{P}/Z$ is the base, a fiber is a ray, and the structural group is $U(1)$. We note that this fiber bundle is nontrivial, i.e., the entire space $\mathcal{H}/Z$ cannot be represented as a direct product of the base and a ray. The equation

$$\xi_5 = 0 \tag{17}$$

defines the complexity on the fiber bundle. According to the conventional terminology of differential geometry, a curve in $\mathcal{H}$ is said to be horizontal (relative to a given complexity) if its velocity vector at every point is directed along the tangent to the surface $\xi_5 = 0$. The complexity defined by Eq. (17) ensures the uniqueness of a horizontal curve in $\mathcal{H}$ that is projected onto a given curve in $\mathcal{P}$ and passes through a given point. For the given type of fiber bundles and for the evolution of the state vector along a horizontal trajectory (relative to the given type of complexity) it has been shown[25,26] that Pancharatnam's phase for two states on a given trajectory $X(t)$ can be expressed in the form

$$\gamma = -\oint_{\Gamma T} A_s ds, \tag{18}$$

where

$$A_s = \mathrm{Im}(\langle X(s)|d/ds|X(s)\rangle)/\langle X(s)|X(s)\rangle, \tag{19}$$

ΓT is the closed contour in $\mathcal{H}$ formed by the segment T of the trajectory between two states and a curve Γ whose projection is geodesic in $\mathcal{P}$.

The integral in (12) has a nonzero value if the state vector of the system in $\mathcal{H}$ does not return to the initial point after transition around the closed contour ΓT in $\mathcal{P}$. This possibility reflects the nontriviality of the fiber bundle.[25,26]

We note that a transformation of the type (2) but with time-dependent ψ

$$\psi(t) = \int_0^t h(\tau) d\tau,$$

where $h(t)$ is a time function, takes (1) into the system of equations

$$\dot{x} = -(\sigma + i h(t))x + \sigma y,$$

$$\dot{y} = -[1 + i(h(t) - \delta)]y + (r - z)x,$$

$$\dot{z} = -bz + \frac{1}{2}(x^* y + x y^*), \tag{20}$$

which is homeomorphic to (1) if $h(t)$ is continuous [in a laser experiment $h(t)$ is the phase difference of the reference signal used in heterodyne measurements[7] from a monochromatic signal having the frequency of the empty-cavity mode].

The horizontal curve whose image is the given trajectory in $\mathcal{P}$ is the trajectory of a dynamical system (20) with $h(t) = \mathrm{Im}(\langle X, F(X)\rangle / \langle X, X\rangle$, where $\mathbf{F}(\mathbf{X})$ is the phase velocity vector of the system of equations (1) [see definition (17)]. It follows from the connection of systems (1) and (20) by the transformation (2) that for arbitrary $h(t)$ the total lead of the common phase of variables x and y in (20) can be represented by the two-term sum

$$\gamma = \gamma_d + \gamma_g, \tag{21}$$

where

$$\gamma_d = \int_0^t [h(\tau) + \mathrm{Im}(\langle X, F(X)\rangle)/\langle X, X\rangle] d\tau \tag{22}$$

is the dynamical phase, and the geometric phase γ_g is given by Eq. (18).

We now show that when $h(t)$ is a given time function or a constant, the phase lead can be determined by solving the system (11) without reference to Eqs. (1). In fact, the dynamical phase is given by the time integral of the function

$$\frac{\mathrm{Im}(\langle X, F(X)\rangle)}{\langle X, X\rangle} = \frac{[\delta(R-u) - (\sigma + r_1 - z)w]}{R},$$

expressed in terms of the coordinates of a point in $\mathcal{P}$.

To prove this statement for the geometrical part, in $\mathcal{P}$ we introduce spherical coordinates

$$u = \rho \cos\theta, \quad v = \rho \sin\theta \cos\phi, \quad w = \rho \sin\theta \sin\phi.$$

According to (8), $x = \rho^{1/2}\cos(\theta/2)\exp[i(\Theta)]$, and $y = \rho^{1/2}\sin(\theta/2)\exp[i(\Theta + \phi)]$, where Θ is the common phase. Expressing A_s in (18) in terms of the spherical coordinates, we obtain

$$\gamma_g = \oint_{\Gamma T} \sin^2(\theta/2) d\phi, \tag{23}$$

where the integral is evaluated in $\mathcal{P}$ around the contour formed by the trajectory and a geodesic.

It is evident that the right side of (23) is just equal to one-half the solid angle subtended by the contour.[27]

This discussion of the characteristics of phase evolution makes it clear that when a trajectory of the system in $\mathcal{H}$ represents the image of a limit cycle in $\mathcal{P}$, it is closed only for a special choice of coordinate system (carrier signal) defined by the function $h(t)$. This result is consistent with the following assertion deduced from the relations between the dimensionalities of the limit sets in $\mathcal{H}$ and $\mathcal{P}$: A periodic attractor in $\mathcal{P}$ must correspond to a torus in $\mathcal{H}$. The projection of the torus onto a limit cycle is but one example of how the analysis of system dynamics is simplified by the introduction of the projective space $\mathcal{P}$. In the next section we employ the representations of the complex Lorenz model in $\mathcal{P}$ (11) and in $\mathcal{P}'$ (12) to reveal some general properties of its solutions.

2. BOUNDEDNESS OF LIMIT SETS IN A HOMOCLINIC BIFURCATION

For the Lorenz model it is a well-known fact[2] that all the limit sets of trajectories in phase space are bounded by the sphere

$$x^2+y^2+(z-\sigma-r)^2-K^2(\sigma+r)^2=0,$$

where

$$K^2\geqslant\frac{1}{4}+\frac{b}{4}\max(\sigma^{-1},1).$$

We now show that this property is preserved in the complex Lorenz model, i.e., the limit sets in $\mathcal{H}$ are bounded by the hypersphere

$$|x|^2+|y|^2+(z-\sigma-r_1)^2-K^2(\sigma+r_1)^2=0 \tag{24}$$

with the same K as for the original Lorenz model.

Let us consider the one-parameter family of spheres

$$V_M\equiv|x|^2+|y|^2+(z-\sigma-r_1)^2-M^2=0, \tag{25}$$

where M is the parameter, and the time derivatives are

$$\dot{V}_M=-2\sigma|x|^2-2|y|^2-2b\left(z-\frac{\sigma+r_1}{2}\right)^2+b\frac{(\sigma+r_1)^2}{2}, \tag{26}$$

since

$$\dot{V}_M|_{V_M=0}=(F,\nabla V_M), \tag{27}$$

where F is the phase velocity vector; the trajectories on the sphere $V_M=0$ are directed into or out of the sphere if the right side of Eq. (26) has positive or negative values, respectively, on the sphere. It is evident from (26) that this function does not depend on the parameters δ and r_2. We can therefore use a result obtained[2] for the Lorenz model: The given function is positive on any sphere $V_M=0$ having a radius greater than the radius of the sphere (24).

An equation describing the bounding surface for limit sets in $\mathcal{P}$, corresponding to the hypersphere (24), can be obtained by making the substitution $|x|^2+|y|^2=2R$ in (24), which gives the equation of a spheroid

$$S{:}2R+(z-\sigma-r_1)^2-K^2(r_1+\sigma)^2=0. \tag{28}$$

We now consider the hypersurface in $\mathcal{P}$ specified by the equation

$$Q(u,v,w,z)\equiv u-\frac{2\sigma}{\delta}w+R=0. \tag{29}$$

In the subspace $\mathcal{R}^3$ of variables u,v,w Eq. (29) specifies a two-dimensional half cone with vertex at the origin, its symmetry axis directed along the unit vector $(\alpha,0,\beta)$, where $\alpha=[1+(2\sigma/\delta)^2]^{-1/2}$ and $\beta=(2\sigma/\delta)\alpha$. The cosine of the angle between the axis and the generatrix of the cone is equal to α. For $\delta>0$ the cone degenerates into the plane $w=0$. For $\delta<0$ the surface is situated in the region of negative w. In space $\mathcal{P}'$ the surface $Q=0$ corresponds to the hyperplane

$$w'=0.$$

Here $Q>(<)\ 0$ corresponds to $w'>(<)\ 0$. It is evident from Eq. (12) that for $w'=0$ we have $\dot{w}'=\nu(R'+u')=\nu|x'|^2$. The derivative $\dot{w}'$ is therefore nonnegative for $\nu>0$ and is nonpositive for negative ν. Consequently, for $\nu>(<)\ 0$ trajectories on the surface are tangent to it or are directed toward the region $\mathcal{P}$, where $Q>(<)\ 0$ [$w'>(<)\ 0$ in $\mathcal{P}'$].

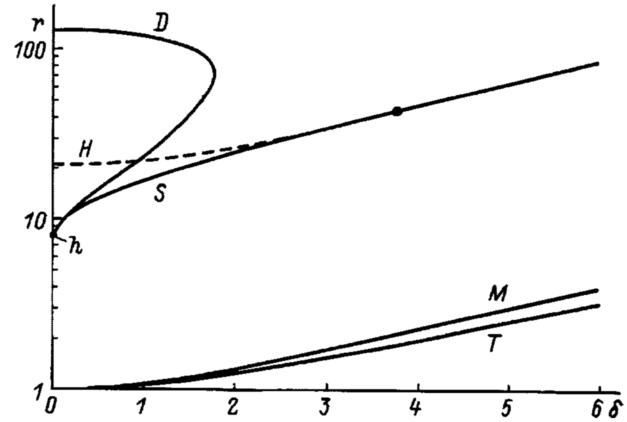

FIG. 1. Bifurcation diagram for the system of equations (11) on the plane of the parameter (δ,r); $\sigma=3$, $r_2=0$, $b=1/9$. The values of the parameter r are plotted in logarithmic scale. T) stability threshold of the trivial equilibrium state; H) Hopf bifurcation curve; S) saddle–node bifurcation curve; D) period-doubling bifurcation curve; M) curve $\sigma(r-1)-\delta^2/4=0$.

It follows from (12) that the surface (29) is globally stable and invariant against flow for $\nu=0$. This condition can be rewritten in the form

$$r_2=r_{2c}=\delta\frac{1-\sigma}{2\sigma}. \tag{30}$$

We now show that for $r_2>(<)r_{2c}$ every trajectory beginning in the region $Q<(>)\ 0$ [$w'<(>)\ 0$] tends toward the region where $Q\geqslant(\leqslant)\ 0$ [$w'\geqslant(\leqslant)\ 0$]. Let $r_2>r_{2c}$ ($\nu>0$). We consider the family of hyperplanes in $\mathcal{P}'$: $w'=C<0$. It is evident from (12) that for $\nu>0$ we have $\dot{w}'>0$ on these surfaces (since μ is always positive). Consequently, every trajectory emanating from the region $w'<0$ ($Q<0$) intersects each of these surfaces in succession and eventually ends up on the surface $w'=0$ ($Q=0$) or in the region $w'>0$ ($Q>0$). More precisely, there exists a set of trajectories, having measure zero, which tends to the origin as $t\to\infty$ (see below). All other trajectories enter the region $w'>0$ ($Q>0$) earlier or later.

To show that for $r_2<r_{2c}$ every trajectory tens to the set of points in $\mathcal{P}$ for which $Q\leqslant0$ ($w'\leqslant0$), we need to analyze the family of surfaces $w'=C>0$. We omit this proof, which is easily done by the same approach as for $r_2>r_{2c}$.

We now look at some important consequences of the existence of the bounding surface $Q=0$. First of all, we note that for a laser ($r_2=0$) all attractors are located in the region of $\mathcal{P}$ where $Q\geqslant0$ or $\delta>0$ and in a symmetric region for $\delta<0$ (see Fig. 1). If the discussion is confined to the subspace of variables (u,v,w), the region in question is the region of the solid angle Ω subtended by the half cone $Q=0$. Consequently, for a trajectory associated with a certain attractor the solid angle subtended by the contour ΓT [see (18)] is not greater than Ω. In the limit $\delta\to\pm0$ the cone is transformed into the flow-invariant plane $w=0$, which is globally stable in this case, so that the solid angle corresponding to ΓT tends to the limiting value $\pm2\pi$. This result explains the resonance jump of the average slope of the phase of a laser field by an amount equal to the characteristic average frequency of the intensity fluctuations $2\pi/\tau$ (τ is the average period of the intensity fluctuations, which coincides with the average time for the representative point to go around the origin on the plane $w=0$). It is interesting to note that this jump was first discovered in numerical calculations[13] and was interpreted as a manifestation of the geometric phase in laser dynamics on the basis of a numerical analysis of the behavior of trajectories in ray space.[16]

Another consequence of the existence of the bounding surface $Q=0$ ($w'=0$) is an additional constraint on the values of the system parameters in homoclinic bifurcation. Sensitive (nonrobust) homoclinic loops of the separatrix are known to form a very important structure responsible for the formation of a chaotic set of trajectories in the original Lorenz model.[3] Since the complex Lorenz model subsumes the original Lorenz model as a special case for $\delta=0$ and $r_2=0$, corresponding homoclinics also occur in the complex model. A necessary condition for their existence is the intersection of stable and unstable invariant manifolds of the saddle point located at the origin.[28,29]

The local structure of the invariant manifolds in the vicinity of the saddle point can be determined from a linear analysis of Eqs. (1) or (6) in the vicinity of the solution $x=y=z=0$ ($x'=y'=z'=0$).

The trivial solution of Eqs. (6) $x'=y'=z'=0$ is unstable and is a saddle point when

$$r_1>1+\frac{\delta^2-r_2\sigma+\delta r_2(1-\sigma)}{(1+\sigma)^2}.$$

Its eigenvalues are

$$\lambda_1=-\frac{\mu}{2}+\sqrt{1+\frac{\mu^2}{4}+i\nu},$$

$$\lambda_2=-\frac{\mu}{2}-\sqrt{1+\frac{\mu^2}{4}+i\nu},$$

$$\lambda_3=-\beta, \tag{31}$$

where μ, ν, and β are defined in Eq. (7).

They correspond to the eigenvectors

$$V_1=N_1^{-1}\begin{pmatrix}1\\ \lambda_1\\ 0\end{pmatrix},\quad V_2=N_2^{-1}\begin{pmatrix}1\\ \lambda_2\\ 0\end{pmatrix},\quad V_3=\begin{pmatrix}0\\ 0\\ 1\end{pmatrix},$$

where $N_{1,2}=\sqrt{1+|\lambda_{1,2}|^2}$.

It follows from Eqs. (31) that for sufficiently small ν we have $\mathrm{Re}\lambda_1>0$, $\mathrm{Re}\lambda_2<0$, and $\lambda_3<0$. Moreover, for $\nu>0$

$$\mathrm{Im}\ \lambda_1=-\mathrm{Im}\ \lambda_2>0. \tag{32}$$

We shall consider only the case $|\lambda_3|<|\mathrm{Re}\lambda_2|$, because it corresponds to the possibility of the occurrence of a Lorenz attractor.

The coordinates of points of space $\mathcal{P}'$ belonging to the unstable linear subspace $E_u=\mathrm{span}\{V_1\}$ and to the stable linear subspace $E^s=\mathrm{span}\{V_2,V_3\}$ satisfy the equations

$$E^u:\quad y'=x'\lambda_1,\quad z'=0,$$

$$E^s:\quad y'=x'\lambda_2.$$

The unstable (stable) manifold W^u (W^s) of the origin is tangent to E^u (E^s) at $x'=y'=z'=0$. Inasmuch as the z' axis is flow-invariant and belongs to W^s, points of these manifolds situated in a sphere of small radius ε satisfy the equations

$$W^u_{\mathrm{loc}}:\quad y'=x'\{\lambda_1+O(\varepsilon)\},\quad z'=O(\varepsilon^2) \tag{33}$$

and

$$W^s_{\mathrm{loc}}:\quad y'=x'\{\lambda_2+O(\varepsilon)\}. \tag{34}$$

We now consider the projections of the invariant manifolds W^u and W^s onto $\mathcal{P}'$. Since they are flow-invariant, they must map into themselves under the action of the group $U(1)$. Consequently, Π maps W^s and W^u onto (respectively) a two-dimensional manifold and a one-dimensional manifold in $\mathcal{P}'$. From Eq. (8), replacing x,y,z,w by x',y',z',w', and from Eqs. (33) and (34) we obtain

$$w'|_{\Pi(W^u_{\mathrm{loc}})}=|x'|^2\{\mathrm{Im}\ \lambda_1+O(\varepsilon)\},$$

$$w'|_{\Pi(W^s_{\mathrm{loc}})}=|x'|^2\{\mathrm{Im}\ \lambda_2+O(\varepsilon)\}. \tag{35}$$

Taking Eq. (32) into account, we conclude that for $\nu>0$ all points of $\Pi(W^u_{\mathrm{loc}})$ lie in the half space $w'\geqslant 0$. Next, we infer from the formal solution of the third equation of the system (12)

$$w'(t')=w'(0)+\nu e^{-\mu t'}\int_0^{t'}(R'(s)+u'(s))ds \tag{36}$$

that all points of $\Pi(W^u)$ lie in this half space. Indeed, since the integrand of Eq. (36) is nonnegative, $w'(t')$ is always nonnegative if $w'(0)>0$. In this regard, it follows from (35) that all points of $\Pi(W^s_{\mathrm{loc}})$ except those on the z' axis lie in the half space $w'<0$. Consequently, $\Pi(W^u)$ and $\Pi(W^s_{\mathrm{loc}})$ can intersect only on the z' axis. But this is impossible, because the z' axis does not belong to W^u. Therefore, for $\nu>0$ Eqs. (1) and (6) do not admit trajectories doubly asymptotic to the origin. Making use of the property of invariance of the system (12) under the substitution

$$\nu\to-\nu,\quad w'\to-w' \tag{37}$$

and proceeding in the same way, one can easily show that such trajectories are also nonexistent for $\nu<0$. Thus, $\nu=0$ or

$$r_2=\frac{\delta(1-\sigma)}{2\sigma}$$

is a necessary condition for the existence of a homoclinic Lorenz butterfly.

3. ONE-DIMENSIONAL MAP

Let the following relation hold for $\nu=0$:

$$k=-\frac{\lambda_3}{\lambda_1}<1,$$

where λ_1 and λ_3 are defined in Eq. (31).

This inequality corresponds to the case where the disintegration of a butterfly for $\delta=r_2=0$ is followed by the emergence of a strange invariant set. In the original Lorenz model the corresponding bifurcation is described by the one-dimensional map[30]

$$\xi \to \text{sign}\ \xi(-\varepsilon_1 + \text{sign}\ A|\xi|^k),$$
$$0 < |\xi| \ll 1, \quad 0 \leqslant \varepsilon_1 \ll 1, \tag{38}$$

where ξ is a real variable, A is a separatrix variable, and ε_1 describes a small deviation from the point of homoclinic bifurcation in parameter space (we shall assume from now on that A is positive).

We wish to construct a similar map for the complex Lorenz model. Let x' and y' be complex and $\nu = 0$ in Eqs. (6). Then, as shown above, a limit set of trajectories of the system belongs to the globally stable hypersurface $x'^*y' - x'y'^* = w = 0$. The solution corresponding to each trajectory on this hypersurface has the form $(x'(t')e^{i\psi}, y'(t')e^{i\psi}, z'(t'))$, where $(x'(t'), y'(t'), z'(t'))$ is the solution of the system (6) for real-valued x' and y', and ψ is a constant that depends on the initial conditions. This result enables us to write the map for complex-valued x' and y' (for $\nu = 0$):

$$\xi \to e^{i \arg \xi}(-\varepsilon_1 + |\xi|^k). \tag{39}$$

Here, in contrast with (38), ξ is complex, and A is set equal to unity, which is made possible by the renormalization of ξ. The change of variables (8) transforms the homoclinic butterfly into a single homoclinic loop in the projective space $\mathcal{P}$. A one-dimensional map describing the dynamics of the system in the vicinity of this loop in the space $\mathcal{P}$ can be obtained from Eq. (39):

$$\Xi \to (-\varepsilon_1 + \Xi^{k/2})^2, \quad 0 < \Xi \ll 1, \quad 0 \leqslant \varepsilon_1 \ll 1. \tag{40}$$

Here $\Xi = |\xi|^2$. As in (39), the map (40) is valid only for $\nu = 0$. For $\nu \neq 0$ we have

$$\Xi \to G(\Xi, \varepsilon_1, \nu), \quad 0 < \Xi \ll 1, \quad 0 \leqslant \varepsilon_1 \ll 1, \tag{41}$$

where $G(\Xi, \varepsilon_1, 0) = (-\varepsilon_1 + \Xi^{k/2})^2$.

Assuming that the derivative

$$G_{\nu\nu}(\Xi, \varepsilon_1, 0) = \left(\frac{\partial^2 G(\Xi, \varepsilon_1, \nu)}{\partial \nu^2} \right)_{\nu=0}$$

exists for small Ξ and small ε_1, we obtain the following from Eq. (41) for small ν:

$$\Xi \to G(\Xi, \varepsilon_1, 0) + \frac{\nu^2}{2} G_{\nu\nu}(\Xi, \varepsilon_1, 0) + O(\nu^4), \tag{42}$$

where $G(\Xi, \varepsilon_1, 0)$ is defined in Eq. (40).

By virtue of the symmetry property (37) of the system (12), Eq. (42) does not contain any terms linear or cubic in ν. Inasmuch as $\Xi, \varepsilon_1 \ll 1$, the dependence of $G_{\nu\nu}(\Xi, \varepsilon, 0)$ on Ξ and ε_1 can be disregarded. Then, omitting small terms $O(\nu^4)$ in Eq. (42), we obtain the map

$$\Xi \to (-\varepsilon_1 + \Xi^{k/2})^2 + \varepsilon_2^2,$$
$$0 < \Xi \ll 1, \quad 0 \leqslant \varepsilon_1, \varepsilon_2 \ll 1, \tag{43}$$

where $\varepsilon_2^2 = (\nu^2/2) G_{\nu\nu}(0,0,0)$.

Since the variable Ξ is nonnegative, we have $G(\Xi, \varepsilon_1, \nu) \geqslant 0$ and $G(0,0,0) = 0$, so that $G_{\nu\nu}(0,0,0) \geqslant 0$. We shall assume that $G_{\nu\nu}(0,0,0) > 0$. We note that the point $\varepsilon_1 = \varepsilon_2 = 0$ corresponds to a homoclinic bifurcation of codimensionality 2, and the parameter ε_2 is proportional to the small quantity ν. The substitution of

$$\Xi = \varepsilon_1^{2/k} \left\{ 1 + \frac{2\lambda}{k^2}(1 - 2\zeta)\varepsilon_1^{2(1-k)/k} \right\}$$

and

$$\varepsilon_2 = \varepsilon_1^{1/k} \left\{ 1 + \frac{\lambda(2-\lambda)}{2k^2} \varepsilon_1^{2(1-k)/k} \right\} \tag{44}$$

into (43) produces the logistic map

$$\zeta \to \lambda\zeta(1-\zeta) + O(\varepsilon_1^{2(1-k)/k}).$$

Consequently, the bifurcations in the map (43) are similar to the bifurcations of the logistic map. Moreover, in a small neighborhood of a bifurcation point of codimensionality 2 ($\varepsilon_1 = 0$, $\varepsilon_2 = 0$) asymptotic expressions can be obtained for the bifurcation sets of the map (43) by substituting the bifurcation values of the parameter λ of the logistic map into Eq. (47). In particular, the first two bifurcations of the logistic map are saddle–node ($\lambda = 1$) and period-doubling ($\lambda = 3$) types. The asymptotic expressions for the bifurcation curves corresponding to these bifurcations on the plane of the parameters $(\varepsilon_1, \varepsilon_2)$ are

$$\varepsilon_2 = \varepsilon_1^{1/k} \left\{ 1 + \frac{1}{2k^2} \varepsilon_1^{2(1-k)/k} + O(\varepsilon_1^{4(1-k)/k}) \right\} \tag{45}$$

and

$$\varepsilon_2 = \varepsilon_1^{1/k} \left\{ 1 - \frac{3}{2k^2} \varepsilon_1^{2(1-k)/k} + O(\varepsilon_1^{4(1-k)/k}) \right\}, \tag{46}$$

respectively.

To verify the conclusions drawn using the one-dimensional map (43), we have numerically plotted several bifurcation curves for the complex Lorenz model with real r, consistent with the model of a single-mode laser. Figure 1 shows these curves on the (δ, r) plane. Inequality (4), which is the condition for replacing the system (1) by the system (6), is satisfied above curve M $[\sigma(r_1 - 1) - \delta^2/4 = 0]$. Curve T represents the stability threshold of the trivial steady-state solution $x' = y' = z' = 0$, which becomes unstable above this curve. Point h corresponds to a homoclinic bifurcation of codimensionality 2. Because the resulting one-dimensional map is identical to the logistic map, this point must be a limit point for an infinite number of bifurcation curves. Some of these curves shown in the figure do in fact go to point h. Curve S corresponds to the saddle–node bifurcation (45). The intersection of this curve with the parametric vector from the right in projective space $\mathcal{P}$ is accompanied by the onset of two limit cycles, one stable and the other unstable. As the parameter δ is further decreased, the stable limit cycle undergoes a series of period-doubling bifurcations, making the transition to chaos. A numerical analysis shows that the corresponding curves (curve D in Fig. 1 corresponds to the first doubling bifurcation) converge to point *h*. This result is in complete agreement with results based on an analysis of the above-derived map (43). Curve *H* in the figure corresponds to a Hopf bifurcation of a ''nonzero'' steady-state solution of the equations of motion in projective space $\mathcal{P}$. The dashed and solid segments of the curve represent its subcritical and supercritical parts, respectively. We note that the intersection of curve *H* with the *r* axis corresponds to a subcritical Hopf bifurcation, which is a well-known occurrence in the original Lorenz model.

CONCLUSION

We have shown that all the dynamical properties of a system, including the salient characteristics of its phase dynamics, can be obtained directly from the representation of the complex Lorenz model in ray space. We note that this representation does not contain singularities for certain values of the parameters, contrary to the analogous representation used in Ref. 10, and it provides an effective, simple method for studying the properties of the complex Lorenz model. We have established a correspondence between the properties of the limit sets in the initial phase space and in ray space, and we have elucidated the boundedness properties of the limit sets in these spaces. We have shown that these properties are responsible for the singular behavior — previously observed in a numerical simulation[13] — of the curve representing the mean slope of the phase of a laser field as a function of the detuning. We have proved that the homoclinic bifurcation of the separatrix of a saddle point in the complex Lorenz model has codimensionality 2. For values of the parameters close to the homoclinic bifurcation point we have constructed a one-dimensional map for points of phase space near the separatrix. For $r_2 \neq r_{2c}$ [see (30)] the resulting map (unlike the Lorenz map) is smooth and equivalent to the logistic map. This result lends support to the assertion that a ''true'' Lorenz attractor, which contains only saddle limit cycles, can exist in the complex Lorenz model only for $r_2 = r_{2c}$. We have demonstrated numerically the correspondence between the hierarchy of bifurcations in the single-mode laser model on the plane of the pump-detuning parameters and the sequence of bifurcations of the logistic map.

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