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Analysis of the stability of laser solitons

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Abstract. An investigation is made of the stability, with respect to small perturbations, of one-dimensional stationary localised radiation structures ('laser autosolitons') in a wide-aperture laser with a saturable absorber. It is shown that an increase in the active-medium gain makes a stationary autosoliton unstable and gives rise to a pulsating localised structure.

The investigation reported below is a continuation of our earlier work [1] and it deals with the stability, in respect of small perturbations, of transversely one-dimensional laser autosolitons, which are localised (soliton-like) radiation structures in a wide-aperture laser with a saturable absorber. We shall use the results and notation of Ref. [1].

Let $A_0(\xi)$ be a stationary localised solution satisfying Eqn (6) of Ref. [1]. We shall study its stability by substituting a perturbed solution

$$A(\xi,\, t) = A_0(\xi) + \delta A(\xi)\mathrm{e}^{\gamma t} \tag{1}$$

into Eqn (4) of Ref. [1]. Linearisation in terms of a small perturbation yields an equation

$$\gamma\begin{pmatrix}\delta A\\ \delta A^*\end{pmatrix} = \hat{L}\begin{pmatrix}\delta A\\ \delta A^*\end{pmatrix} \tag{2}$$

for finding the spectrum of a linear operator $\hat{L}$, which can be written in the form (see Ref. [2])

$$\hat{L} = \mathrm{i}\left(-\alpha_0 + \frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\right)\sigma_3 + f\left(|A_0|^2\right) + \left[|A_0|^2 + m(A_0)\right]\frac{\mathrm{d}f(I)}{\mathrm{d}I}\bigg|_{I=|A_0|^2}\,, \tag{3}$$

$$m(A_0) = \begin{pmatrix} 0 & A_0^2 \\ A_0^{*2} & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Here, α_0 is the value of the parameter α for which there exists the solution $A_0(\xi)$. The continuous spectrum of the operator $\hat{L}$ is determined by the behaviour of this operator in the limit $\xi \to \pm\infty$. In the bistability region, where the spatially homogeneous solution $E = 0$ (nonlasing regime) is stable, the spectrum of the operator $\hat{L}$ lies in the left half-plane of a complex plane and does not lead to an instability of the solution $A_0(\xi)$. Therefore, the stability of the solution $A_0(\xi)$ is governed by the discrete spectrum of the operator $\hat{L}$.

In view of the symmetry of Eqn (4) of Ref. [1],

$$A(\xi,\, t) \to A(\xi,\, t)\mathrm{e}^{\mathrm{i}\eta}\,, \qquad A(\xi,\, t) \to A(\xi + h,\, t)\,,$$

$$A(\xi,\, t) \to A(\xi - vt,\, t)\exp\left(\frac{\mathrm{i}v\xi}{2} - \frac{\mathrm{i}v^2 t}{4}\right),$$

the discrete spectrum of the operator $\hat{L}$ includes a triple zero eigenvalue. It corresponds to two eigenvectors

$$\hat{L}\boldsymbol{\psi}_{1,2}(\xi) = 0\,,$$

which are described by the relationships

$$\boldsymbol{\psi}_1(\xi) = \begin{pmatrix} \mathrm{i}A_0(\xi) \\ -\mathrm{i}A_0^*(\xi) \end{pmatrix}, \qquad \boldsymbol{\psi}_2(\xi) = \begin{pmatrix} \mathrm{d}A_0(\xi)/\mathrm{d}\xi \\ \mathrm{d}A_0^*(\xi)/\mathrm{d}\xi \end{pmatrix}, \tag{4}$$

and one associated vector

$$\hat{L}\boldsymbol{\varphi}_2(\xi) = \boldsymbol{\psi}_2(\xi)\,,$$

which is of the form

$$\boldsymbol{\varphi}_2(\xi) = \begin{pmatrix} -\mathrm{i}\xi A_0(\xi)/2 \\ \mathrm{i}\xi A_0^*(\xi)/2 \end{pmatrix}.$$

It should be noted that for a localised structure defined by an even (odd) function $A_0(\xi)$, the function $\boldsymbol{\psi}_1(\xi)$ is even (odd), and the functions $\boldsymbol{\psi}_2(\xi)$ and $\boldsymbol{\varphi}_2(\xi)$ are odd (even). For both even and odd functions $A_0(\xi)$ the operator $\hat{L}$ is invariant under the substitution $\xi \to -\xi$.

We shall consider a family of stationary localised solutions $A_0(\xi,\, \alpha)$ corresponding to a certain part of the curve shown in Fig. 1 of Ref. [1]. At the bifurcation points S and S′ and at other points of this curve where its slope becomes infinite ($\mathrm{d}g_0/\mathrm{d}\alpha = 0$), the operator $\hat{L}$ has a four-dimensional and not three-dimensional root subspace. An autosoliton is then stable (unstable) on the part of the curve adjoining from below (from above) the point S (S′). An additional zero eigenvalue, which appears at the point S, corresponds to an associated vector $\boldsymbol{\varphi}_1(\xi)$, which is even in respect of ξ:

$$\hat{L}\boldsymbol{\varphi}_1(\xi) = \boldsymbol{\psi}_1(\xi)\,.$$

This vector is given by the expression

$$\boldsymbol{\varphi}_1(\xi) = \begin{pmatrix} \dfrac{\mathrm{d}A_0(\xi,\,\alpha)}{\mathrm{d}\alpha}\bigg|_{\alpha=\alpha_\mathrm{S}} \\ \dfrac{\mathrm{d}A_0^*(\xi,\,\alpha)}{\mathrm{d}\alpha}\bigg|_{\alpha=\alpha_\mathrm{S}} \end{pmatrix}.$$

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Here, α_S corresponds to the point S. A quadruple zero eigenvalue was always obtained for the nonlinear Schrödinger equation, which is conservative. It corresponds to two eigenvectors and two associated vectors [2].

The stability of stationary localised structures calculated in Ref. [1] was found by numerical solution of Eqn (2) dealing separately with stability against even and odd perturbations. The second derivative in Eqn (3) was approximated by the required function at five points and the infinite interval along ξ was replaced by a sufficiently long but finite interval. As a result, the problem was reduced to finding the eigenvalues of a matrix whose maximum order in our calculations was 1024×1024.

The autosoliton solutions corresponding to those parts of the curve in Fig. 1 of Ref. [1] which are identified by a dashed curve are unstable. The stable autosolitons correspond to the continuous parts of this curve. These parts, of length which decreases on transition to the spiral turns which are closer to its centre (which is why only two such parts are shown in Fig. 1 of Ref. [1]), lie between the points S and S′ of a saddle–node bifurcation and the points H and H′ of an Andronov–Hopf bifurcation [3, 4]. Fig. 1 in the present paper shows, for two values of the parameter g_0 close to the point H, the positions of several eigenvalues which belong to the discrete spectrum of the operator $\hat{L}$ and lie closest to the imaginary axis. The left-hand (right-hand) parts of Fig. 1 in the present paper give the eigenvalues corresponding to even (odd) eigenfunctions. Two upper (lower) parts of Fig. 1 correspond to a subcritical (supercritical) region of g_0. In full agreement with the above conclusions, our numerical calculation predicts a simple zero eigenvalue for even perturbations and a doubly degenerate zero eigenvalue for odd perturbations.

Fig. 1 demonstrates the existence of the Andronov–Hopf bifurcation, which appears on increase in the parameter g_0. It can be seen that a pair of complex-conjugate eigenvalues, corresponding to even eigenfunctions $\boldsymbol{\psi}_{3,4}(\xi)$, intersects the imaginary axis. The distribution of these eigenfunctions is shown in Fig. 2. For comparison, this figure includes also the modulus of the eigenfunction $\boldsymbol{\psi}_1(\xi)$ corresponding to zero eigenvalue and described by expression (4). Reduction of the saturation parameter reduces the width of the stability region of the autosoliton solution lying between the points S and H (see Fig. 3). Between curves *2* and *3* in Fig. 3 the solution in the form of a single autosoliton is stable. Direct numerical integration of Eqn (1) demonstrated that the Andronov–Hopf bifurcation at the point H is supercritical and it leads to soft excitation of a pulsating localised structure (see also Ref. [5]). This structure is shown in Fig. 4.

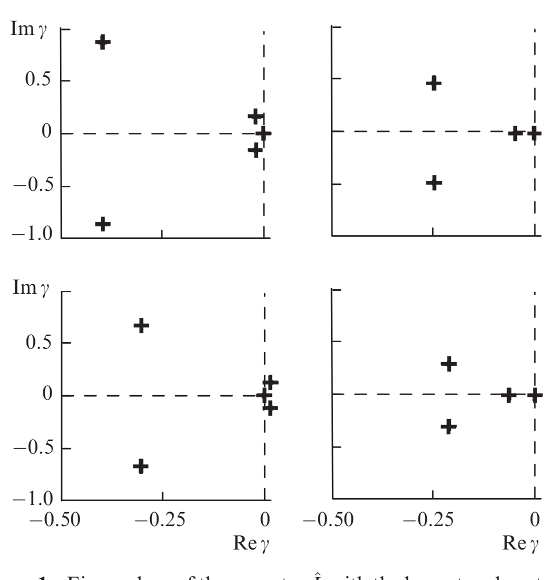

Figure 1. Eigenvalues of the operator $\hat{L}$ with the largest real parts, shown for $a_0 = 2.0$ and $I_g = 10.0$. The upper (lower) parts of the figure correspond to $g_0 = 2.08$ (above) and 2.10 (below) for even (on the left) and odd (on the right) perturbations.

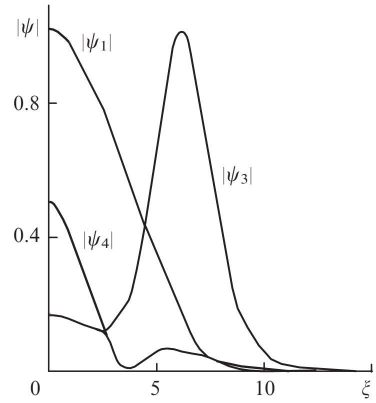

Figure 2. Moduli of the eigenfunctions $\psi_{1,3,4}(\xi)$ (since these functions are even, they are shown only for $\xi \geqslant 0$).

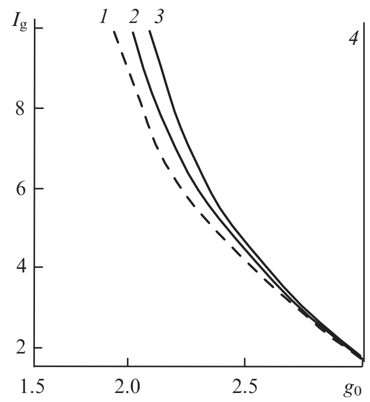

Figure 3. Bifurcation curves for a single autosoliton in the plane of the parameters g_0, I_g, plotted for $a_0 = 2.0$: (*1*, *4*) limits of the bistability region; (*2*) saddle–node bifurcation curve; (*3*) Andronov–Hopf bifurca-tion curve.

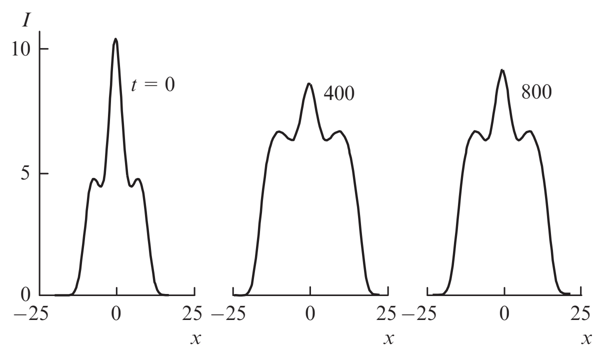

Figure 4. Oscillatory (period 1200) localised structure predicted for $g_0 = 2.102$, $a_0 = 2.0$, and $I_g = 10.0$ at various moments in time.

Let us now assume that the unperturbed solution $A_0(\xi)$ of Eqn (3) represents a stationary localised structure formed by two coupled autosolitons. It is natural to assume that if these autosolitons are sufficiently far from one another, then the spectrum of the operator $\hat{L}$ is close to the spectrum obtained for a single autosoliton. Hence, we can readily conclude that instability of a single autosoliton leads to instability of 'two-soliton' stationary structures with two autosolitons sufficiently far apart. However, it does not follow in general from the stability of a single autosoliton that 'two-soliton' structures are stable.

In fact, since each single autosoliton has three zero eigenvalues, a 'two-soliton' structure should have six eigenvalues located on a complex plane in a small region near the origin of the coordinates. Only three out of these six eigenvalues vanish identically and the other three are displaced from the origin of the coordinates because of the interaction between autosolitons. If all three eigenvalues are shifted to the left half-plane, a 'two-soliton' structure is stable; otherwise, it is unstable. Therefore, the problem of stability of stationary 'two-soliton' structures requires separate consideration. Here, we shall report only some of the results of our numerical calculations relating to the stability of 'two-soliton' structures. For example, if $g_0 = 2.04$, $a_0 = 2.0$, and $I_g = 10.0$, there is a stable single autosoliton with $\alpha_0 = 0.06725$. For the same parameters of a laser a stationary structure described by the odd function $A_0(\xi)$ and corresponding to two coupled autosolitons with a minimal distance between them ($\alpha_0 = 0.06419$) is stable. A similar structure described by the even function $A_0(\xi)$ exists for $\alpha_0 = 0.06706$ and is unstable.

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