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Bifurcation analysis of laser autosolitons

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Abstract. An investigation is reported of one-dimensional stationary localised radiation structures ('laser autosolitons'), which appear in a wide-aperture laser with a saturable absorber under conditions of hard lasing excitation. Bifurcation theory methods are used to find the range of existence of single autosolitons and of their coupled states. It is shown that such structures correspond to a transverse dependence of the field amplitude which is either symmetric and node-free or antisymmetric and has a single node. An infinite set of single autosolitons with various widths and transverse amplitude profiles is predicted. It is shown that there is an infinite set of 'two-soliton' structures differing in respect of the distance between the component autosolitons.

1. Introduction

The term 'localised (soliton-like) structures of laser radiation' is used for structures formed in a limited lasing region of laser systems with an arbitrarily large extent of the active medium in the transverse direction, for example, in wide-aperture lasers. They are of interest because they represent the case of self-organisation in coherent optical systems and have promising applications in optical data processing.

To the best of our knowledge, such structures have not yet been observed experimentally, but in recent years they have been the subject of intensive theoretical investigations. Such localised structures are predicted [1 – 3] specifically for active single-mode nonlinear waveguides and for 'slab' lasers described by similar equations with a strong dependence of the field on just one transverse coordinate. However, such structures prove to be unstable and, therefore, physically unattainable because an excess of the unsaturated gain above the losses induces growth of perturbations in the low-intensity wings of such a structure.

A decisive condition of the stability of localised structures is hard excitation of lasing (bistable lasers). These stable localised laser structures, 'laser autosolitons', were first predicted in Ref. [4] and were then investigated in detail [5 – 8] (see also the literature cited there). Mathematical aspects of a theory of similar structures were studied without specific reference to lasers [9, 10] (see also the literature cited there). Laser autosolitons, representing islands of lasing against the background of a stable nonlasing regime and formed by hard excitation, are largely analogous to localised structures in bistable passive nonlinear-optical systems, such as wide-aperture nonlinear interferometers excited by external radiation investigated earlier. Stationary and pulsating 'diffraction autosolitons' were predicted [11, 12] and detected experimentally [13, 14] (see also reviews [7, 8]). 'Diffusion autosolitons' were investigated intensively earlier in various physical, chemical, and biological systems [15, 16].

Laser solitons of various types are now known: stationary and pulsating, immobile (in a transverse direction), moving and rotating, single and coupled. The main method for the analysis of these autosolitons involves numerical computations and approximate calculations, including the approximate method of moments [19] similar to that proposed in Refs [17, 18]. In view of the difficulties encountered in solving nonlinear partial differential equations describing laser autosolitons, it would be desirable to establish a systematic classification of such structures and to demonstrate more rigorously their stability. This is the task we set ourselves: we shall consider these topics for transversely one-dimensional bistable laser systems.

2. Model and the initial relationships

We shall consider a wide-aperture (characterised by a large Fresnel number) laser with a saturable absorber. The relaxation times of the active and absorbing media inside the cavity are assumed to be sufficiently short compared with the time taken to establish the field in the cavity (class A laser). The slowly varying amplitude E of the electric field, averaged along the longitudinal (axial) direction, obeys the quasi-optical equation proposed by A F Suchkov [21]:

$$\frac{\partial E}{\partial t} = \mathrm{i}\frac{\partial^2 E}{\partial x^2} + Ef(|E|^2) \ . \qquad (1)$$

The time t and the transverse coordinate x are dimensionless, and the function f represents the difference between saturated gain and the total (linear and nonlinear) losses:

$$f(|E|^2) = -1 + \frac{g_0}{1+|E|^2/I_{\mathrm{g}}} - \frac{a_0}{1+|E|^2} \ , \qquad (2)$$

where a_0 and g_0 are the unsaturated gain and the absorption coefficient, normalised to the nonresonant loss factor; $I = |E|^2$ and I_{g} are, respectively the radiation intensity, and the intensity needed to saturate the absorbing medium, expressed in units of the gain-saturation intensity. Such a

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model describes a laser with a homogeneous medium and an unbounded (along one of the transverse directions) aperture (in the other transverse direction the lasing is single-mode). We shall assume that the various frequency detunings are small, because if they are included the function $f(I)$ would be complex.

Laser autosolitons represent transversely inhomogeneous distributions of the field with the asymptotics $E \to 0$ in the limit $x \to \pm\infty$. Therefore, the existence of stable laser autosolitons demands stability of the lasing-free regime, which is achieved for $f_0 = f(0) = g_0 - a_0 - 1 < 0$. Eqn (1) is symmetric under the 'Galilean transformation' (transformation to a moving system of coordinates). Therefore, the presence of one laser autosoliton is evidence of the existence of a family of autosolitons with a continuously varying parameter, which is the velocity υ of transverse motion of the structure as a whole [22].

3. Stationary localised structures

If the complex amplitude of the electric field is represented in the form

$$E(x,\,t) = A(\xi,\,t)\mathrm{e}^{-\mathrm{i}\nu t + \mathrm{i}\upsilon\xi/2}\,, \qquad \xi = x - \upsilon t\,, \tag{3}$$

we obtain the following equation for $A(\xi,\,t)$:

$$\frac{\partial A}{\partial t} = +\mathrm{i}\alpha A + \mathrm{i}\frac{\partial^2 A}{\partial \xi^2} + Af\left(|A|^2\right)\,, \tag{4}$$

where $\alpha = \nu + \upsilon^2/4$ represents the lasing-frequency shift.

Since Eqn (1) is invariant under reversal of the signs of E and x, Eqn (4) is invariant under the transformations

$$A \to -A\,, \qquad \xi \to -\xi\,. \tag{5}$$

Stationary (time-dependent) solutions of Eqn (4), $A(\xi,\,t) = A(\xi)$, satisfy the equation

$$\frac{\mathrm{d}^2 A}{\mathrm{d}\xi^2} + \alpha A - \mathrm{i}Af(|A|^2) = 0\,, \tag{6}$$

where α plays the role of an eigenvalue. We shall represent the complex variable A in the form $A(\xi) = a(\xi)\mathrm{e}^{\mathrm{i}\Phi(\xi)}$, where $a \geqslant 0$ and Φ are real. Then, in terms of real variables

$$a,\, q = \frac{\mathrm{d}\Phi}{\mathrm{d}\xi}\,, \qquad k = \frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}\xi}\,, \tag{7}$$

Eqn (6) can be rewritten as a system of three first-order real equations [10]:

$$\begin{aligned} &\frac{\mathrm{d}a}{\mathrm{d}\xi} = ak\,, \\ &\frac{\mathrm{d}q}{\mathrm{d}\xi} = -2qk + f\left(a^2\right)\,, \\ &\frac{\mathrm{d}k}{\mathrm{d}\xi} = -\alpha + q^2 - k^2\,. \end{aligned} \tag{8}$$

In view of the symmetry of expressions (5), the vector field described by the system of equations (8) is invariant under the transformation

$$(\xi,\, a,\, q,\, k) \to (-\xi,\, a,\, -q,\, -k)\,. \tag{9}$$

The three-dimensional phase space of the system of equations (8) includes two fixed points which correspond to zero intensity of the laser field ($I = a^2 = 0$). These points, denoted by L_- and L_+, are defined, respectively, by the following respective relationships:

$$\begin{aligned} &a = 0\,, \quad q = -\left[\frac{(\alpha^2 + f_0^2)^{1/2} + \alpha}{2}\right]^{1/2}\,, \\ &k = \left[\frac{(\alpha^2 + f_0^2)^{1/2} - \alpha}{2}\right]^{1/2}\,, \end{aligned} \tag{10}$$

$$\begin{aligned} &a = 0\,, \quad q = \left[\frac{(\alpha^2 + f_0^2)^{1/2} + \alpha}{2}\right]^{1/2}\,, \\ &k = -\left[\frac{(\alpha^2 + f_0^2)^{1/2} - \alpha}{2}\right]^{1/2}\,. \end{aligned} \tag{11}$$

In addition to the fixed points described by the above expressions, the phase space defined by the system of equations (8) may include a further two or four fixed points, corresponding to a nonzero intensity of laser radiation ($I = a^2 > 0$, bistability). It follows from Ref. [23] that stable laser autosolitons can exist only in the bistability region.

The fixed point L_- is a saddle – focus. It has a one-dimensional unstable manifold [24] and a two-dimensional stable manifold, lying in the invariant (relative to the radiation flux) plane $a = 0$. The fixed point L_+ is transformed to L_- by substitution (9) and, therefore, it has a one-dimensional stable manifold and a two-dimensional unstable manifold, which lies in the $a = 0$ plane. Stationary localised structures of Eqn (4) correspond to heteroclinic trajectories lying in the region $a \geqslant 0$ of the phase space of the system of equations (8) and, as ξ is increased, these trajectories propagate from the fixed point L_- ($\xi = -\infty$) to the fixed point L_+ ($\xi = +\infty$). It follows that the process of finding stationary localised structures includes identifying the bifurcation points in the space of the parameters for which the system (8) had heteroclinic trajectories of the type just described.

Classification of heteroclinic trajectories can be used to describe stationary localised structures of various types (at this stage independently of their stability). In particular, the simplest ('single-pass') heteroclinic trajectory, which propagates directly from L_- to L_+, corresponds to the solution of Eqn (4) in the form of a single autosoliton, whereas 'multipass' heteroclinic trajectories which pass repeatedly the vicinities of the fixed points L_- and L_+ correspond to stationary structures in the form of coupled autosolitons. Moreover, heteroclinic trajectories with the same number of passes n may differ in respect of the number of passes near the fixed points L_- and L_+. Such trajectories correspond to stationary structures formed by n coupled autosolitons separated by various distances.

When the parameters α, g_0, a_0, and I_{g} are fixed, there may be only one heteroclinic trajectory from L_- to L_+ and this trajectory is therefore invariant under substitution (9) and it consists of two symmetric parts which transform into one another as a result of this substitution. A point C, which separates a heteroclinic trajectory into two symmetric parts (and belongs to both), lies in the phase space of the system of equations (8) on the a axis ($a > 0$, $q = k = 0$). If we take the point C to be the origin from which the variable ξ is measured, we find that stationary localised solutions of Eqn (4), corresponding to heteroclinic trajectories of the system (8), are even functions of the variable ξ: $A(\xi) = A(-\xi)$. In addition to C, such a heteroclinic trajectory has no other points on the $q = k = 0$ axis.

A heteroclinic trajectory corresponding to a single autosoliton emerges from the fixed point L_- ($\xi = -\infty$), intersects the a axis at a relatively large value of a, and terminates at the point L_+ ($\xi = +\infty$). Fig. 1 shows a curve, obtained

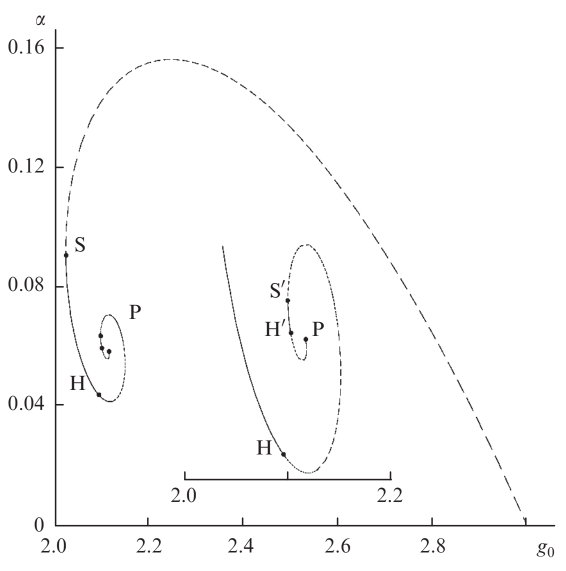

Figure 1. Stationary solution of Eqn (4) in the form of a single auto-soliton calculated for $a_0 = 2.0$ and $I_g = 10.0$ (the inset shows the vicinity of the point P on an enlarged scale). The autosoliton solution is stable (unstable) in the parts of the curve identified by the continuous (dashed) line [20]; S and S′ are the points of a saddle – node bifurcation; H and H′ are the Andronov – Hopf bifurcation points of an autosoliton solution; P is the bifurcation point of codimensionality 2 of the system of equations (8).

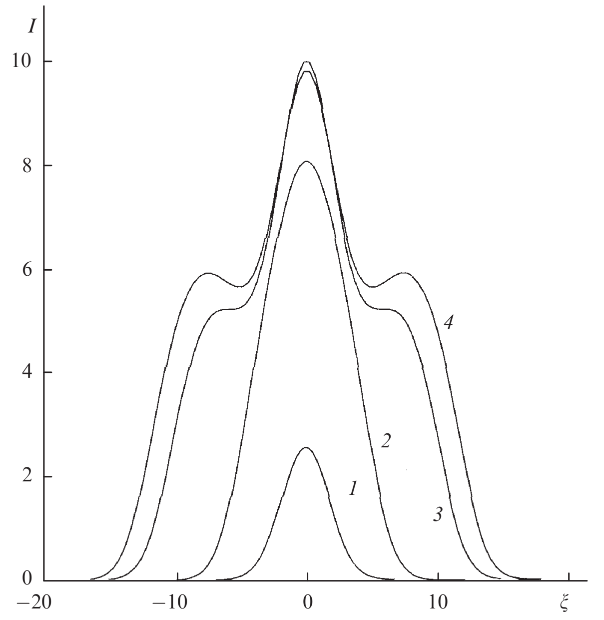

Figure 2. Solutions in the form of a single autosoliton obtained for $g_0 = 2.102$, $a_0 = 2.0$, $I_g = 10.0$, and the following values of α: 0.14175 (1), 0.04218 (2), 0.06663 (3), and 0.05934 (4). Only the autosoliton solution represented by curve 4 is stable [20].

by applying the INTSEP suite of programs [25], demonstrating the relationship between the parameters g_0 and α for which there is such a heteroclinic trajectory in the system (8). This curve lies entirely in the bistability region where the equation $f(a^2) = 0$ has two positive roots. It begins at the point on the upper boundary of the bistability region $g_0 = 1 + a_0$ and passes to the region of lower values of g_0; it then turns back and finally winds itself into a spiral. A point P, on which this spiral is wound, is the bifurcation point of codimensionality 2 of the system (8). In the vicinity of the point P, for the same parameter g_0, there may be several solutions in the form of a single autosoliton with different values of α. The closer the point on the spiral to its centre, the greater the width of the autosoliton solution corresponding to this point (Fig. 2) and the greater the number of oscillations of the laser field intensity on the autosoliton envelope.

The existence, at $\alpha = \alpha_0$, of a solution of Eqn (4) in the form of a single autosoliton implies the existence of an infinite set of different coupled stationary 'multisoliton' solutions with values of α close to α_0 and, in particular, the existence of a denumerably infinite number of solutions formed by two coupled autosolitons. A 'two-pass' heteroclinic trajectory corresponding to two coupled autosolitons passes once near the fixed point L_+ before it crosses the a axis at low values of a. The closer the value of α for a 'two-soliton' solution to α_0, the larger the number of passes and, consequently, the longer the 'time' ξ spent in the vicinity of the point L_+. The longer the time, the greater the distance between coupled autosolitons and the closer they resemble two separate autosolitons.

Since the variable k defined by expression (7) has a singularity at $a = 0$, the system of equations (8) is convenient for describing only those localised stationary structures for which the amplitude a is finite when ξ is finite (we demonstrated above that such structures are described by even functions of the variable ξ). However, if this condition is not satisfied, it is more convenient to use a system of four first-order differential equations for the variables

$$x_1 = |A|^2, \quad x_2 = \left|\frac{dA}{d\xi}\right|^2, \quad x_3 = A\frac{dA^*}{d\xi} + A^*\frac{dA}{d\xi}, \quad x_4 = \frac{1}{i}\left(A\frac{dA^*}{d\xi} - A^*\frac{dA}{d\xi}\right). \tag{12}$$

This system follows from Eqn (6) and is of the form

$$\frac{dx_1}{d\xi} = x_3, \quad \frac{dx_2}{d\xi} = -\alpha x_3 - x_4 f(x_1), \quad \frac{dx_3}{d\xi} = -2\alpha x_1 + 2x_2, \quad \frac{dx_4}{d\xi} = -2x_1 f(x_1). \tag{13}$$

It follows from the definitions given by the set of expressions (12) that the trajectories of the system of equations (13) of interest to us lie on a three-dimensional hypersurface:

$$4x_1x_2 - x_3^2 - x_4^2 = 0, \quad x_1 \geq 0, \quad x_2 \geq 0. \tag{14}$$

The vector field described by the system of equations (13) is invariant under the transformation

$$(\xi, x_1, x_2, x_3, x_4) \to (-\xi, x_1, x_2, -x_3, -x_4). \tag{15}$$

The system (13) has one fixed point L, which corresponds to zero laser radiation intensity:

$$x_1 = x_2 = x_3 = x_4 = 0.$$

Localised stationary structures of Eqn (4) correspond to homoclinic trajectories of the system of equations (13), which pass — on increase in ξ — from the fixed point L back to this point, and which satisfy the set of relationships (14). Among these trajectories there are such for which x_1 does not vanish for finite values of ξ. We investigated them already with the aid of the system of equations (8) so that now we shall consider those homoclinic trajectories which contain a point Z with $x_1 = 0$, that does not coincide with the fixed point L. It follows from the set of expressions (14) that the point Z lies on the x_2 axis ($x_1 = x_3 = x_4 = 0,\ x_2 \geqslant 0$).

Let us consider a segment Γ_+ of a homoclinic trajectory Γ which originates from the fixed point L and terminates at the point Z on the x_2 axis. Transformation (15) transfers Γ_+ to a different segment Γ_- of a trajectory corresponding to the system of equations (13) and this segment forms, together with Γ_+, a complete homoclinic trajectory Γ. Consequently, the point Z is the only point on the homoclinic trajectory which lies on the x_2 axis and is characterised by $x_2 \neq 0$. This point divides the trajectory into two symmetric parts which are converted into one another by transformation (15). We shall select the point Z to be the origin for measuring the variables ξ. Then, a homoclinic trajectory which contains this point and is described by the system of equations (13) corresponds to a pair of stationary localised structures of Eqn (4) described by odd functions $A(\xi) = -A(-\xi)$. One of these structures is converted into the other one by the substitution $A \to -A$. It follows that any odd function $A(\xi)$ which describes a stationary localised structure of Eqn (4) has just one root $A(0) = 0$.

The simplest odd stationary localised structures are of 'two-soliton' type and they are formed by two coupled antiphase autosolitons (Fig. 3). They can be described as follows within the framework of the system of equations (8). A one-dimensional unstable manifold of the fixed point L_- is transferred to the vicinity of the point L_+ and hence to infinity, where the point Z is located. The second half of the trajectory is obtained from the first by substitution (9) and is identical with a one-dimensional stable manifold of the fixed point L_+. It arrives from infinity reaching the vicinity of the point L_- and then passes to the point L_+ in the limit $\xi \to +\infty$. There is also a denumerably infinite number of odd 'two-soliton' solutions with different distances between coupled autosolitons, as found earlier for even 'two-soliton' solutions when the intensity I does not vanish for finite values of ξ. The larger the distance between the coupled autosolitons, the closer the parameter α corresponding to a 'two-soliton' solution to α_0, which corresponds to a single autosoliton.

It follows that finding localised stationary structures which appear in a wide-aperture laser with a saturable absorber reduces, in the case of one spatial variable, to identification of heteroclinic (or homoclinic) trajectories of a system of ordinary differential equations. Stationary localised structures may be of two types. Classification of 'multipass' and 'multirevolution' heteroclinic (homoclinic) trajectories provides the basis for classifying stationary solutions in the form of single and coupled autosolitons. There is an infinite number of solutions predicting a single autosoliton with a variety of widths. For given laser parameters, the existence of a single autosoliton solution corresponding to a 'single-pass' heteroclinic trajectory implies the existence of a denumerably infinite set of stationary states formed by two coupled autosolitons and of corresponding 'two-pass' heteroclinic (homoclinic) trajectories. Such 'two-soliton' solutions can be the two types described above, such that various solutions of one type differ in respect of the distance between coupled autosolitons, which is governed by the number of revolutions of a trajectory near fixed points of a system of ordinary differential equations.

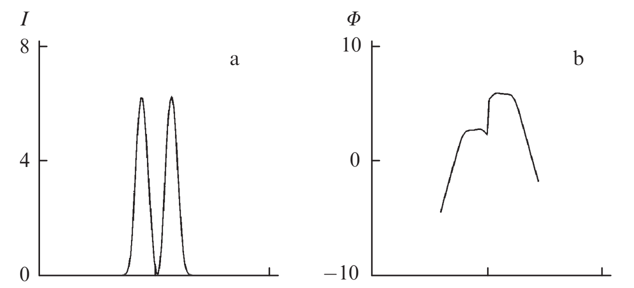

Figure 3. Stationary solution of Eqn (4) in the form of two antiphase coupled autosolitons described by an odd complex function $A(\xi) = -A(-\xi)$ with a single root $\xi = 0$ of the equation $A(\xi) = 0$.

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