Statistical inference for time-changed Lévy processes via Mellin transform approach

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Abstract

Given a Lévy process $(L_t)_{t\geq 0}$ and an independent nondecreasing process (time change) $(\mathcal{T}(t))_{t\geq 0}$, we consider the problem of statistical inference on $\mathcal{T}$ based on low-frequency observations of the time-changed Lévy process $L_{\mathcal{T}(t)}$. Our approach is based on the genuine use of Mellin and Laplace transforms. We propose a consistent estimator for the density of the increments of $\mathcal{T}$ in a stationary regime, derive its convergence rates and prove the optimality of the rates. It turns out that the convergence rates heavily depend on the decay of the Mellin transform of $\mathcal{T}$. Finally, the performance of the estimator is analysed via a Monte Carlo simulation study.

Keywords: time-changed Lévy processes, low-frequency observations, Mellin transform, Laplace transform.

1 Introduction

Let $L = (L_t)_{t\geq 0}$ be a one-dimensional Lévy process with a Lévy triplet (μ, σ^2, ν) and let $\mathcal{T} = (\mathcal{T}(s))_{s\geq 0}$ be a non-negative, non-decreasing stochastic process independent of L with $\mathcal{T}(0) = 0$. A time-changed Lévy process $Y = (Y_s)_{s\geq 0}$ is then defined via $Y_s = L_{\mathcal{T}(s)}$. The process $\mathcal{T}$ is usually referred to as time change. Here we consider the problem of statistical inference on the distribution of the time change $\mathcal{T}$ based on low-frequency observations of the time-changed Lévy process (Y_t). Suppose that n observations of the time-changed Lévy process (Y_t) at times $t_j = j\Delta$, $j = 0, \ldots, n$, are available. If the sequence $\mathcal{T}(t_j) - \mathcal{T}(t_{j-1})$, $j = 1, \ldots, n$, is ergodic, strictly stationary with invariant stationary distribution p_Δ, then for any bounded "test function" f,

$$(1) \qquad \frac{1}{n}\sum_{j=1}^{n} f\left(L_{\mathcal{T}(t_j)} - L_{\mathcal{T}(t_{j-1})}\right) \xrightarrow{\text{a.s.}} \mathbb{E}_{p_\Delta}[f(L_{\mathcal{T}(\Delta)})], \quad n \to \infty.$$

The limiting expectation in (1) is then given by

$$\mathbb{E}_{p_\Delta}[f(L_{\mathcal{T}(\Delta)})] = \int_0^\infty \mathbb{E}[f(L_s)]\, p_\Delta(ds).$$

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Taking $f(z) = f_u(z) = \exp(\mathrm{i}uz)$, $u \in \mathbb{R}$, and using the well-known Lévy–Khintchine formula:

$$\phi(u) := \mathbb{E}\left[e^{\mathrm{i}uL_t}\right] = e^{-t\psi(u)}, \quad u \in \mathbb{R}, \quad t \geq 0$$

with

$$\psi(u) := \frac{1}{2}\sigma^2 u^2 - \mathrm{i}\mu u - \int_{\mathbb{R}} \left(e^{ixu} - 1 - \mathrm{i}xu 1_{\{|x| \leq 1\}}(x)\right) \nu(dx),$$

we arrive at the following representation for the characteristic function of Y_Δ:

$$\mathbb{E}\left[\exp\left(\mathrm{i}uY_\Delta\right)\right] = \int_0^\infty \exp(-t\psi(u))\, p_\Delta(dt) = \mathcal{L}[p_\Delta](\psi(u)), \tag{2}$$

where $\mathcal{L}[p_\Delta]$ stands for the Laplace transform of p_Δ. Hence the problem of statistical inference on p_Δ is related to the problem of Laplace transform inversion based on noisy and indirect (due to the presence of ψ) observations. The resulting statistical inverse problem is known to be highly nonlinear and ill-posed, see [11]. Here we propose a novel and general approach for the estimation of p_Δ, which is based on the genuine use of Laplace and Mellin transforms.

The problem of estimating the parameters of a discretely observed Lévy process has recently got much attention in the literature (see, e.g., a recent monograph [4]). Time-changed Lévy processes have been recently studied in Belomestny [3], where it is shown how to estimate the Lévy triplet of the underlying Lévy process L from low-frequency observations of the process (Y_t) without knowledge of the time change $\mathcal{T}$. The results in [3] rely on the fact that the process L is essentially multidimensional. To the best of our knowledge, the problem of estimating the time change $\mathcal{T}$ has not yet been studied in the literature except in some special cases. For example, the case of stopped Poisson process was considered in a recent paper of Comte and Genon-Catalot [9]. The case of the time changed Brownian motion (the so-called statistical Skorohod embedding problem) has recently been studied in Belomestny and Schoenmakers [5]. Note that the latter problem can be transformed to a kind of deconvolution problem using time scalability of Brownian motion. Unfortunately, such a transformation is not possible in the case of general Lévy processes. Statistical inference for time-changed Lévy processes based on high-frequency observations of (Y_t) was the subject of many studies, see, e.g. Bull, [8] and Todorov and Tauchen, [17] and the references therein. Although the problem of estimating the density of $\mathcal{T}$ from discrete (low-frequency) observations of the corresponding time-changed Lévy process Y is related to the problem of non-parametric mixture estimation (see, e.g. Zhang [19] for continuous case or Roueff and Rydén [16] for discrete mixtures), it does not, in general, fit into the existing literature on this topic for several reasons. First, the mixtures encountered here are not, in general, of mean-variance type and therefore can not be handled by an application of one single transform (Fourier or Mellin). Next we have to consider complex-valued functions of complex arguments. Last but not least, the observations are not i.i.d. and so the standard techniques of non-parametric statistics based on the independence assumption would fail here.

The paper is organised as follows. In Section 2 we recall the definition and the basic properties of Mellin transforms. Section 3 describes the construction of our estimator for p_Δ. The convergence of the estimator is studied in Section 4. In particular, we prove upper and lower bounds on the expected pointwise risk. Numerical examples are presented in Section 5.1, which also contains some discussion on adaptive choice of tuning parameters.

2 Mellin transform

Our approach towards estimating the density of the stationary distribution p_Δ makes use of the Mellin transform technique. In this section we introduce the Mellin transform and discuss its main properties.

Definition 2.1. Let ξ be a non-negative random variable with a probability density p_ξ, then the *Mellin transform* of p_ξ is defined via

$$\mathcal{M}[p_\xi](z) := \mathbb{E}[\xi^{z-1}] = \int_0^\infty p_\xi(x)x^{z-1}\,dx \tag{3}$$

for all $z \in \mathcal{S}_\xi$ with $\mathcal{S}_\xi := \big\{z \in \mathbb{C} : \mathbb{E}[\xi^{\mathsf{Re} z-1}] < \infty\big\}$.

Discussion Since p_ξ is a density, it is integrable and so at least $\{z \in \mathbb{C} : \mathsf{Re}(z) = 1\} \subset \mathcal{S}_\xi$. Under mild assumptions on the growth of p_ξ near the origin, one obtains

$$\{z \in \mathbb{C} : 0 \le a_\xi < \mathsf{Re}(z) < b_\xi\} \subset \mathcal{S}_\xi$$

for some $0 \le a_\xi < 1 \le b_\xi$. Then the Mellin transform (3) exists and is analytic in the strip $a_\xi < \mathrm{Re}\, z < b_\xi$. For example, if p_ξ is essentially bounded in a right-hand neighborhood of zero, we may take $a_\xi = 0$. More generally, if p_ξ satisfies

$$p_\xi(x) = \begin{cases} O(x^{-a_\xi}), & x \to +0, \\ O(x^{-b_\xi}), & x \to +\infty, \end{cases} \tag{4}$$

for some $a_\xi < b_\xi$, then the Mellin transform $\mathcal{M}[p_\xi](z)$ is a holomorphic function of z in the strip $a_\xi < \mathsf{Re}(z) < b_\xi$. The role of Mellin transform in probability theory is mainly related to the product of independent random variables: for any two independent r. v. ξ_1 and ξ_2, we have

$$\mathbb{E}[(\xi_1\xi_2)^{z-1}] = \mathbb{E}[\xi_1^{z-1}] \cdot \mathbb{E}[\xi_2^{z-1}], \quad z \in \mathcal{S}_{\xi_1} \cap \mathcal{S}_{\xi_2}.$$

The inversion formula for (3) follows directly from the corresponding formula for the bilateral Laplace transform and is of the form:

$$p_\xi(x) = \frac{1}{2\pi\mathrm{i}} \int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} \mathcal{M}[p_\xi](s)\, x^{-s}\, ds, \quad a_\xi < \gamma < b_\xi \tag{5}$$

at all points $x \ge 0$ where $p_\xi(x)$ is continuous. Note that although the r.h.s. of (5) formally doesn't depend on γ, the choice of γ may be important for numerical evaluation of the integral. A fundamental result in Mellin transform theory is the so-called Parseval formula. Suppose the functions $f(x)$ and $g(x)$ are such that the integral

$$\int_0^\infty f(x)g(x)\,dx$$

exists. Assume that the Mellin transforms $\mathcal{M}[f]$ and $\mathcal{M}[g]$ are both analytical in a strip $\mathcal{S}$, then

$$\int_0^\infty f(x)g(x)\,dx = \frac{1}{2\pi\mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathcal{M}[f](1-s)\mathcal{M}[g](s)\,ds, \tag{6}$$

provided that there exists c such that $\{\mathsf{Re}(z) = c\} \in \mathcal{S}$, $\mathcal{M}[f](1-c-\mathrm{i}\cdot) \in L_1(\mathbb{R})$ and $x^{c-1}g(x) \in L_1(\mathbb{R}_+)$. Let us now discuss some results on the asymptotic behaviour of $\mathcal{M}[p_\xi](z)$. The behaviour of $\mathcal{M}[p_\xi](z)$ in the strip of analyticity is given by the following lemma.

Lemma 2.2. *For $a_\xi < \mathsf{Re}(z) < b_\xi$, we have*

$$\mathcal{M}[p_\xi](z) \to 0, \quad \mathsf{Im}(z) \to \pm\infty.$$

This result is readily established by application of the Riemann-Lebesgue lemma. More precise statements about the rate of decay of $\mathcal{M}[p_\xi](z)$ in the strip $a_\xi < \mathsf{Re}(z) < b_\xi$ can be made with additional information on the behaviour of the density p_ξ, see, e.g., [6].

3 Construction of estimator

Define a curve in $\mathbb{C}$

$$\ell := \Big\{ \mathsf{Re}(\psi(u)) + \mathrm{i}\,\mathsf{Im}(\psi(u)),\ u \in \mathbb{R}_+ \Big\}, \tag{7}$$

where $\psi(u) = -\log(\mathbb{E}(\exp(\mathrm{i}uL_1)))$ and log function is the continuous principal branch of the complex logarithm. Our approach to reconstruct the density p_Δ from discrete time observations of the process Y is based on the following simple identity (cf. (2))

$$\mathcal{F}[p_{Y_\Delta}](\lambda) = \mathbb{E}[\exp(\mathrm{i}\lambda L_{\mathcal{T}(\Delta)})] = \mathcal{L}[p_\Delta](\psi(\lambda)) \tag{8}$$

where $\mathcal{F}[p_{Y_\Delta}]$ stands for the Fourier transform of p_{Y_Δ} and

$$\mathcal{L}[p_\Delta](u) := \int_0^\infty e^{-us} p_\Delta(s)\, ds.$$

Since the Laplace transform $\mathcal{L}[p_\Delta](u)$ is analytic in the domain $\{\mathsf{Re}(u) > 0\}$, the function

$$\mathcal{M}[\mathcal{L}[p_\Delta]](z) = \int_0^\infty u^{z-1} \mathcal{L}[p_\Delta](u)\, du$$

is well defined for $z \in \mathbb{C}$ with $0 < \mathsf{Re}(z) < 1$. The reason to consider the quantity $\mathcal{M}[\mathcal{L}[p_\Delta]](z)$ lies in the following relation

$$\begin{aligned} \mathcal{M}[\mathcal{L}[p_\Delta]](z) &= \int_0^\infty \mathcal{M}[e^{-\cdot\, s}](z)\, p_\Delta(s)\, ds \\ &= \Gamma(z) \int_0^\infty s^{-z} p_\Delta(s)\, ds \\ &= \mathcal{M}[p_\Delta](1-z)\Gamma(z), \quad 0 < \mathsf{Re}(z) < 1, \end{aligned} \tag{9}$$

which relates the Mellin transform of p_Δ to the quantity $\mathcal{M}[\mathcal{L}[p_\Delta]](z)$. On the other hand, the identity (8) implies that $\mathcal{M}[\mathcal{L}[p_\Delta]](z)$ can be connected to the Fourier transform of p_{Y_Δ}, if the contour integral $\int_\ell w^{z-1}\mathcal{L}[p_\Delta](w)dw$ with ℓ defined in (7), can be connected to the integral $\int_0^\infty u^{z-1}\mathcal{L}[p_\Delta](u)du$. The latter connection can be readily established via the well known integral Cauchy theorem. Indeed, under some rather weak assumptions, the difference of the above two contour integrals will be zero by the Cauchy theorem (see Figure 1). The next proposition makes this heuristic explanation more precise.

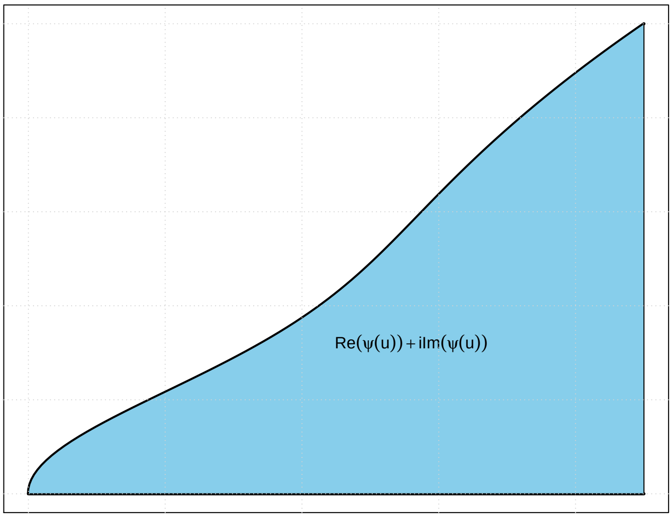

Figure 1: The contour ℓ in the case of a Lévy process L with parameters $\mu = 0$, $\sigma^2 = 1$ and $\nu(dx) = \frac{10}{\sqrt{2\pi}} e^{-x^2/2}$.

Proposition 3.1. *Let us assume that* $\int_{|x|>1} |x|\nu(dx) < \infty$, $\mathsf{Re}(\psi(u)) > 0$ *for* $u > 0$, $\mathsf{Re}(\psi(u)) \to \infty$ *as* $u \to \infty$, *and*

$$\frac{|\mathsf{Im}(\psi(u))|}{\mathsf{Re}(\psi(u))} < A < \infty \tag{10}$$

for all $u > 1$ *and some* $A > 0$. *Moreover, let* p_Δ *be (essentially) bounded. Then, for* $0 < \operatorname{Re} z < 1$ *it holds that*

$$\mathcal{M}[\mathcal{L}[p_\Delta]](z) = \int_0^\infty u^{z-1}\mathcal{L}[p_\Delta](u)du$$

$$= \lim_{\delta \searrow 0} \int_\delta^\infty [\psi(u)]^{z-1}\, \mathcal{L}[p_\Delta](\psi(u))\psi'(u)\, du \tag{11}$$

$$= \lim_{\delta \searrow 0} \int_\delta^\infty \left[\overline{\psi(u)}\right]^{z-1} \mathcal{L}[p_\Delta](\overline{\psi(u)})\overline{\psi'(u)}\, du, \tag{12}$$

where $w^{z-1} := w^{-1}\exp[z\log w]$ *with* $\log w$ *denoting the principal branch of the logarithm defined on* $\operatorname{Re} w > 0$.

Remark 3.2. Note that if $\sigma^2 > 0$, then the condition $\mathsf{Re}(\psi(u)) \to \infty$, $|u| \to \infty$ holds true. If $\sigma = 0$ and $\nu(]x, +\infty[) = C/x^\gamma + o(x^{-\gamma})$, $x \to +0$ for some $C > 0$ and $\gamma \in (0, 2)$, then $\mathsf{Re}(\psi(u)) \to \infty$, $|u| \to \infty$, too. Due to the identity

$$\frac{|\mathsf{Im}(\psi(u))|}{\mathsf{Re}(\psi(u))} = \frac{\left|\mu u + \int_{\mathbb{R}} (\sin(xu) - xu 1_{\{|x|\le 1\}}(x))\, \nu(dx)\right|}{\sigma^2 u^2/2 - \int_{\mathbb{R}} \big(\cos(xu) - 1\big)\, \nu(dx)},$$

the condition (10) is fulfilled if, for example, the diffusion part of L is nonzero ($\sigma > 0$) or if ψ is real. The latter property of ψ always holds in the case of $\mu = 0$ and symmetric Lévy measures ν.

Under the assumptions of Proposition 3.1, we may write

$$\mathcal{M}[\mathcal{L}[p_\Delta]](z) = \int_0^\infty [\psi(\lambda)]^{z-1} \mathcal{L}[p_\Delta](\psi(\lambda))\psi'(\lambda)d\lambda,$$

where $\mathcal{L}[p_\Delta](\psi(\lambda)) = \mathcal{F}[p_{Y_\Delta}](\lambda)$ due to (8). Hence we deduce from (9)

$$\mathcal{M}[p_\Delta](z) = \frac{\mathcal{M}[\mathcal{L}[p_\Delta]](1-z)}{\Gamma(1-z)} = \frac{\int_0^\infty [\psi(\lambda)]^{-z} \mathcal{F}[p_{Y_\Delta}](\lambda)\psi'(\lambda)d\lambda}{\Gamma(1-z)}, \quad 0 < \mathsf{Re}(z) < 1. \tag{13}$$

In principle, one can now replace the Fourier transform of p_{Y_Δ} in (13) by its empirical counterpart based on low-frequency observations of Y. However, in this case we need to regularize the resulting estimate of $\mathcal{M}[p_\Delta](z)$ to perform the inverse Mellin transform. Suppose that

$$\mathsf{Im}(\psi(\lambda)) \geq 0 \text{ or } \mathsf{Im}(\psi(\lambda)) \leq 0 \text{ for all } \lambda > 0. \tag{14}$$

Consider the following regularised approximation for $\mathcal{M}[\mathcal{L}[p_\Delta]]$:

$$\mathcal{M}_n[\mathcal{L}[p_\Delta]](z) := \frac{1}{n}\sum_{k=1}^{n} \Phi_n(1-z, Y_{t_k} - Y_{t_{k-1}}), \tag{15}$$

where

$$\Phi_n(z,y) := \begin{cases} \int_0^{A_n} [\psi(\lambda)]^{z-1} e^{\mathrm{i}\lambda y}\, \psi'(\lambda)\, d\lambda, & \mathsf{Im}(z)\mathsf{Im}(\psi(\lambda)) > 0, \\ \int_0^{A_n} \left[\overline{\psi(\lambda)}\right]^{z-1} e^{-\mathrm{i}\lambda y}\, \overline{\psi'(\lambda)}\, d\lambda, & \text{otherwise} \end{cases} \tag{16}$$

and $A_n \to \infty$, $n \to \infty$. Such a definition ensures that $\mathcal{M}_n[\mathcal{L}[p_\Delta]](z)$ has the property $\mathcal{M}_n[\mathcal{L}[p_\Delta]](z) = \overline{\mathcal{M}_n[\mathcal{L}[p_\Delta]](\bar{z})}$, which holds for the original Mellin transform. More importantly, the function

$$f(z) = \begin{cases} [\psi(\lambda)]^{z-1}, & \mathsf{Im}(z)\mathsf{Im}(\psi(\lambda)) > 0, \\ \left[\overline{\psi(\lambda)}\right]^{z-1}, & \mathsf{Im}(z)\mathsf{Im}(\psi(\lambda)) \leq 0, \end{cases}$$

is bounded by $|\psi(\lambda)|^{\mathrm{Re}\, z-1}$ for all $z \in \mathbb{C}$ leading to a covariance structure of the estimate $\mathcal{M}_n[\mathcal{L}[p_\Delta]](z)$ which is bounded on any line $\mathsf{Re}(z) = \text{const}$. Moreover, due to Proposition 3.1, the estimate $\mathcal{M}_n[\mathcal{L}[p_\Delta]](z)$ is asymptotically unbiased for $A_n \to \infty$. In view of the remark below, the condition (14) is not an essential restriction.

Remark 3.3. The condition (14) can be relaxed in the following way. By continuity of $\mathsf{Im}(\psi)$, its graph can be decomposed in terms of excursions from zero (cf. excursion theory for Brownian motion). That is, there exist countable, open and disjoint intervals I_k^+, $0 \leq k \leq k_\infty^+ \leq \infty$, and I_k^-, $0 \leq k \leq k_\infty^- \leq \infty$, such that the positive excursions of $\mathsf{Im}(\psi(u))$ are supported on $\overline{I_k^+}$, (i.e., $\mathsf{Im}(\psi(u)) > 0$ on each I_k^+) the negative excursions of $\mathsf{Im}(\psi(u))$ are supported by $\overline{I_k^-}$ (i.e., $\mathsf{Im}(\psi(u)) < 0$ on each I_k^-) and $\mathsf{Im}(\psi(u)) = 0$ on

$$\mathbb{R}_{\geq 0} \setminus \bigcup_{k=1}^{k_\infty^+} I_k^+ \cup \bigcup_{k=1}^{k_\infty^-} I_k^-. \tag{17}$$

Then by Proposition 3.1, the Cauchy theorem and smoothness of ψ (ψ is smooth due to $\int_{|x|>1}|x|\nu(dx)<\infty$), we get

$$\begin{aligned}\mathcal{M}[\mathcal{L}[p_\Delta]](z) &= \int_{\mathbb{R}_+\setminus\cup_{k=1}^{k_\infty^-}I_k^-}[\psi(u)]^{z-1}\,\mathcal{L}[p_\Delta](\psi(u))\psi'(u)\,du \\ &\quad+\sum_{k=1}^{k_\infty^-}\int_{I_k^-}\left[\overline{\psi(u)}\right]^{z-1}\mathcal{L}[p_\Delta](\overline{\psi(u)})\overline{\psi'(u)}\,du \\ &= \int_{\mathbb{R}_+\setminus\cup_{k=1}^{k_\infty^+}I_k^+}[\psi(u)]^{z-1}\,\mathcal{L}[p_\Delta](\psi(u))\psi'(u)\,du \\ &\quad+\sum_{k=1}^{k_\infty^+}\int_{I_k^+}\left[\overline{\psi(u)}\right]^{z-1}\mathcal{L}[p_\Delta](\overline{\psi(u)})\overline{\psi'(u)}\,du.\end{aligned}$$

Therefore, instead of (16), we can take

$$\begin{aligned}\Phi_n(z,y) &:= \int_{\mathbb{R}_+\cap[0,A_n]\setminus\cup_{k=1}^{k_\infty^-}I_k^-}[\psi(u)]^{z-1}\,e^{\mathrm{i}uy}\psi'(u)\,du \\ &\quad+\sum_{k=1}^{k_\infty^-}\int_{I_k^-\cap[0,A_n]}\left[\overline{\psi(u)}\right]^{z-1}e^{-\mathrm{i}uy}\overline{\psi'(u)}\,du, \quad \text{if} \quad \operatorname{Im} z\geq 0,\end{aligned}$$

and

$$\begin{aligned}\Phi_n(z,y) &:= \sum_{k=1}^{k_\infty^+}\int_{I_k^+\cap[0,A_n]}\left[\overline{\psi(u)}\right]^{z-1}e^{-\mathrm{i}uy}\overline{\psi'(u)}\,du \\ &\quad+\int_{\mathbb{R}_+\cap[0,A_n]\setminus\cup_{k=1}^{k_\infty^+}I_k^+}[\psi(u)]^{z-1}\,e^{\mathrm{i}uy}\psi'(u)\,du, \quad \text{if} \quad \operatorname{Im} z<0,\end{aligned}$$

in the estimator (15).

By using a regularised version of the inversion formula (5), we define in view of (13),

$$\begin{aligned}(18)\qquad p_{n,\gamma}(x) &:= \frac{1}{2\pi}\int_{-U_n}^{U_n}\frac{\mathcal{M}_n[\mathcal{L}[p_\Delta]](1-\gamma-\mathrm{i}v)}{\Gamma(1-\gamma-\mathrm{i}v)}x^{-\gamma-\mathrm{i}v}\,dv \\ &= \frac{1}{2\pi\, n}\sum_{k=1}^{n}\int_{-U_n}^{U_n}\frac{\Phi_n(1-\gamma-\mathrm{i}v, Y_{t_k}-Y_{t_{k-1}})}{\Gamma(1-\gamma-\mathrm{i}v)}x^{-\gamma-\mathrm{i}v}\,dv \quad \text{for} \quad 0<\gamma<1,\end{aligned}$$

where $U_n, A_n\to\infty$ in a suitable way as $n\to\infty$. Note that in many cases the function Φ_n can be found in closed form. For example, consider the case of a subordinated stable Lévy process with $\psi(\lambda) = |\lambda|^\alpha$ and $1 < \alpha < 2$. It then holds for $\mathsf{Re}(z) > 0$,

$$\begin{aligned}\Phi_n(z,x) &= \int_0^{A_n} [\psi(\lambda)]^{z-1} e^{\mathrm{i}x\lambda}\psi'(\lambda)d\lambda \\ &= \alpha\int_0^{A_n} \lambda^{\alpha(z-1)}e^{\mathrm{i}x\lambda}\lambda^{\alpha-1}\,d\lambda \\ &= \alpha\int_0^{A_n} \lambda^{\alpha z-1}e^{\mathrm{i}x\lambda}d\lambda \\ &= \frac{A_n^{\alpha z}}{z}F_1(\alpha z; 1+\alpha z; \mathrm{i}A_n x),\end{aligned}$$

where F_1 is Kummer's function. In the next section we shall prove that the estimate $p_{n,\gamma}(x)$ converges to $p_\Delta(x)$ and derive the corresponding convergence rates.

4 Convergence

In this section we analyse the convergence properties of the density estimator $p_{n,\gamma}(x)$. Throughout the section we make the following assumption.

(ATS) The sequence $T_k := \mathcal{T}(k\,\Delta) - \mathcal{T}((k-1)\,\Delta)$, $k \in \mathbb{N}$, is strictly stationary, α-mixing with mixing coefficients $(\alpha_T(j))_{j\in\mathbb{N}}$ satisfying

$$\sum_{j=0}^{\infty} \alpha_T(j) < \infty. \tag{19}$$

(ATM) The stationary distribution of the sequence $T_k := \mathcal{T}(k\,\Delta) - \mathcal{T}((k-1)\,\Delta)$, $k \in \mathbb{N}$, possesses a density p_Δ, which is essentially bounded and fulfils

$$\int_0^\infty u\,p_\Delta(u)\,du < \infty.$$

Now we are prepared to derive the minimax upper bounds for the expected pointwise risk of the estimate $p_{n,\gamma}(x)$.

4.1 Upper bounds

For any $\beta > 0$, $\rho > 0$, $0 < \gamma_\circ < \gamma^\circ$ and $L > 0$ introduce two classes of probability densities

$$\mathcal{C}(\beta,\gamma_\circ,\gamma^\circ,L) := \left\{ f : f \in \mathcal{P},\ \sup_{\gamma_\circ\le c\le\gamma^\circ} \int_{-\infty}^{\infty} |\mathcal{M}[f](c+\mathrm{i}v)|\,e^{\beta|v|}\,dv \le L \right\}$$

and

$$\mathcal{D}(\rho,\gamma_\circ,\gamma^\circ,L) := \left\{ f : f \in \mathcal{P},\ \sup_{\gamma_\circ\le c\le\gamma^\circ} \int_{-\infty}^{\infty} |\mathcal{M}[f](c+\mathrm{i}v)|\,(1+|v|^\rho)\,dv \le L \right\},$$

where $\mathcal{P}$ stands for the class of all probability densities. While the class $\mathcal{C}(\beta,\gamma_\circ,\gamma^\circ,L)$ contains densities with exponentially decaying Mellin transforms, the Mellin transform of densities from $\mathcal{D}(\rho,\gamma_\circ,\gamma^\circ,L)$ decays only polynomially fast. Let us turn to some examples.

Example 4.1. Consider a class of Gamma densities

$$p_\Delta(x;\alpha) = \frac{x^{\alpha\,\Delta-1}\cdot e^{-x}}{\Gamma(\alpha\,\Delta)}, \quad x \geq 0, \quad \alpha > 0, \tag{20}$$

corresponding to Gamma subordinators $\mathcal{T}(t)$, $t \geq 0$. Since

$$\mathcal{M}[p_\Delta](z) = \frac{\Gamma(z + \alpha\,\Delta - 1)}{\Gamma(\alpha\,\Delta)}, \quad \mathsf{Re}(z) > 0,$$

we derive that $p_\Delta \in \mathcal{C}(\beta, \gamma_\circ, \gamma^\circ, L)$ for all $0 < \beta < \pi/2$, $0 < \gamma_\circ < \gamma^\circ < \infty$ and some $L = L(\beta, \gamma_\circ, \gamma^\circ)$ due to the asymptotic properties of the Gamma function (see Lemma 7.3 in Appendix).

Example 4.2. Let us look at the family of densities

$$p_\Delta(x;q) = \frac{q\sin(\pi/q)}{\pi}\frac{1}{1+x^q}, \quad q \geq 2, \quad x \geq 0.$$

We have

$$\mathcal{M}[p_\Delta](z) = \frac{\sin(\pi/q)}{\sin(\pi z/q)}, \quad 0 < \mathsf{Re}(z) < q.$$

Therefore

$$p_\Delta \in \mathcal{C}(\beta, \gamma_\circ, \gamma^\circ, L)$$

for all $0 < \beta < \pi/q$, $0 < \gamma_\circ < \gamma^\circ < \infty$ and some $L = L(\beta, \gamma_\circ, \gamma^\circ)$.

Theorem 4.3. *Suppose that $\sigma^2 > 0$, $\int_{\{|x|>1\}} |x|\nu(dx) < \infty$ and (19) holds. Furthermore, suppose that there are two numbers $\gamma_\circ, \gamma^\circ$ with $0 < \gamma_\circ < \gamma^\circ \leq 1$ such that $p_\Delta \in \mathcal{C}(\beta, \gamma_\circ, \gamma^\circ, L)$ for some $\beta > 0$. Then under the choice*

$$A_n = n^{1/4}, \quad U_n = \frac{\gamma^\circ}{2\beta+\pi}\log n - \frac{2\gamma^\circ - 1}{2\beta+\pi}\log\log n, \tag{21}$$

we get for the estimator $p_{n,\gamma}$ in (18) with $\gamma = \gamma^\circ$

$$\sup_{x\geq 0}\sqrt{\mathrm{E}\left[w(x)\,|p_{n,\gamma}(x) - p_\Delta(x)|^2\right]} \lesssim n^{-\frac{\beta\gamma^\circ}{2\beta+\pi}}\log^{\beta\frac{2\gamma^\circ-1}{2\beta+\pi}} n, \quad n \to \infty, \tag{22}$$

where $w(x) := \min\{1, x^2\}$ and the notation $\lesssim$ means that the above inequality is valid up to a multiplicative constant that does not depend on the unknown density.

Corollary 4.4. *Under conditions of Theorem 4.3,*

$$\sup_{p_\Delta\in\mathcal{C}(\beta,\gamma_\circ,\gamma^\circ,L)}\sup_{x\geq 0}\sqrt{\mathrm{E}\left[w(x)\,|p_{n,\gamma}(x) - p_\Delta(x)|^2\right]} \lesssim n^{-\frac{\beta\gamma^\circ}{2\beta+\pi}}\log^{\beta\frac{2\gamma^\circ-1}{2\beta+\pi}} n, \quad n \to \infty. \tag{23}$$

In the case $p_\Delta \in \mathcal{D}(\rho, \gamma_\circ, \gamma^\circ, L)$, we get logarithmic rates.

Theorem 4.5. *Suppose that* $\sigma^2 > 0$, $\int_{\{|x|>1\}} |x|\nu(dx) < \infty$ *and* (19) *holds. Suppose that* $p_\Delta \in \mathcal{D}(\rho, \gamma_\circ, \gamma^\circ, L)$ *for some* $\rho > 0$ *and* $0 < \gamma_\circ < \gamma^\circ \le 1$. *Then under the choice*

$$A_n = n^{1/4}, \quad U_n = \frac{\gamma}{\pi}\log n - \frac{2}{\pi}\left(\rho + \gamma^\circ - 1/2\right)\log\log n, \tag{24}$$

we get for the estimator $p_{n,\gamma}$ *in* (18) *with* $\gamma = \gamma^\circ$,

$$\sup_{x\ge 0}\sqrt{\mathrm{E}\left[w(x)\left|p_{n,\gamma}(x) - p_\Delta(x)\right|^2\right]} \lesssim \log^{-\rho}(n), \quad n \to \infty,$$

where $w(x) := \min\{1, x^2\}$ *and the notation* $\lesssim$ *means that the above inequality is valid up to a multiplicative constant that does not depend on the unknown density.*

Corollary 4.6. *Under conditions of Theorem 4.5,*

$$\sup_{p_\Delta\in\mathcal{D}(\rho,\gamma_\circ,\gamma^\circ,L)}\sup_{x\ge 0}\sqrt{\mathrm{E}\left[w(x)\left|p_{n,\gamma}(x) - p_\Delta(x)\right|^2\right]} \lesssim \log^{-\rho}(n), \quad n \to \infty. \tag{25}$$

Discussion

- Due to the relation

  $$\mathcal{M}[p_\Delta](\gamma + \mathrm{i}v) = \mathcal{F}[e^{\gamma\cdot}p_\Delta(e^\cdot)](v), \quad a_\Delta < \gamma < b_\Delta,$$

  the conditions $p_\Delta \in \mathcal{C}(\beta, \gamma_\circ, \gamma^\circ, L)$ and $p_\Delta \in \mathcal{D}(\rho, \gamma_\circ, \gamma^\circ, L)$ are closely related to the smoothness properties of the function $e^{\gamma^\circ x}p_\Delta(e^x)$. For example, if $p_\Delta \in \mathcal{C}(\beta, \gamma_\circ, \gamma^\circ, L)$, then

  $$\int_{-\infty}^{\infty}\left|\mathcal{F}[e^{\gamma^\circ\cdot}p_\Delta(e^\cdot)](v)\right| e^{\beta|v|}\,dv \le L$$

  and the function $e^{\gamma^\circ x}p_\Delta(e^x)$ is called supersmooth in this case, see Meister [14] for a discussion on different smoothness classes in the context of the additive deconvolution problems.

4.2 Lower bounds

Now let us turn to the question of optimality of the rates in Theorem 4.3 and Theorem 4.5. It turns out that the above rates are already optimal (up to a logarithmic factor and for $\gamma^\circ = 1$ in Theorem 4.3) in minimax sense for the case of time-changed Brownian motions and i.i.d observations (see Belomestny and Schoenmakers [5], where the lower bounds were already announced without proof). This entails the optimality for a larger class of time-changed Lévy processes and dependent observations.

Theorem 4.7. *Fix some* $\beta > 1$. *There are* $\varepsilon > 0$ *and* $x > 0$ *such that*

$$\liminf_{n\to\infty}\inf_{p_n}\sup_{p_\Delta\in\mathcal{C}(\beta,\gamma_\circ,1,L)} \mathrm{P}_{p_\Delta}^{\otimes n}\left(\left|p_\Delta(x) - p_n(x)\right| \ge \varepsilon\, n^{-\frac{\beta}{\pi+2\beta}}\log^{-\kappa}(n)\right) > 0,$$

$$\liminf_{n\to\infty}\inf_{p_n}\sup_{p_\Delta\in\mathcal{D}(\rho,\gamma_\circ,\gamma^\circ,L)} \mathrm{P}_{p_\Delta}^{\otimes n}\left(\left|p_\Delta(x) - p_n(x)\right| \ge \varepsilon\log^{-\rho}(n)\right) > 0,$$

for some $\kappa > 0$, $0 < \gamma_\circ < \gamma^\circ \leq 1$, *where the infimum is taken over all estimators (i.e. all measurable functions of* $X_1, \ldots, X_n$*) of* p_Δ *and* $\mathrm{P}_{p_\Delta}^{\otimes n}$ *is the distribution of the i.i.d. sample* $X_1, \ldots, X_n$ *with* $X_1 \sim W_T$*, where* W *is a Brownian motion and* T *is a independent random variable with distribution* p_Δ.

5 Numerical example

5.1 Gamma subordinator

We focus our numerical analysis on time-changed Brownian motion with drift, i.e., we consider the process Y of the form $Y_t = T_t + W_{T_t}$, $t \geq 0$, where (T_t) is chosen to follow a Gamma process with marginal densities

$$p_\Delta(x) = \frac{x^{2\,t-1} \cdot e^{-x}}{\Gamma(2\,t)}, \quad x \geq 0, \quad t \geq 0. \tag{26}$$

We fix $\Delta = 1$ and generate a time series $Y_0, Y_\Delta, \ldots, Y_{n\Delta}$ from Y of the length n. The estimate (18) is constructed as follows. First note that $\psi(\lambda) = -\mathrm{i}\lambda + \lambda^2/2$. In order to numerically compute the function $\Phi_n(1-z, X_k)$ for $z = \gamma + \mathrm{i}v$ with $\gamma < 1$, we use the decomposition

$$\begin{aligned} \frac{1}{n}\sum_{k=1}^{n} \Phi_n(1-z, Y_{\Delta k} - Y_{\Delta(k-1)}) &= \int_0^{A_n} [\psi(\lambda)]^{-z}\, [\phi_n(\lambda) - e^{-m_n\psi(\lambda)}]\psi'(\lambda)\, d\lambda \\ &\quad + m_n^{z-1}\Gamma(1-z) + O\big(m_n^{-(1-\gamma)} \exp(-m_n A_n^2/2)\big), \end{aligned} \tag{27}$$

where $\phi_n(\lambda) = \frac{1}{n}\sum_{k=1}^{n} e^{\mathrm{i}\lambda(Y_{\Delta k} - Y_{\Delta(k-1)})}$ is the empirical characteristic function and

$$m_n = \frac{1}{n}\sum_{k=1}^{n}(Y_{\Delta k} - Y_{\Delta(k-1)}) \to 2.$$

This decomposition follows from a Cauchy argument similar to one used in the proof of Proposition 3.1 and is quite useful to reduce the cost of computing the integral in (27), since the integral on the r.h.s. of (27) is much easier to compute numerically due to the asymptotic relation $\phi_n(\lambda) - e^{-m_n\psi(\lambda)} = O(\lambda^2)$, $\lambda \to 0$. Next we take $\gamma = 0.7$ and compute the estimate $p_{n,\gamma}$ for $n = 10000$ and different values of the cut-off parameter U_n (A_n is fixed by the asymptotic formula (21)). On the left-hand side of Figure 3, the loss $\sup_{x\in[0,10]}\big\{|p_{n,\gamma}(x) - p_\Delta(x)|\big\}$ is shown as function of U_n with the minimum attained for $U_n \approx 2.2$.

As can be seen from Figure 3, the choice of the cut-off parameter U_n is crucial for a good performance of the estimate $p_{n,\gamma}$ and a data-driven choice of U_n would be desirable. To this end, we adopt the so called "quasi-optimality" approach proposed in [2]. This approach is aimed to perform a model selection in inverse problems without taking into account the noise level. Although one can prove the optimality of this criterion on average only, it leads in many situations to quite reasonable results. In order to implement the "quasi-optimality" algorithm in our situation, we first fix a sequence of bandwidths $U_1, \ldots, U_L$ and construct the estimates $p_n^{(1)}, \ldots, p_n^{(L)}$ using the formula (18) with cut-off parameters $U_1, \ldots, U_L$, respectively. Then one finds $l^\star = \operatorname{argmin}_l f(l)$ with

$$f(l) := \|p_n^{(l+1)} - p_n^{(l)}\|_{L_1([0,10])}, \quad l = 1, \ldots, L.$$

Denote by $\widetilde{p}_n = p_n^{(l^\star)}$ a new adaptive estimate for p_Δ. In our implementation of the "quasi-optimality" approach we take $U_l = 0.1 \times l$, $l = 1, \ldots, 40$. On the right-hand side of Figure 3 one can see the objective function $f(l)$ and the location of its minimum ($U_l \approx 1.8$).

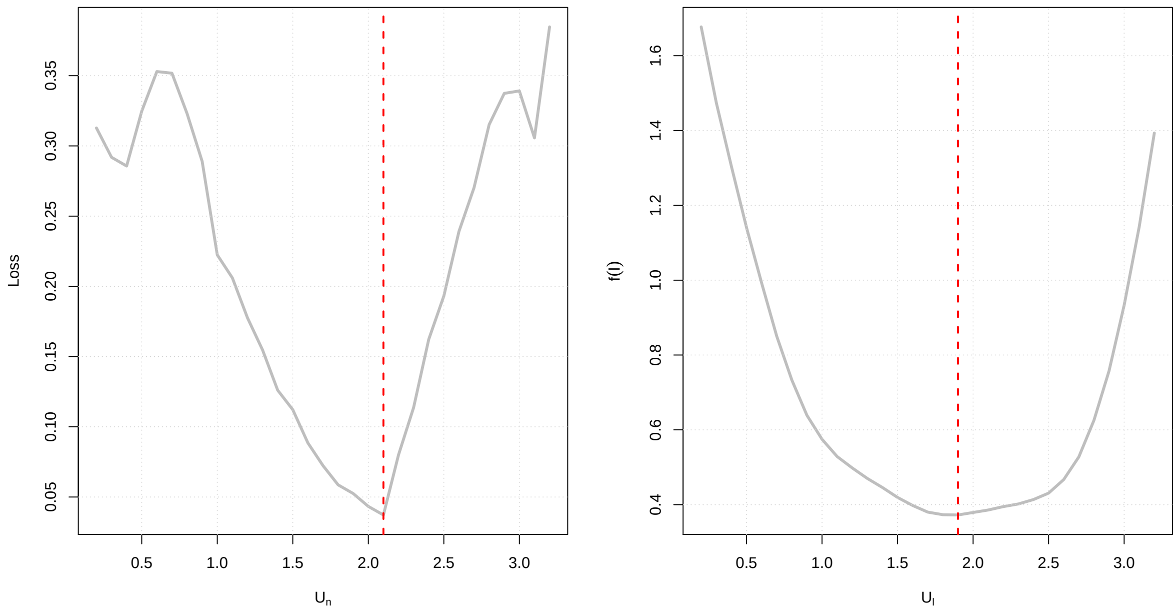

Figure 2: Left: the loss $\sup_{x\in[0,10]}\{|p_{n,\gamma}(x) - p_\Delta(x)|\}$ as a function of the cut-off parameter U_n. Right: the objective function f used in the adaptive "quasi-optimality" algorithm.

In order to assess the finite sample performance of the "quasi-optimality" algorithm, we conduct 100 runs of the estimation algorithm for different sample sizes with the optimal (oracle) choice of U_n, which minimises the loss $\sup_{x\in[0,10]}\{|p_{n,\gamma}(x) - p_\Delta(x)|\}$. The results in form of box plots of the loss are shown on the l.h.s. of Figure 3. Then we repeat 100 runs and use the data-driven procedure described above to choose the cut-off parameter U_n, $n \in \{1000, 5000, 10000, 50000\}$. The corresponding box-plots of the loss are shown on the r.h.s. of Figure 3. By comparing these two graphs, we conclude that the performance of the "quasi-optimality" algorithm is quite reasonable in our situation.

5.2 Integrated CIR process

Another candidate for the time change process is given by the integrated Cox-Ingersoll-Ross (CIR) process. The CIR process is defined as a solution of the following SDE:

$$dZ_t = (a - bZ_t)dt + \zeta\sqrt{Z_t}\,dW_t, \tag{28}$$

where a, b and ζ are positive numbers, and W_t is a Wiener process. If Z_0 is sampled from the stationary invariant distribution π and $2a \geq \zeta^2$, then Z_t is strictly stationary and ergodic. The

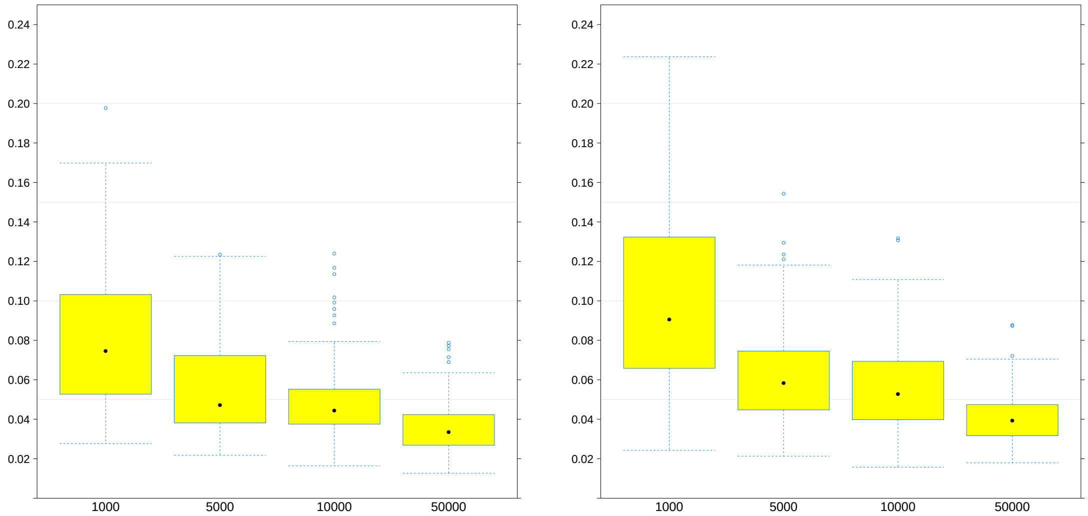

Figure 3: Left: box plots of the loss $\sup_{x\in[0,10]}\left\{|p_{n,\gamma}(x)-p_\Delta(x)|\right\}$ for different sample sizes under the oracle choice of the cut-off parameter U_n. Right: box plots of the loss $\sup_{x\in[0,10]}\left\{|p_{n,\gamma}(x)-p_\Delta(x)|\right\}$ for different sample sizes under adaptive choice of the cut-off parameter U_n.

time change process $\mathcal{T}(s)$ is then defined as

$$\mathcal{T}(s) = \int_0^s Z_t\,dt.$$

The Laplace transform of $\mathcal{T}(\Delta)$ under π is given by

$$\begin{aligned} \mathcal{L}_\Delta^{ICIR}(u) &= \mathbb{E}_\pi\left[\mathbb{E}\left[e^{-u\mathcal{T}(\Delta)}|Z_0\right]\right] \\ &= \frac{\exp\left\{ab\Delta/\zeta^2\right\}}{\left(\cosh(\Lambda\Delta/2)+\frac{b}{\Lambda}\sinh(\Lambda\Delta/2)\right)^{2a/\zeta^2}}\,\mathbb{E}_\pi\left[\exp\left\{\frac{-2Z_0u}{b+\Lambda\coth(\Lambda\Delta/2)}\right\}\right], \end{aligned} \tag{29}$$

where $\Lambda(u)=\sqrt{b^2+2\zeta^2u}$, see Chapter 15.1.2 from Cont and Tankov, [10]. Since the stationary distribution of the CIR process is the Gamma distribution with parameters $2a/\zeta^2$ and $2b/\zeta^2$, the Laplace transform of Z_0 under π has a form

$$\mathbb{E}_\pi\left[e^{-hZ_0}\right] = \left(1+h\frac{\zeta^2}{2b}\right)^{-2a/\zeta^2},$$

and therefore

$$\mathbb{E}_\pi\left[\exp\left\{\frac{-2Z_0u}{b+\Lambda\coth(\Lambda\Delta/2)}\right\}\right] = \left(1+B_1(\Lambda)\frac{\zeta^2}{b}\frac{u}{\Lambda}\right)^{-2a/\zeta^2}, \tag{30}$$

where $B_1(\Lambda)=\left(b/\Lambda+\coth(\Lambda\Delta/2)\right)^{-1}\to 1$ as $\Lambda\to\infty$. Using an inverse Fourier transform, we obtain the corresponding probability density function of the integrated CIR process as

$$p_\Delta(x) = \frac{1}{\pi}\int_0^\infty \mathsf{Re}[e^{-\mathrm{i}ux}\mathcal{L}_\Delta^{ICIR}(-\mathrm{i}u)]\,du. \tag{31}$$

We apply the adaptive Gauss-Lobatto quadrature to compute $p_\Delta(x)$ numerically via (31). The following set of parameters was used

$$\zeta^2 = 0.5,\ a = 0.5,\ b = 0.25.$$

We fix $\Delta = 1$ and generate a time series $Y_0, Y_\Delta, \ldots, Y_{n\Delta}$ from the process $Y_t = T_t + W_{T_t}$, $t \geq 0$ of the length n, where for simulation of the time change (T_t) on the time grid $\Delta, 2\Delta, \ldots, n\Delta$ via the (28) the Milstein scheme with time step 0.001 is used. Next we take $\gamma = 0.5$ and compute the estimate $p_{n,\gamma}$ as described in the previous section. In the Figure 4 the box plots of the error $\sup_{x\in[0,10]}\left\{|p_{n,\gamma}(x) - p_\Delta(x)|\right\}$ based on 100 runs are shown for different values of n.

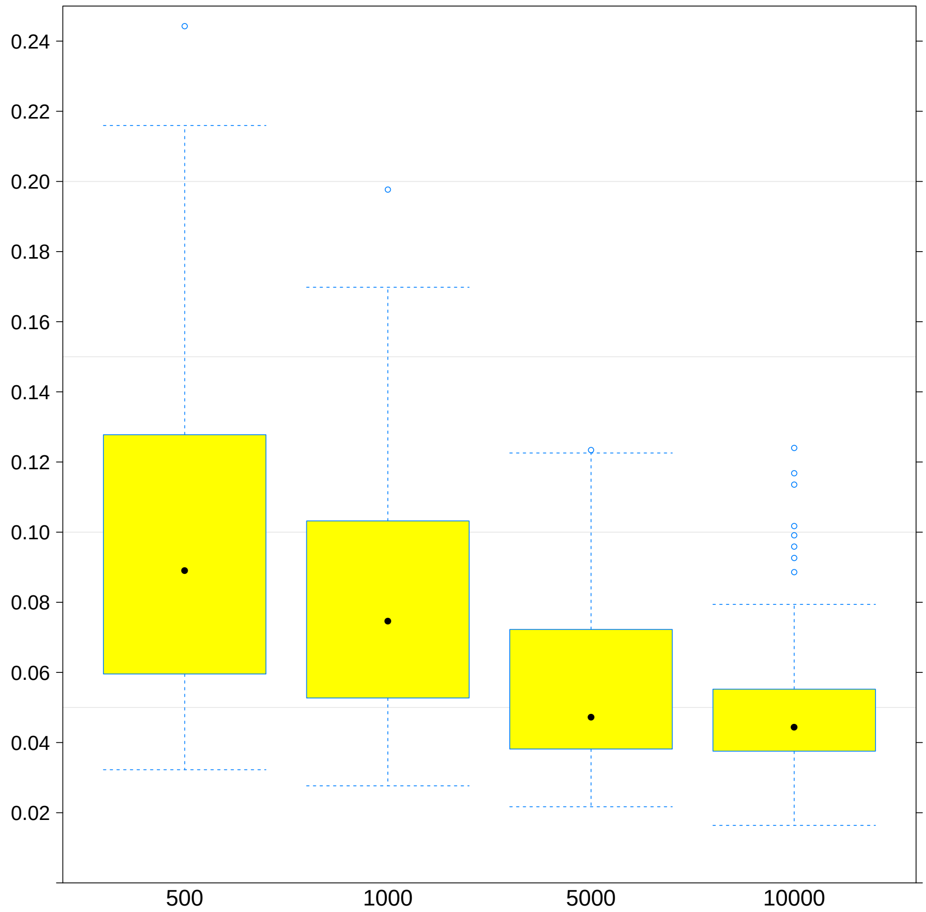

Figure 4: Left: box plots of the loss $\sup_{x\in[0,10]}\left\{|p_{n,\gamma}(x) - p_\Delta(x)|\right\}$ for different sample sizes under the oracle choice of the cut-off parameter U_n.

6 Proofs

6.1 Proof of Proposition 3.1

First note that the condition $\int_{|x|>1} |x|\nu(dx) < \infty$ ensures that $\psi(u)$ is smooth. It is enough to prove (11), (12) goes analogously. Let $\theta_{\max}$ be such that $A = \tan\theta_{\max}$. Take an arbitrary $U > 1$ and let $\psi(U) = R_U\, e^{\mathrm{i}\theta_U}$. Assume w.l.o.g. that $\theta_U > 0$, hence by assumption (10), $0 < \theta_U \le \theta_{\max} < \pi/2$. For fixed $0 < \operatorname{Re} z < 1$ the map

$$w \to w^{z-1}\mathcal{L}[p_\Delta](w)$$

is holomorphic in the region $\operatorname{Re} w > 0$. Note that for $\operatorname{Re} w > 0$ it holds that

$$\left|w^{z-1}\right| = |w|^{-1}\, |w|^{\operatorname{Re} z}\, e^{-\operatorname{Im} z \,\arg w} \le |w|^{\operatorname{Re} z-1}\, e^{\pi|\operatorname{Im} z|/2} =: C_z\, |w|^{\operatorname{Re} z-1}. \tag{32}$$

At the arc $K_U : w = R_U\, e^{\mathrm{i}\theta}$, $0 \le \theta \le \theta_U$, it then holds that

$$\begin{aligned} |I_U| &:= \left|\int_{K_U} w^{z-1}\mathcal{L}[p_\Delta](w)dw\right| \le C_z R_U \theta_{\max} \cdot R_U^{\operatorname{Re} z-1} \int e^{-xR_U\cos\theta_{\max}} p_\Delta(x)dx \\ &\le C_z B\theta_{\max} R_U^{\operatorname{Re} z} \int e^{-xR_U\cos\theta_{\max}} dx \\ &= C_z B\theta_{\max} \frac{R_U^{\operatorname{Re} z-1}}{\cos\theta_{\max}}, \end{aligned}$$

with $\sup_{x>0} p_\Delta(x) := B$. So when $U \to \infty$, we have $R_U \ge \operatorname{Re}(\psi(U)) \to \infty$, and then by $\operatorname{Re} z < 1$, $I_U \to 0$. Now let K_δ denote the arc between $|\psi(\delta)|$ and $\psi(\delta)$ parametrized by $w = |\psi(\delta)|\, e^{it}$. For $\operatorname{Re} w \ge 0$, we have that

$$|\mathcal{L}[p_\Delta](w)| \le \int_0^\infty e^{-x\operatorname{Re} w} p_\Delta(x)dx \le 1.$$

Therefore we have for $\delta \downarrow 0$ by (32),

$$|I_\delta| := \left|\int_{K_\delta} w^{z-1}\mathcal{L}[p_\Delta](w)dw\right| \le C_z\, |\psi(\delta)|^{\operatorname{Re} z-1}\, |\psi(\delta)|\, \pi/2 = C_z\, |\psi(\delta)|^{\operatorname{Re} z}\, \pi/2.$$

Since $\psi(\delta) \to 0$ for $\delta \downarrow 0$ it thus holds by $\operatorname{Re} z > 0$ that $I_\delta \to 0$ for $\delta \downarrow 0$. Now by Cauchy's theorem we so have for any $\delta > 0$, $U > 1$,

$$\int_\delta^U (\psi(u))^{z-1}\, \mathcal{L}[p_\Delta](\psi(u))\psi'(u)du - I_U - \int_{|\psi(\delta)|}^{R_U} u^{z-1}\mathcal{L}[p_\Delta](u)du + I_\delta = 0$$

and the theorem is proved by sending $U \to \infty$ and $\delta \downarrow 0$, respectively.

6.2 Proof of Theorem 4.7

Our construction relies on the following basic result (see [18] for the proof).

Theorem 6.1. *Suppose that for some $\varepsilon > 0$ and $n \in \mathbb{N}$ there are two densities $p_{0,n}, p_{1,n} \in \mathcal{G}$ such that*

$$d(p_{0,n}, p_{1,n}) > 2\varepsilon v_n.$$

If the observations in model n follow the product law $\mathsf{P}_{p,n} = \mathsf{P}_p^{\otimes n}$ under the density $p \in \mathcal{G}$ and

$$\chi^2(p_{1,n} \,|\, p_{0,n}) \leq n^{-1} \log(1 + (2 - 4\delta)^2)$$

holds for some $\delta \in (0, 1/2)$, then the following lower bound holds for all density estimators $\hat{p}_n$ based on observations from model n:

$$\inf_{\hat{p}_n} \sup_{p \in \mathcal{G}} \mathrm{P}_p^{\otimes n} \left(d(\hat{p}_n, p) \geq \varepsilon v_n \right) \geq \delta.$$

If the above holds for fixed $\varepsilon, \delta > 0$ and all $n \in \mathbb{N}$, then the optimal rate of convergence in a minimax sense over $\mathcal{G}$ is not faster than v_n.

6.2.1 Proof of a lower bound for the class $\mathcal{C}(\beta, \gamma_\circ, \gamma^\circ, L)$

Let us start with the construction of the densities $p_{0,n}$ and $p_{1,n}$. Define for any $\nu > 1$ and $M > 0$ two auxiliary functions

$$q(x) = \frac{\nu \sin(\pi/\nu)}{\pi} \frac{1}{1 + x^\nu}, \quad x \geq 0$$

and

$$\rho_M(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{\log^2(x)}{2}} \frac{\sin(M \log(x))}{x}, \quad x \geq 0.$$

The properties of the functions q and ρ_M are collected in the following lemma.

Lemma 6.2. *The function q is a probability density on $\mathbb{R}_+$ with the Mellin transform*

$$\mathcal{M}[q](z) = \frac{\sin(\pi/\nu)}{\sin(\pi z/\nu)}, \quad 0 < \mathsf{Re}[z] < \nu.$$

The Mellin transform of the function ρ_M is given by

$$\mathcal{M}[\rho_M](u + \mathrm{i}v) = \frac{1}{2\mathrm{i}} \left[e^{(u+\mathrm{i}(v+M))^2/2} - e^{(u+\mathrm{i}(v-M))^2/2} \right], \tag{33}$$

hence

$$\int_0^\infty \rho_M(x) dx = \mathcal{M}[\rho_M](1) = 0.$$

Proof. The formula for $\mathcal{M}[q](z)$ can be found in [15]. We have

$$\begin{aligned}
\mathcal{M}[\rho_M](z) &= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{\log^2(x)}{2}} \sin(M \log(x)) x^z d\log(x) \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-\frac{y^2}{2}} \sin(My) e^{yz} dy \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-\frac{y^2}{2}} \left[\frac{e^{y(z+\mathrm{i}M)} - e^{y(z-\mathrm{i}M)}}{2\mathrm{i}} \right] dy \\
&= \frac{e^{(z+\mathrm{i}M)^2/2} - e^{(z-\mathrm{i}M)^2/2}}{2\mathrm{i}}.
\end{aligned}$$

□

Set now for any $M > 0$ and some $\delta > 0$,

$$q_{0,M}(x) := q(x), \quad q_{1,M}(x) := q(x) + \delta(q \vee \rho_M)(x),$$

where $f \vee g$ stands for the multiplicative convolution of two functions f and g on $\mathbb{R}_+$ defined as

$$(f \vee g)(x) := \int_0^\infty \frac{f(t)g(x/t)}{t} dt, \quad x \geq 0. \tag{34}$$

The following lemma describes some properties of $q_{0,M}$ and $q_{1,M}$.

Lemma 6.3. *For any $M > 0$ and some $\delta > 0$ not depending on M, the function $q_{1,M}$ is a probability density satisfying*

$$\|q_{0,M} - q_{1,M}\|_\infty = \sup_{x \in \mathbb{R}_+} |q_{0,M}(x) - q_{1,M}(x)| \gtrsim \exp(-M\pi/\nu), \quad M \to \infty.$$

Moreover, $q_{0,M}$ and $q_{1,M}$ are in $\mathcal{C}(\beta, \gamma_\circ, \gamma^\circ, L)$ for all $0 < \beta < \pi/\nu$ and $\gamma^\circ > \gamma_\circ > 0$ with L depending on $\gamma_\circ$ and γ°.

Proof. It holds with $c_\nu := \frac{\nu \sin(\pi/\nu)}{\pi}$,

$$\begin{aligned}
|(q \vee \rho_M)(y)| &\leq c_\nu \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{\log^2(x)}{2}} \frac{1}{x^2} \frac{1}{1 + (y/x)^\nu} dx \\
&= c_\nu \int_0^1 \frac{1}{\sqrt{2\pi}} e^{-\frac{\log^2(x)}{2}} \frac{1}{x^2} \frac{1}{1 + (y/x)^\nu} dx \\
&\quad + c_\nu \int_1^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{\log^2(x)}{2}} \frac{1}{x^2} \frac{1}{1 + (y/x)^\nu} dx \\
&=: c_\nu I_1 + c_\nu I_2,
\end{aligned}$$

where

$$I_1 \leq \frac{1}{\sqrt{2\pi}} \frac{1}{1 + y^\nu} \int_0^\infty e^{-\frac{(y-1)^2}{2} + \frac{1}{2}} dy \leq \frac{e^{1/2}}{1 + y^\nu}$$

and

$$I_2 \leq \frac{1}{\sqrt{2\pi} y} \int_0^1 e^{-\frac{\log^2(y/z)}{2}} \frac{1}{1 + z^\nu} dz \leq \frac{1}{\sqrt{2\pi} y} e^{-\frac{\log^2(y)}{2}}.$$

Note that for any $\nu > 1$, there is a constant $c_1 = c_1(\nu)$ such that

$$\frac{1}{\sqrt{2\pi} y} e^{-\frac{\log^2(y)}{2}} \leq \frac{c_1}{1 + y^\nu}, \quad y \geq 0.$$

Hence we have with $\delta = 1/(\sqrt{e} + c_1)$,

$$\delta\, |(q \vee \rho_M)(y)| \leq q(y), \quad y \geq 0.$$

Moreover

$$\int_0^\infty q_{1,M}(x) dx = 1 + \int_0^\infty (q \vee \rho_M)(x) = 1 + \mathcal{M}[q](1)\mathcal{M}[\rho_M](1) = 1.$$

Furthermore, due to the Parseval identity

$$
\begin{aligned}
(q \vee \rho_M)(y) &= \frac{c_\nu}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{\log^2(x)}{2}} \frac{\sin(M\log(x))}{x^2} \frac{1}{1+(y/x)^\nu} dx \\
&= \frac{c_\nu}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-\frac{v^2}{2}} \sin(Mv) \frac{e^{-v}}{1+e^{-\nu(v-\log(y))}} dv \\
&= \frac{c_\nu e^{-\log(y)}}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-\frac{v^2}{2}} \sin(Mv) \frac{e^{(\log(y)-v)}}{1+e^{\nu(\log(y)-v)}} dv \\
&= \frac{c_\nu e^{-\log(y)}}{2\pi} \int_{-\infty}^\infty e^{-\mathrm{i}u\log(y)} \left[\frac{H(u+M)-H(u-M)}{2i}\right] \mathcal{F}[R](u) du,
\end{aligned}
$$

$R(x) = \frac{e^x}{1+e^{\nu x}}$ and $H(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$. Note that

$$
\mathcal{F}[R](u) = \int_{-\infty}^\infty \frac{e^{x+\mathrm{i}ux}}{1+e^{\nu x}} dx = \frac{1}{\nu} \int_{-\infty}^\infty \frac{e^{v/\nu+\mathrm{i}uv/\nu}}{1+e^v} dv = \frac{1}{\nu} \Gamma\left(\frac{1+\mathrm{i}u}{\nu}\right) \Gamma\left(1-\frac{1+\mathrm{i}u}{\nu}\right),
$$

since

$$
\int_{-\infty}^\infty \frac{e^{vz}}{1+e^v} dv = /y = (1+e^v)^{-1}/ = \int_0^1 (1-y)^{z-1} y^{1-z-1} dy = \Gamma(z)\Gamma(1-z)
$$

for any z with $0 < \mathsf{Re}(z) < 1$. Hence due to (57)

$$
\sup_{y\in\mathbb{R}_+} |q_{0,M}(y) - q_{1,M}(y)| = \sup_{y\in\mathbb{R}_+} |(q \vee \rho_M)(y)| \gtrsim \exp(-M\pi/\nu), \quad M \to \infty.
$$

The second statement of the lemma follows from Lemma 6.2 and the fact that $\mathcal{M}[q \vee \rho_M] = \mathcal{M}[q]\mathcal{M}[\rho_M]$. Indeed, the Mellin transform $\mathcal{M}[\rho_M](u+\mathrm{i}v)$ is of order $O(e^{-v^2/2})$ for $|v| \to \infty$ and so $\mathcal{M}[q \vee \rho_M](u+\mathrm{i}v)$ has the same order. □

Let $T_{0,M}$ and $T_{1,M}$ be two random variables with densities $q_{0,M}$ and $q_{1,M}$, respectively. Then the density of the r.v. $|W_{T_{i,M}}|$, $i = 0,1$, is given by

$$
p_{i,M}(x) := \frac{2}{\sqrt{2\pi}} \int_0^\infty \lambda^{-1/2} e^{-\frac{x^2}{2\lambda}} q_{i,M}(\lambda)\, d\lambda, \quad i = 0,1.
$$

For the Mellin transform of $p_{i,M}$ we get

$$
\begin{aligned}
\mathcal{M}[p_{i,M}](z) &= \mathbb{E}\big[|W_1|^{z-1}\big]\mathbb{E}\big[T_{i,M}^{(z-1)/2}\big] \\
&= \mathbb{E}\big[|W_1|^{z-1}\big]\mathcal{M}[q_{i,M}]((z+1)/2) \\
&= \frac{2^{z/2}}{\sqrt{2\pi}} \Gamma(z/2)\mathcal{M}[q_{i,M}]((z+1)/2), \quad i = 0,1. \qquad (35)
\end{aligned}
$$

Lemma 6.4. *The χ^2-distance between the densities $p_{0,M}$ and $p_{1,M}$ fulfills*

$$
\chi^2(p_{1,M}|p_{0,M}) = \int \frac{(p_{1,M}(x) - p_{0,M}(x))^2}{p_{0,M}(x)} dx \lesssim M^{\nu-1} e^{-M\pi(1+2/\nu)}, \quad M \to \infty.
$$

Proof. First note that $p_{0,M}(x) > 0$ on $[0,\infty)$. Since

$$\begin{aligned}
p_{0,M}(x) &= \frac{2}{\sqrt{2\pi}} \frac{\nu \sin(\pi/\nu)}{\pi} \int_0^\infty \lambda^{-1/2} e^{-\frac{x^2}{2\lambda}} \frac{1}{1+\lambda^\nu} d\lambda \\
/y = 1/\lambda/ &= \frac{2}{\sqrt{2\pi}} \frac{\nu \sin(\pi/\nu)}{\pi} \int_0^\infty y^{1/2} e^{-y\frac{x^2}{2}} \frac{1}{y^2(1+y^{-\nu})} dy \\
&= \frac{2}{\sqrt{2\pi}} \frac{\nu \sin(\pi/\nu)}{\pi} \int_0^\infty e^{-y\frac{x^2}{2}} \frac{y^{\nu+1/2-2}}{(1+y^\nu)} dy \\
&= \frac{2}{\sqrt{2\pi}} \frac{\nu \sin(\pi/\nu)}{\pi} \Gamma(\nu - 1/2) x^{-2\nu+1} + O(x^{-2\nu}), \quad x \to \infty,
\end{aligned}$$

we have $p_{0,M}(x) \gtrsim x^{-2\nu+1}$, $x \to \infty$. Furthermore, due to (35), the Parseval identity (6) and the identity $\mathcal{M}[(\cdot)^a\, p(\cdot)](z) = \mathcal{M}[p(\cdot)](z+a)$, we get

$$\begin{aligned}
&\int_0^\infty x^{2\nu-1} \left| p_{0,M}(x) - p_{1,M}(x) \right|^2 dx = \\
&\frac{2^{-4+2\nu}}{\pi} \int_{\gamma - i\infty}^{\gamma + i\infty} \mathcal{M}[q \vee \rho_M] \left(\frac{z+1}{2} \right) \Gamma\left(\frac{z}{2} \right) \mathcal{M}[q \vee \rho_M] \left(\frac{2\nu - z + 1}{2} \right) \Gamma\left(\frac{2\nu - z}{2} \right) dz,
\end{aligned} \tag{36}$$

where $\mathcal{M}[q \vee \rho_M](z) = \mathcal{M}[q](z)\mathcal{M}[\rho_M](z)$. Due to (33)

$$\left| \mathcal{M}[\rho_M](u + \mathrm{i}v) \right| \leq e^{\frac{(u-1)^2}{2}} \frac{\phi(v+M) + \phi(v-M)}{2} \tag{37}$$

with $\phi(v) = e^{-\frac{v^2}{2}}$. Combining (57) (see Appendix), (36) and (37), we derive

$$\begin{aligned}
\chi^2(p_{1,M} | p_{0,M}) &= \int \frac{(p_{1,M}(x) - p_{0,M}(x))^2}{p_{0,M}(x)} dx \\
&\lesssim \int_0^\infty (p_{1,M}(x) - p_{0,M}(x))^2 dx + \int_0^\infty x^{2\nu-1} (p_{1,M}(x) - p_{0,M}(x))^2 dx \\
&\lesssim \int_{-\infty}^\infty |v|^{\nu-1} e^{-|v|\pi/2 - |v|\pi/\nu} \left(\phi(v/2 + M) + \phi(v/2 - M) \right)^2 dv \\
&\lesssim M^{\nu-1} e^{-M\pi(1+2/\nu)}, \quad M \to \infty.
\end{aligned}$$

□

Fix some $\kappa \in (0, 1/2)$. Due to Lemma 6.4, the inequality

$$n\chi^2(p_{1,M} | p_{0,M}) \leq \kappa$$

holds for M large enough, provided

$$M = \frac{1 + \varepsilon_1}{\pi(1 + 2/\nu)} (\log(n) + (\nu - 1) \log\log(n))$$

for arbitrary small $\varepsilon_1 > 0$. Hence Lemma 6.3 and Theorem 6.1 imply

$$\inf_{\hat{p}_n} \sup_{p \in \mathcal{C}(\beta, \gamma_\circ, 1, L)} \mathsf{P}_{p,n}\left(\|\hat{p}_n - p\|_\infty \geq c v_n \right) \geq \delta \tag{38}$$

with some constants $c > 0$ and $\delta > 0$, $\beta = (1-\varepsilon_2)\pi/\nu$ for arbitrary small $\varepsilon_2 \in (0,1)$ and

$$v_n = \exp(-M\pi/\nu) = n^{-\frac{\pi/\nu}{(\pi+2\pi/\nu)}(1+\varepsilon_1)} \log^{-\frac{\pi(\nu-1)/\nu}{\pi(1+2/\nu)}(1+\varepsilon_1)}(n).$$

Hence

$$v_n = n^{-(1+\breve{\varepsilon})\beta/(\pi+2\beta)} \log^{-(1+\breve{\varepsilon})\beta(\nu-1)/(\pi+2\beta)}(n),$$

where $\breve{\varepsilon} = \breve{\varepsilon}(\varepsilon_1, \varepsilon_2) \to 0$ as $\max\{\varepsilon_1, \varepsilon_2\} \to 0$.

6.2.2 Proof of a lower bound for the class $\mathcal{D}(\rho, \gamma_\circ, \gamma^\circ, L)$

Define for any $\nu > 1$ and $M > 0$,

$$q(x) = [2\Gamma(\nu)]^{-1} \times \begin{cases} \log^{\nu-1}(1/x), & 0 \le x \le 1, \\ x^{-2}\log^{\nu-1}(x), & x > 1 \end{cases}$$

and

$$\rho_M(x) = \frac{1}{2\pi} e^{-\frac{\log^2(x)}{2}} \frac{\sin(M\log(x))}{x\log(x)}, \quad x \ge 0.$$

The properties of the functions q and ρ_M can be found in the next lemma.

Lemma 6.5. *The function q is a probability density on $\mathbb{R}_+$ with Mellin transform*

$$\mathcal{M}[q](z) = \frac{1}{2}\left[z^{-\nu} + (2-z)^{-\nu}\right], \quad 0 < \mathsf{Re}[z] < 2.$$

The Mellin transform of the function ρ_M is given by

$$\mathcal{M}[\rho_M](u + \mathrm{i}v) = e^{\frac{(u-1)^2}{2}} \frac{G(u, v+M) - G(u, v-M)}{2}, \tag{39}$$

where $G(u,v) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{v} e^{-\frac{x^2}{2}+\mathrm{i}x(u-1)}dx$, hence

$$\zeta_M := \int_0^\infty \rho_M(x)dx = \mathcal{M}[\rho_M](1) = \frac{1}{\sqrt{2\pi}}\int_{-M}^{M} e^{-\frac{x^2}{2}}dx.$$

Proof. We have

$$\begin{aligned}
\mathcal{M}[q](z) &= [2\Gamma(\nu)]^{-1}\left[\int_0^1 x^{z-1}\log^{\nu-1}(1/x)\,dx + \int_1^\infty x^{z-1}x^{-2}\log^{\nu-1}(x)\,dx\right] \\
&= [2\Gamma(\nu)]^{-1}\left[\int_0^\infty e^{-yz}y^{\nu-1}\,dy + \int_0^\infty e^{y(z-2)}y^{\nu-1}\,dy\right] \\
&= \frac{z^{-\nu} + (2-z)^{-\nu}}{2}.
\end{aligned}$$

Furthermore

$$\begin{aligned}
\mathcal{M}[\rho_M](u+\mathrm{i}v) &= \frac{1}{2\pi}\int_0^\infty e^{-\frac{\log^2(x)}{2}} \frac{\sin(M\log(x))}{\log(x)} x^{u+\mathrm{i}v-1}\,d\log(x) \\
&= \frac{1}{2\pi}\int_{-\infty}^\infty e^{-\frac{y^2}{2}} \frac{\sin(My)}{y} e^{y(u+\mathrm{i}v-1)}\,dy \\
&= \frac{1}{2\pi}\int_{-\infty}^\infty e^{-\frac{y^2}{2}+y(u-1)} \left[\frac{e^{\mathrm{i}y(v+M)} - e^{\mathrm{i}y(v-M)}}{2\mathrm{i}y}\right] dy
\end{aligned}$$

and hence

$$\begin{aligned}\frac{\partial}{\partial v}\mathcal{M}[\rho_M](u+\mathrm{i}v) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{y^2}{2}+y(u-1)}\left[\frac{e^{\mathrm{i}y(v+M)}-e^{\mathrm{i}y(v-M)}}{2}\right]dy \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-(y-u+1)^2/2+(u-1)^2/2}\left[\frac{e^{\mathrm{i}y(v+M)}-e^{\mathrm{i}y(v-M)}}{2}\right]dy \\ &= \frac{e^{(u-1)^2/2}}{\sqrt{2\pi}}\frac{e^{\mathrm{i}(u-1)(v+M)-(v+M)^2/2}+e^{\mathrm{i}(u-1)(v-M)-(v-M)^2/2}}{2}.\end{aligned}$$

□

Set now for any $M>0$ and some $\delta>0$,

$$q_{0,M}(x) := q(x), \quad q_{1,M}(x) := (1-\delta\zeta_M)q(x) + \delta(q\vee\rho_M)(x),$$

where $f\vee g$ is defined in (34).

Lemma 6.6. *For any $M>0$ and some $\delta>0$ not depending on M, the function $q_{1,M}$ is a probability density satisfying*

$$\sup_{x\in(1-\varrho,1+\varrho)}|q_{0,M}(x)-q_{1,M}(x)| = |\cos(\pi\nu/2)|M^{-\nu+1}+O(M^{-\nu}), \quad M\to\infty,$$

where $\varrho>0$ is a fixed number. Moreover, $q_{0,M}$ and $q_{1,M}$ are in $\mathcal{D}(\rho,\gamma_\circ,\gamma^\circ,L)$ for all $\rho<\nu-1$ and $\gamma_\circ,\gamma^\circ\in(0,2)$.

Proof. The nonnegativity of $q_{1,M}$ for some $\delta>0$ can be proved along the same lines as in the proof of Lemma 6.3. Next

$$\int_0^\infty q_{1,M}(x)dx = 1+\delta\int_0^\infty(q\vee\rho_M)(x)-\delta\zeta_M = 1+\delta\mathcal{M}[\rho_M](1)\times\mathcal{M}[q](1)-\delta\zeta_M = 1.$$

So $q_{1,M}$ is indeed a probability density. Furthermore, $(q\vee\rho_M)(y) = \frac{1}{2\pi}\left[2\Gamma(\nu)\right]^{-1}\left[I_1(y)+I_2(y)\right]$ with

$$\begin{aligned}I_1(y) &= \int_y^\infty e^{-\frac{\log^2(x)}{2}}x^{-2}\frac{\sin(M\log(x))}{\log(x)}\log^{\nu-1}(x/y)dx \\ &= \int_{\log(y)}^\infty e^{-\frac{z^2}{2}-z}\frac{\sin(Mz)}{z}(z-\log(y))^{\nu-1}dz\end{aligned}$$

and

$$\begin{aligned}I_2(y) &= \int_0^y e^{-\frac{\log^2(x)}{2}}y^{-2}\frac{\sin(M\log(x))}{\log(x)}\log^{\nu-1}(y/x)dx \\ &= \int_{-\infty}^{\log(y)} e^{-\frac{z^2}{2}+z}y^{-2}\frac{\sin(Mz)}{z}(\log(y)-z)^{\nu-1}dz.\end{aligned}$$

By taking $y = \exp(A)$, we get for $I_1(y)$

$$\begin{aligned} I_1(y) &= \int_0^\infty e^{-\frac{(z+A)^2}{2}-(z+A)} \frac{\sin(M(z+A))}{z+A} z^{\nu-1} dz \\ &= \cos(AM) \int_0^\infty \frac{e^{-\frac{(z+A)^2}{2}-(z+A)}}{z+A} \sin(Mz) z^{\nu-1} dz \\ &\quad + \sin(AM) \int_0^\infty \frac{e^{-\frac{(z+A)^2}{2}-(z+A)}}{z+A} \cos(Mz) z^{\nu-1} dz. \end{aligned}$$

The well known Erdélyi lemma (see Appendix) implies

$$\int_0^\infty \frac{e^{-\frac{(z+A)^2}{2}-(z+A)}}{z+A} \sin(Mz) z^{\nu-1} dz = \frac{e^{-\frac{A^2}{2}-A}}{A} \Gamma(\nu) \sin(\pi\nu/2) M^{-\nu} + O(M^{-1-\nu}), \quad M \to \infty$$

and

$$\int_0^\infty \frac{e^{-\frac{(z+A)^2}{2}-(z+A)}}{z+A} \cos(Mz) z^{\nu-1} dz = \frac{e^{-\frac{A^2}{2}-A}}{A} \Gamma(\nu) \cos(\pi\nu/2) M^{-\nu} + O(M^{-1-\nu}), \quad M \to \infty,$$

since the function $z \mapsto \frac{e^{-\frac{(z+A)^2}{2\alpha}-(z+A)}}{z+A}$ is infinitely smooth with all derivatives vanishing at infinity. Hence

$$I_1(e^A) = \frac{e^{-\frac{A^2}{2}-A}}{A} \Gamma(\nu) \sin(AM + \pi\nu/2) M^{-\nu} + O(M^{-1-\nu}), \quad M \to \infty. \tag{40}$$

Analogously

$$\begin{aligned} I_2(e^A) &= e^{-2A} \int_{-\infty}^A e^{-\frac{z^2}{2}+z} \frac{\sin(Mz)}{z} (A-z)^{\nu-1} dz \\ &= e^{-2A} \int_0^\infty e^{-\frac{(A-z)^2}{2}+A-z} \frac{\sin(M(A-z))}{A-z} z^{\nu-1} dz \\ &= e^{-2A} \sin(AM) \int_0^\infty e^{-\frac{(A-z)^2}{2}+A-z} \frac{\cos(Mz)}{A-z} z^{\nu-1} dz \\ &\quad - e^{-2A} \cos(AM) \int_0^\infty e^{-\frac{(A-z)^2}{2}+A-z} \frac{\sin(Mz)}{A-z} z^{\nu-1} dz \\ &= \frac{e^{-\frac{A^2}{2}-A}}{A} \Gamma(\nu) \sin(AM - \pi\nu/2) M^{-\nu} + O(M^{-1-\nu}). \end{aligned}$$

Combining the previous estimates, we arrive at

$$I_2(e^A) + I_1(e^A) = 2\frac{e^{-\frac{A^2}{2}-A}}{A} \Gamma(\nu) \sin(AM) \cos(\pi\nu/2) M^{-\nu} + O(M^{-1-\nu}).$$

It remains to note that the maximum of the main term in (40) is attained for $A \in \{\pi/2M, 3\pi/2M\}$ and

$$\sup_{A \in \{\pi/2M, 3\pi/2M\}} [I_2(e^A) + I_1(e^A)] = \Gamma(\nu) |\cos(\pi\nu/2)| M^{-\nu+1} + O(M^{-\nu}).$$

The property $q_{1,M} \in \mathcal{D}(\rho, \gamma_\circ, \gamma^\circ, L)$ for all $\rho < \nu - 1$ and $\gamma_\circ, \gamma^\circ \in (0, 2)$ with L depending on $\gamma_\circ$ and γ° follows from the identity $\mathcal{M}[q_{1,M}](z) = \mathcal{M}[q](z)(1 - \delta\zeta_M) + \delta\mathcal{M}[\rho_M](z)\mathcal{M}[q](z)$ and (39). $\square$

Let $T_{0,M}$ and $T_{1,M}$ be two random variables with densities $q_{0,M}$ and $q_{1,M}$ respectively. The density of the r.v. $|W_{T_{i,M}}|$, $i = 0, 1$, is given by

$$p_{i,M}(x) := \frac{2}{\sqrt{2\pi}} \int_0^\infty \lambda^{-1/2} e^{-\frac{x^2}{2\lambda}} q_{i,M}(\lambda) d\lambda, \quad i = 0, 1.$$

For the Mellin transform of $p_{i,M}$, we have

$$\begin{aligned} \mathcal{M}[p_{i,M}](z) &= \mathbb{E}\big[|W_1|^{z-1}\big]\mathbb{E}\big[T_{i,M}^{(z-1)/2}\big] \\ &= \mathbb{E}\big[|W_1|^{z-1}\big]\mathcal{M}[q_{i,M}]((z+1)/2) \\ &= \frac{2^{z/2}}{\sqrt{2\pi}}\Gamma(z/2)\mathcal{M}[q_{i,M}]((z+1)/2). \end{aligned} \tag{41}$$

Lemma 6.7. *The χ^2-distance between the densities $p_{0,M}$ and $p_{1,M}$ satisfies*

$$\chi^2(p_{1,M}|p_{0,M}) := \int \frac{(p_{1,M}(x) - p_{0,M}(x))^2}{p_{0,M}(x)} dx \lesssim e^{-M\pi/2}, \quad M \to \infty.$$

Proof. First note that $p_{0,M}(x) > 0$ on $[0, \infty)$. Since

$$\begin{aligned} \int_0^1 \lambda^{-1/2} e^{-\frac{x^2}{2\lambda}} \log^{\nu-1}(1/\lambda) d\lambda &= \int_0^1 \lambda^{-1/2} e^{-\frac{x^2}{2\lambda}} \log^{\nu-1}(1/\lambda) d\lambda \\ /y = 1/\lambda, \lambda = 1/y/ \quad &= \int_1^\infty y^{-3/2} e^{-x^2 y/2} \log^{\nu-1}(y) dy \\ &= \int_{x^2}^\infty x^{-2} (y/x^2)^{-3/2} e^{-y/2} \log^{\nu-1}(y/x^2) dy \\ &= x \int_{x^2}^\infty y^{-3/2} e^{-y/2} \log^{\nu-1}(y/x^2) dy \lesssim e^{-x^2/2} \end{aligned}$$

and

$$\begin{aligned} \int_1^\infty \lambda^{-3/2} e^{-\frac{x^2}{2\lambda}} \log^{\nu-1}(\lambda) d\lambda &= \int_0^1 y^{-1/2} e^{-\frac{x^2}{2} y} \log^{\nu-1}(1/y) dy \\ &= \frac{\Gamma(1/2)}{\sqrt{2}} x^{-1} \log^{\nu-1}(x^2) + O(1/x), \end{aligned}$$

we have $p_{0,M}(x) \gtrsim x^{-1}$, $x \to \infty$. Furthermore, due to (41) and the Parseval identity (6),

$$\int_0^\infty x^{a-1} \left|p_{0,M}(x) - p_{1,M}(x)\right|^2 dx = \tag{42}$$

$$\frac{2^{-4+a}}{\pi} \int_{\gamma - \mathrm{i}\infty}^{\gamma + \mathrm{i}\infty} \mathcal{M}[q \vee \rho_M]\left(\frac{z+1}{2}\right) \Gamma\left(\frac{z}{2}\right) \mathcal{M}[q \vee \rho_M]\left(\frac{a-z+1}{2}\right) \Gamma\left(\frac{a-z}{2}\right) dz,$$

where $\mathcal{M}[q \vee \rho_M](z) = \mathcal{M}[q](z)\mathcal{M}[\rho_M](z)$. Due to (39),

$$|\mathcal{M}[\rho_M](u + \mathrm{i}v)| \leq e^{\frac{(u-1)^2}{2}} \frac{\Phi(v + M) + \Phi(v - M)}{2} \tag{43}$$

with $\Phi(v) = \int_{-\infty}^{v} e^{-\frac{x^2}{2}} dx$. Combining (42) with properly chosen $\gamma > 0$, (43) and Lemma 7.3 (see Appendix), we derive

$$\begin{aligned} \chi^2(p_1|p_0) &= \int \frac{(p_1(x) - p_0(x))^2}{p_0(x)} dx \lesssim \int_0^{\infty} (p_1(x) - p_0(x))^2 dx + \int_0^{\infty} x(p_1(x) - p_0(x))^2 dx \\ &\lesssim \int_{-\infty}^{\infty} e^{-|v|\pi/2} \left(\Phi(v/2 + M) + \Phi(v/2 - M)\right)^2 dv \lesssim e^{-M\pi/2}, \quad M \to \infty. \end{aligned}$$

□

Fix some $\kappa \in (0, 1/2)$. Due to Lemma 6.7, the inequality

$$n\chi^2(p_{1,M}|p_{0,M}) \leq \kappa$$

holds for M large enough, provided

$$M = \frac{2(1 + \varepsilon_1)}{\pi} \log(n)$$

for an arbitrary small $\varepsilon_1 > 0$. Fix $\rho = (1 - \varepsilon_2)(\nu - 1)$ for some $\varepsilon_2 \in (0, 1)$. Then Lemma 6.6 and Theorem 6.1 imply

$$\inf_{\hat{p}_n} \sup_{p \in \mathcal{D}(\rho, \gamma_\circ, \gamma^\circ, L)} \mathsf{P}_{p,n}\left(\|\hat{p}_n - p\|_\infty \geq c v_n\right) \geq \delta$$

for any $\gamma_\circ, \gamma^\circ \in (0, 2)$, some constants $c > 0$, $\delta > 0$ and $v_n = \log^{-(\nu-1)}(n) = \log^{-\rho/(1-\varepsilon_2)}(n)$. Since ε_2 can be taken to be arbitrary close to 0, we get the desired statement.

6.3 Proof of Theorem 4.3

For simplicity we assume that $\mathsf{Im}(\psi(\lambda)) \geq 0$ for all $\lambda \geq 0$ (cf. (14)). The proof in the more general case of Remark 3.3 can be done in a similar way. Denote $X_k := Y_{t_k} - Y_{t_{k-1}}$, $k = 1, \ldots, n$, and fix some $\gamma \in [\gamma_\circ, \gamma^\circ]$. By (13), we derive for the bias of $p_{n,\gamma}(x)$, $x > 0$,

$$\begin{aligned} |\mathbb{E}[p_{n,\gamma}(x)] - p_\Delta(x)| &= \left| \frac{1}{2\pi} \int_{-U_n}^{U_n} \frac{\mathrm{E}\left[\Phi_n(1 - \gamma - \mathrm{i}v, X_1)\right]}{\Gamma(1 - \gamma - \mathrm{i}v)} x^{-\gamma - \mathrm{i}v} dv - \int_{-\infty}^{\infty} \mathcal{M}[p_\Delta](\gamma + \mathrm{i}v) x^{-\gamma - \mathrm{i}v} dv \right| \\ &\leq \left| \frac{1}{2\pi} \int_0^{U_n} \frac{\int_{A_n}^{\infty} \left[\overline{\psi(\lambda)}\right]^{-\gamma - \mathrm{i}v} \mathcal{F}[p_X](-\lambda) \overline{\psi'(\lambda)} d\lambda}{\Gamma(1 - \gamma - \mathrm{i}v)} x^{-\gamma - \mathrm{i}v} dv \right| \\ &\quad + \left| \frac{1}{2\pi} \int_{-U_n}^{0} \frac{\int_{A_n}^{\infty} [\psi(\lambda)]^{-\gamma - \mathrm{i}v} \mathcal{F}[p_X](\lambda) \psi'(\lambda) d\lambda}{\Gamma(1 - \gamma - \mathrm{i}v)} x^{-\gamma - \mathrm{i}v} dv \right| \\ &\quad + \frac{|x|^{-\gamma}}{2\pi} \int_{\{|v| > U_n\}} |\mathcal{M}[p_\Delta](\gamma + \mathrm{i}v)| \, dv \\ &=: (*)_{1a} + (*)_{1b} + (*)_2. \end{aligned}$$

Since $p_\Delta \in \mathcal{C}(\beta, \gamma_\circ, \gamma^\circ, L)$, we have for the third term

$$(*)_2 \le \frac{|x|^{-\gamma}}{2\pi} e^{-\beta U_n} \int_{\{|v|>U_n\}} |\mathcal{M}[p_\Delta](\gamma + \mathrm{i}v)|\, e^{\beta|v|} dv \le e^{-\beta U_n} \frac{|x|^{-\gamma} L}{2\pi}.$$

Furthermore, by Lemma 7.2

$$|\mathcal{F}[p_X](\lambda)| = |\mathcal{L}[p_\Delta](\psi(\lambda))| \lesssim \frac{1}{\mathsf{Re}(\psi(\lambda))} \lesssim \frac{1}{\lambda^2}$$

and by Lemma 7.3

$$(*)_{1a} \lesssim \frac{|x|^{-\gamma}}{2\pi} \left| \int_0^{U_n} \frac{\int_{A_n}^{\infty} \left[\overline{\psi(\lambda)}\right]^{-\gamma - \mathrm{i}v} \mathcal{F}[p_X](-\lambda) \overline{\psi'(\lambda)} d\lambda}{\Gamma(1-\gamma-\mathrm{i}v)} dv \right|$$

$$\lesssim |x|^{-\gamma} \int_0^{U_n} \frac{\int_{A_n}^{\infty} \lambda^{-2\gamma-1} d\lambda}{|\Gamma(1-\gamma-\mathrm{i}v)|} dv \lesssim \frac{|x|^{-\gamma}}{2\gamma} U_n^{\gamma-1/2} e^{U_n \pi/2} A_n^{-2\gamma}.$$

Similarly we get the same estimate for $(*)_{1b}$. For the variance we have

$$\mathrm{Var}(p_{n,\gamma}(x)) = \frac{1}{(2\pi)^2 n} \mathrm{Var}\left[\int_{-U_n}^{U_n} \frac{\Phi_n(1-\gamma-\mathrm{i}v, X_1)}{\Gamma(1-\gamma-\mathrm{i}v)} x^{-\gamma-\mathrm{i}v} dv\right] + \mathcal{R}_n(x)$$

$$\le \frac{1}{(2\pi)^2 n} |x|^{-2\gamma} \left[\int_{-U_n}^{U_n} \frac{\sqrt{\mathrm{Var}[\Phi_n(1-\gamma-\mathrm{i}v, X_1)]}}{|\Gamma(1-\gamma-\mathrm{i}v)|} dv\right]^2 + \mathcal{R}_n(x), \tag{44}$$

where

$$\mathcal{R}_n(x) = \frac{1}{n^2(2\pi)^2} \sum_{1\le|j-k|\le n-1} \int_{-U_n}^{U_n}\int_{-U_n}^{U_n} \frac{\mathrm{Cov}\left[\Phi_n(1-\gamma-\mathrm{i}v, X_k), \Phi_n(1-\gamma-\mathrm{i}u, X_j)\right]}{\Gamma(1-\gamma-\mathrm{i}v)\overline{\Gamma(1-\gamma-\mathrm{i}u)}} x^{-2\gamma+\mathrm{i}(u-v)}\, du\, dv.$$

For example let us take $u \ge 0$ and $v \ge 0$, then

$$\mathrm{Cov}\left[\Phi_n(1-\gamma-\mathrm{i}v, X_k), \Phi_n(1-\gamma-\mathrm{i}u, X_j)\right] = \int_0^{A_n}\int_0^{A_n} \left[\overline{\psi(\lambda_1)}\right]^{-\gamma-\mathrm{i}v} [\psi(\lambda_2)]^{-\gamma+\mathrm{i}u}$$
$$\overline{\psi'(\lambda_1)}\psi'(\lambda_2)\, \mathrm{Cov}\left[e^{-\mathrm{i}\lambda_1 X_k}, e^{-\mathrm{i}\lambda_2 X_j}\right] d\lambda_1 d\lambda_2.$$

Due to Lemma 7.1, the sequence $X_1, \ldots, X_n$ is α-mixing with mixing coefficients satisfying (19). The Billingsley's inequality (see, e.g., [7]) implies $\left|\mathrm{Cov}\left[e^{-\mathrm{i}\lambda_1 X_k}, e^{-\mathrm{i}\lambda_2 X_j}\right]\right| \le 4\,\alpha(|j-k|)$ and consequently

$$\left|\mathrm{Cov}\left[\Phi_n(1-\gamma-\mathrm{i}v, X_k), \Phi_n(1-\gamma-\mathrm{i}u, X_j)\right]\right| \le 4\alpha(|j-k|) \left[\int_0^{A_n} |\psi(\lambda)|^{-\gamma} \left|\psi'(\lambda)\right| d\lambda\right]^2. \tag{45}$$

By respecting definition (16) for different signs of u and v, we obtain the estimate (45) for all u and v. Hence

(46)

$$\left|\mathcal{R}_n(x)\right| \le \frac{4|x|^{-2\gamma}}{(2\pi)^2 n^2} \sum_{1\le|j-k|\le n-1} \alpha(|j-k|) \left[\int_0^{A_n} |\psi(\lambda)|^{-\gamma} \left|\psi'(\lambda)\right| d\lambda\right]^2 \left[\int_{-U_n}^{U_n} \frac{1}{|\Gamma(1-\gamma-\mathrm{i}v)|} dv\right]^2$$

$$\le \frac{2|x|^{-2\gamma}}{\pi^2 n} \left[\int_0^{A_n} |\psi(\lambda)|^{-\gamma} \left|\psi'(\lambda)\right| d\lambda\right]^2 \left[\int_{-U_n}^{U_n} \frac{1}{|\Gamma(1-\gamma-\mathrm{i}v)|} dv\right]^2 \sum_{k=0}^{\infty} \alpha(k),$$

where $\sum_{k=0}^{\infty} \alpha(k) < \infty$ due to (19). On the other hand, e.g. for $v \geq 0$ we have

$$\sqrt{\mathrm{Var}[\Phi_n(1-\gamma-\mathrm{i}v, X_1)]} \leq \int_0^{A_n} \sqrt{\mathrm{Var}[[\overline{\psi(\lambda)}]^{-\gamma-\mathrm{i}v} e^{-\mathrm{i}X_1\lambda} \overline{\psi'(\lambda)}]}\, d\lambda \tag{47}$$

$$\leq \int_0^{A_n} |\psi(\lambda)|^{-\gamma} \sqrt{\left(1 - |\mathcal{F}[p_{X_1}](\lambda)|^2\right)} \left|\psi'(\lambda)\right| d\lambda$$

and the same estimate applies for $v \leq 0$. Hence combining (44), (46), and (47), we get

$$\mathrm{Var}(p_{n,\gamma}(x)) \lesssim \frac{|x|^{-2\gamma}}{n} \left[\int_0^{A_n} |\psi(\lambda)|^{-\gamma} \left|\psi'(\lambda)\right| d\lambda\right]^2 \left[\int_{-U_n}^{U_n} \frac{1}{|\Gamma(1-\gamma-\mathrm{i}v)|} dv\right]^2. \tag{48}$$

Due to Lemma 7.2, we have

$$\int_1^{A_n} |\psi(\lambda)|^{-\gamma} \left|\psi'(\lambda)\right| d\lambda \lesssim \int_1^{A_n} \lambda^{(1-2\gamma)} d\lambda \leq C_0 \frac{A_n^{2(1-\gamma)}}{1-\gamma}$$

for $\gamma < 1$ and

$$\int_1^{A_n} |\psi(\lambda)|^{-\gamma} \left|\psi'(\lambda)\right| d\lambda \lesssim \int_1^{A_n} \lambda^{(1-2\gamma)} d\lambda \leq C_0 \log(A_n)$$

for $\gamma = 1$. Furthermore, it holds due to $|\mathcal{F}[p_{X_1}](\lambda)| \geq 1 - \mathrm{Var}(X_1)\lambda^2/2$,

$$\int_0^1 |\psi(\lambda)|^{-\gamma} \sqrt{\left(1 - |\mathcal{F}[p_{X_1}](\lambda)|^2\right)} \left|\psi'(\lambda)\right| d\lambda \leq \int_0^1 |\psi(\lambda)|^{-\gamma+1} \left|\psi'(\lambda)\right| d\lambda \leq \frac{C_1}{2-\gamma}$$

for some constants $C_0, C_1 > 0$. Hence from (48) we get by (57),

$$|x|^{2\gamma} \mathrm{Var}(p_{n,\gamma}(x)) \lesssim \frac{1}{n} \left(U_n^{\gamma-1/2} e^{U_n\pi/2} A_n^{2(1-\gamma)}\right)^2 =: (*)_3,$$

and by gathering $(*)_{1a}$, $(*)_{1b}$, $(*)_2$, and $(*)_3$,

$$\sqrt{\mathrm{E}\left[x^{2\gamma} \left|p_{n,\gamma}(x) - p_\Delta(x)\right|^2\right]} \lesssim \frac{1}{\sqrt{n}} U_n^{\gamma-1/2} e^{U_n\pi/2} A_n^{2(1-\gamma)} + U_n^{\gamma-1/2} e^{U_n\pi/2} A_n^{-2\gamma} + e^{-\beta U_n}.$$

Next, the choice (21) leads to the desired result.

6.4 Proof of Theorem 4.5

This proof is similar to the above one. Only $(*)_2$ in the bias term is different and now becomes

$$(*)_2 \leq (1 + |U_n|^\rho)^{-1} \frac{|x|^{-\gamma}}{2\pi} \int_{\{|v|>U_n\}} |\mathcal{M}[p_\Delta](\gamma + \imath v)| \, (1 + |v|^\rho) dv \leq (1 + |U_n|^\rho)^{-1} \frac{|x|^{-\gamma} L}{2\pi},$$

whence

$$\sqrt{\mathrm{E}\left[x^{2\gamma} \left|p_{n,\gamma}(x) - p_\Delta(x)\right|^2\right]} \lesssim \frac{1}{\sqrt{n}} U_n^{\gamma-1/2} e^{U_n\pi/2} A_n^{2(1-\gamma)} + U_n^{\gamma-1/2} e^{U_n\pi/2} A_n^{-2\gamma} + (1 + |U_n|^\rho)^{-1},$$

and the choice (24) leads to the desired result.

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7 Appendix

7.1 Some results on time-changed Lévy processes

Lemma 7.1. *Let L_t be a Lévy process with the Lévy measure ν and let $\mathcal{T}(t)$ be a time change independent of L_t. Fix some $\Delta > 0$ and consider two sequences $T_k = \mathcal{T}(\Delta k) - \mathcal{T}(\Delta(k-1))$ and $X_k = Y_{\Delta k} - Y_{\Delta(k-1)}$, $k = 1, 2, \ldots$ where $Y_t = L_{\mathcal{T}(t)}$. If the sequence $(T_k)_{k\in\mathbb{N}}$ is strictly stationary and α-mixing with mixing coefficients $(\alpha_T(j))_{j\in\mathbb{N}}$, then the sequence $(X_k)_{k\in\mathbb{N}}$ is also strictly stationary and α-mixing with mixing coefficients $(\alpha_X(j))_{j\in\mathbb{N}}$, satisfying*

$$\alpha_X(j) \leq \alpha_T(j), \quad j \in \mathbb{N}. \tag{49}$$

Proof. Fix two functions ϕ and ψ mapping $\mathbb{R}$ to $\mathbb{R}$ such that $\mathbb{E}[\phi(X_k)]^2 < \infty$ and $\mathbb{E}[\psi(X_k)]^2 < \infty$. Using the independence of increments of the Lévy process L_t and the fact that $\mathcal{T}$ is a non-decreasing process, we get

$$\mathbb{E}[\phi(X_k)] = \mathbb{E}[\widetilde{\phi}(T_k)], \quad \mathbb{E}[\phi(X_k)\psi(X_{k+l})] = \mathbb{E}[\widetilde{\phi}(T_k)\widetilde{\psi}(T_{k+l})], \quad k, l \in \mathbb{N},$$

where $\widetilde{\phi}(t) = \mathbb{E}[\phi(L_t)]$ and $\widetilde{\psi}(t) = \mathbb{E}[\psi(L_t)]$. Since T_k is strictly stationary, $\mathbb{E}[\widetilde{\phi}(T_k)]$ is independent of k and $\mathbb{E}[\widetilde{\phi}(T_k)\widetilde{\psi}(T_{k+l})]$ depends on l only. Furthermore, for any $i, j \in \mathbb{N}$ and any two functions $\phi : \mathbb{R}^i \to [0,1]$ and $\psi : \mathbb{R}^j \to [0,1]$, we have

$$\mathbb{E}[\phi(X_1, \ldots, X_i)\psi(X_{i+k+1}, \ldots, X_{i+k+j})] = \mathbb{E}[\widetilde{\phi}(T_1, \ldots, T_i)\widetilde{\psi}(T_{i+k+1}, \ldots, T_{i+k+j})],$$

where $\widetilde{\phi}(t_1, \ldots, t_i) = \mathbb{E}[\phi(L_{t_1}, \ldots, L_{t_i})]$ and $\widetilde{\psi}(t_1, \ldots, t_j) = \mathbb{E}[\psi(L_{t_1}, \ldots, L_{t_j})]$. By noting that

$$\alpha_X(k) = \sup\left\{\left|\mathrm{Cov}\left(\phi(X_1, \ldots, X_i), \psi(X_{i+k+1}, \ldots, X_{i+k+j})\right)\right|; \, \phi : \mathbb{R}^i \to [0,1], \, \psi : \mathbb{R}^j \to [0,1]\right\}$$

and that $\widetilde{\phi}, \widetilde{\psi} \in [0,1]$, we get (49). □

Lemma 7.2. *Let $(L_t, \, t \geq 0)$ be a Lévy process with the triplet (μ, σ^2, ν). Suppose that $\int_{\{|x|>1\}} |x|\nu(dx) < \infty$, and that σ and ν are not both zero. It then holds for $\psi(u) = -\log(\mathbb{E}(\exp(iuL_t)))$*

$$(i): \ |\psi(u)| \lesssim u^2 \quad \text{and} \quad (ii): \ \left|\psi'(u)\right| \lesssim u, \quad u \to \infty. \tag{50}$$

Further, if

$$d = \mu + \int_{\{|x|>1\}} x\nu(dx) \neq 0 \tag{51}$$

we have

$$(i): \ |\psi(u)| \gtrsim u \quad \text{and} \quad (ii): \ \left|\psi'(u)\right| \lesssim 1, \quad u \downarrow 0. \tag{52}$$

If $d = 0$ we have in the case $\nu(\{|x| > 1\} \cap dx) \equiv 0$,

$$(53) \qquad (i): \ |\psi(u)| \gtrsim u^2, \quad \textit{and} \quad (ii): \ \left|\psi'(u)\right| \lesssim u, \quad u \downarrow 0,$$

and in the case $\nu(\{|x| > 1\} \cap dx) \neq 0$,

$$(54) \qquad (i): \ |\psi(u)| \gtrsim u, \quad \textit{and} \quad (ii): \ \left|\psi'(u)\right| = o(1), \quad u \downarrow 0.$$

Proof. In general we have

$$(55) \qquad \psi(u) = -\mathrm{i}u\mu + \frac{u^2\sigma^2}{2} + \int_{\mathbb{R}} (1 - e^{\mathrm{i}ux} + \mathrm{i}ux1_{|x|\le 1})\nu(dx),$$

where

$$(56) \qquad \int_{\mathbb{R}} (1 - e^{\mathrm{i}ux} + \mathrm{i}ux1_{|x|\le 1})\nu(dx) = u^2 \int_{\{|x|\le 1\}} \frac{1 - e^{\mathrm{i}ux} + \mathrm{i}ux}{(ux)^2} x^2\nu(dx) + \int_{\{|x|>1\}} \left(1 - e^{\mathrm{i}ux}\right)\nu(dx).$$

Note that

$$0 < \frac{\left|1 - e^{\mathrm{i}y} + \mathrm{i}y\right|}{y^2} < c \ \text{ for } \ y \in \mathbb{R},$$

with $c > 0$, and that

$$\int_{\{|x|>1\}} \left(1 - e^{\mathrm{i}ux}\right) x\nu(dx) \longrightarrow \int_{\{|x|>1\}} x\nu(dx) \quad \text{for } u \to \infty$$

by Riemann-Lebesgue. This yields (50)-(i). It is not difficult to show by standard arguments that due to the integrability condition we have

$$\psi'(u) = -\mathrm{i}\mu + u\sigma^2 - \mathrm{i}\int_{\mathbb{R}} (e^{\mathrm{i}ux} - 1_{|x|\le 1})x\nu(dx).$$

Next, (50)-(ii) follows by observing that

$$\int_{\{|x|\le 1\}} (e^{\mathrm{i}ux} - 1)x\nu(dx) = u\int_{\{|x|\le 1\}} \frac{e^{\mathrm{i}ux} - 1}{ux} x^2\nu(dx),$$

where $\left(e^{\mathrm{i}y} - 1\right)/y$ is bounded for $y \in \mathbb{R}$. Suppose $d \neq 0$. By (51), $\psi'(0) = -\mathrm{i}d \neq 0$, and since $\psi(0) = 0$ we have (52)-(i), and (52)-(ii) is obvious. Next suppose $d = 0$, i.e. $\psi'(0) = 0$. We then have,

$$\begin{aligned}
\psi(u) &= \psi(u) - u\psi'(0) = \psi(u) + \mathrm{i}ud \\
&= \frac{u^2\sigma^2}{2} + \int_{\mathbb{R}} (1 - e^{\mathrm{i}ux} + \mathrm{i}ux1_{|x|\le 1})\nu(dx) + \mathrm{i}u\int_{\{|x|>1\}} x\nu(dx) \\
&= \frac{u^2\sigma^2}{2} + \int_{\{|x|\le 1\}} (1 - e^{\mathrm{i}ux} + \mathrm{i}ux)\nu(dx) \\
&+ \int_{\{|x|>1\}} (1 - e^{\mathrm{i}ux})\nu(dx) + \mathrm{i}u\int_{\{|x|>1\}} x\nu(dx)
\end{aligned}$$

and

$$\begin{aligned}\psi'(u) &= \psi'(u) - \psi'(0)\\ &= u\sigma^2 - \mathrm{i}\int_{\mathbb{R}}(e^{\mathrm{i}ux} - 1_{|x|\le 1})x\nu(dx) + \mathrm{i}\int_{\{|x|>1\}} x\nu(dx).\end{aligned}$$

If $\nu(\{|x|>1\}\cap dx) \equiv 0$ we thus have

$$\begin{aligned}\psi(u) &= \frac{u^2\sigma^2}{2} + \int_{\{|x|\le 1\}}(1 - e^{\mathrm{i}ux} + \mathrm{i}ux)\nu(dx)\\ &= \frac{u^2\sigma^2}{2} + u^2\int_{\{|x|\le 1\}}\frac{1 - e^{\mathrm{i}ux} + \mathrm{i}ux}{(ux)^2}x^2\nu(dx)\end{aligned}$$

and we observe that

$$\operatorname{Re}\left(1 - e^{\mathrm{i}ux} + \mathrm{i}ux\right) = 1 - \cos(ux) \ge 0.$$

So in particular $\operatorname{Re}\psi(u) \gtrsim u^2$ while $|\psi(u)| \lesssim u^2$. Hence (53)-(i) is shown. Then, since

$$\begin{aligned}\psi'(u) &= u\sigma^2 - \mathrm{i}\int_{\{|x|\le 1\}}(e^{\mathrm{i}ux} - 1)x\nu(dx)\\ &= u\sigma^2 - \mathrm{i}u\int_{\{|x|\le 1\}}\frac{e^{\mathrm{i}ux} - 1}{ux}x^2\nu(dx)\end{aligned}$$

and again $\left(e^{\mathrm{i}y} - 1\right)/y$ is bounded, we have (53)-(ii). Finally, if $d = 0$ and $\nu(\{|x|>1\}\cap dx) \ne 0$, let us write

$$\begin{aligned}\psi(u) &= \frac{u^2\sigma^2}{2} + u^2\int_{\{|x|\le 1\}}\frac{1 - e^{\mathrm{i}ux} + \mathrm{i}ux}{(ux)^2}x^2\nu(dx)\\ &+ \int_{\{|x|>1\}}(1 - \cos(ux))\nu(dx) + \mathrm{i}\int_{\{|x|>1\}}(ux - \sin(ux))\,\nu(dx),\end{aligned}$$

where

$$0 \le \int_{\{|x|>1\}}(ux - \sin(ux))\,\nu(dx) \le u\int_{\{|x|>1\}} x\nu(dx) \lesssim u,$$

but due to dominated convergence also

$$\int_{\{|x|>1\}}(ux - \sin(ux))\,\nu(dx) = u\int_{\{|x|>1\}} x\nu(dx) + o(1).$$

Hence,

$$\int_{\{|x|>1\}}(ux - \sin(ux))\,\nu(dx) = u + o(u), \quad u \downarrow 0,$$

and from this (54)-(i). For the derivative we have,

$$\begin{aligned}\psi'(u) &= u\sigma^2 - \mathrm{i}\int_{\mathbb{R}}(e^{\mathrm{i}ux} - 1_{|x|\le 1})x\nu(dx) + \mathrm{i}\int_{\{|x|>1\}} x\nu(dx)\\ &= u\sigma^2 - \mathrm{i}u\int_{\{|x|\le 1\}}\frac{e^{\mathrm{i}ux} - 1}{ux}x^2\nu(dx) - \mathrm{i}\int_{\{|x|>1\}}\left(e^{\mathrm{i}ux} - 1\right)x\nu(dx)\\ &= o(1), \quad u \downarrow 0,\end{aligned}$$

by similar arguments, i.e. (54)-(ii). □

Lemma 7.3. *For any $\alpha \geq -2$, there exist positive constants C_1 and $C_2(\alpha)$ such that uniformly for $|\beta| \geq 2$,*

$$C|\beta|^{\alpha-1/2}e^{-|\beta|\pi/2} \leq |\Gamma(\alpha + \mathrm{i}\beta)| \leq C_\alpha|\beta|^{\alpha-1/2}e^{-|\beta|\pi/2}. \tag{57}$$

Proof. See, for example, Theorem 1.4.2 in [1]. □

Corollary 7.4. *For all $0 < \alpha < 1/2$ and all $U > 2$, it holds*

$$\int_{-U}^{U} \frac{d\beta}{|\Gamma(\alpha + \mathrm{i}\beta)|} \leq CU^{1/2-\alpha}e^{U\pi/2} \tag{58}$$

for a constant $C > 0$. For $\alpha > 1/2$, we have

$$\int_{-U}^{U} \frac{d\beta}{|\Gamma(\alpha + \mathrm{i}\beta)|} \leq C_1(\alpha) + C_2e^{U\pi/2}, \tag{59}$$

where C_2 does not depend on α.

7.2 Lemma of Erdélyi

Lemma 7.5 ([12],[13])**.** *Let $\alpha \geq 1$, $\nu > 0$, function $f \in C^\infty([0,\infty))$ be such that f and all its derivatives vanish at infinity, then*

$$\int_0^\infty x^{\nu-1}f(x)e^{\mathrm{i}\lambda x^\alpha}\,dx = \sum_{k=0}^{\infty} a_k\lambda^{-\frac{k+\nu}{\alpha}}, \quad \lambda \to \infty,$$

where $a_k = \frac{f^{(k)}(0)}{\alpha\, k!}\Gamma\left(\frac{k+\nu}{\alpha}\right)\exp\left[\frac{\mathrm{i}\pi(k+\nu)}{2\alpha}\right]$.

References

[1] George E Andrews, Richard Askey, and Ranjan Roy. Special Functions, volume 71 of Encyclopedia of Mathematics and its Applications, 1999.

[2] Frank Bauer and Markus Reiß. Regularization independent of the noise level: an analysis of quasi-optimality. *Inverse Problems*, 24(5):055009, 2008.

[3] Denis Belomestny. Statistical inference for time-changed Lévy processes via composite characteristic function estimation. *The Annals of Statistics*, 39(4):2205–2242, 2011.

[4] Denis Belomestny, Fabienne Comte, Valentine Genon-Catalot, Hiroki Masuda, and Markus Reiß. *Lévy Matters IV, Estimation for Discretely Observed Lévy Processes.* Springer, 2014.

[5] Denis Belomestny and John Schoenmakers. Statistical skorohod embedding problem: Optimality and asymptotic normality. *Statistics & Probability Letters*, 104(9):169–180, 2015.

[6] Norman Bleistein and Richard A Handelsman. *Asymptotic expansions of integrals.* Courier Dover Publications, 1975.

[7] Denis Bosq. *Nonparametric statistics for stochastic processes.* Springer, 1996.

[8] Adam D Bull. Estimating time-changes in noisy Lévy models. *Annals of Statistics*, 42(4):2026–2057, 2014.

[9] Fabienne Comte and Valentine Genon-Catalot. Adaptive Laguerre density estimation for mixed poisson models. *Archives-Ouvertes*, 2014.

[10] Cont, Rama and Tankov, Peter. *Financial modelling with jump process.* CRC Press UK, 2004.

[11] Heinz Werner Engl, Martin Hanke, and Andreas Neubauer. *Regularization of inverse problems*, volume 375. Springer, 1996.

[12] Arthur Erdélyi. *Asymptotic expansions.* Number 3. Courier Dover Publications, 1956.

[13] Mikhail Fedoryuk. *Asimptotika: integraly i ryady.* Mathematical Reference Library. "Nauka", Moscow, 1987.

[14] Alexander Meister. *Deconvolution problems in nonparametric statistics*, volume 193. Springer, 2009.

[15] Fritz Oberhettinger. *Tables of Mellin transforms.* Springer Science & Business Media, 2012.

[16] Francois Roueff and Tobias Rydén. Nonparametric estimation of mixing densities for discrete distributions. *Annals of statistics*, pages 2066–2108, 2005.

[17] Viktor Todorov and George Tauchen. Realized Laplace transforms for pure-jump semimartingales. *The Annals of Statistics*, 40(2):1233–1262, 2012.

[18] Alexandre B Tsybakov. *Introduction to nonparametric estimation*, volume 11. Springer, 2009.

[19] Cun-Hui Zhang. Fourier methods for estimating mixing densities and distributions. *The Annals of Statistics*, pages 806–831, 1990.