Numerically stable computation of CreditRisk$^+$

Hermann Haaf
Commerzbank AG, Risk Control (ZRC), Methods & Policies, 60261 Frankfurt am Main, Germany

Oliver Reiß
IKB Deutsche Industriebank AG, Wilhelm-Bötzkes-Strasse 1, D-40474 Düsseldorf, Germany

John Schoenmakers
Weierstrass Institute for Applied Analysis and Stochastics, Mohrenstrasse 39, 10117 Berlin, Germany

The CreditRisk$^+$ model launched by Credit Suisse First Boston in 1997 is widely used by practitioners in the banking sector as a simple means for the quantification of credit risk, primarily of the loan book. We present an alternative numerical recursion scheme for CreditRisk$^+$, equivalent to an algorithm recently proposed by Giese, that is based on well-known expansions of the logarithm and the exponential of a power series. We show that it is advantageous for the Panjer recursion advocated in the original CreditRisk$^+$ document, in that it is numerically stable. The crucial stability arguments are explained in detail. We explain how to apply the suggested recursion scheme to incorporate stochastic exposures into the CreditRisk$^+$ model as introduced by Tasche (2004). Finally, the computational complexity of the resulting algorithm is stated and compared with other methods for computing the CreditRisk$^+$ loss distribution.

1 Introduction

A widely used model to describe the credit loss distribution of a loan portfolio is the CreditRisk$^+$ model presented by Credit Suisse First Boston in 1997. CreditRisk$^+$ is a *default-mode* model which distinguishes between two states, *default* or *survival* of an obligor within a one-year period. The popularity of CreditRisk$^+$ is due to the following features: the input data and parameters are readily available. For instance, default probabilities and recovery rates are required in the context of the internal ratings-based approach of the Basel II framework on the regulatory treatment of credit risk. Affiliation to economic sectors and sector variabilities can be obtained from the information provided by rating agencies and economic research institutes. Furthermore, CreditRisk$^+$ is very efficient from a computational point of view due to its analytical tractability. In particular, the probability-generating function of the loss distribution is explicitly known, and as a result the distribution can be computed by fast methods.

This paper was developed in cooperation with the project “Effiziente Methoden zur Bestimmung von Risikomaßen”, which was supported by bmbf and Bankgesellschaft Berlin AG. O.R. was also supported by the DFG research center “Mathematics for key technologies” (FZT 86) in Berlin. The authors thank D. Tasche for helpful suggestions.

There are also some limitations to CreditRisk$^+$. On a finer scale than default or survival, a change in the credit quality of an obligor that is captured as a transition of its *internal* or *external rating* is not reflected. Further, we mention the deterministic description of recoveries and the fact that large loss probabilities may lead to a distortion of the loss distribution due to multiple defaults arising from the Poisson approximation. On the other hand, however, more sophisticated models typically require more statistical input information, which in practice is often hard to identify.

In the CreditRisk$^+$ documentation (Credit Suisse First Boston, 1997), an algorithm for computing the CreditRisk$^+$ distribution is proposed which relies on a general recursion method for compound distributions given by Panjer (1981). However, it is well known that this algorithm suffers from problems of numerical stability. In this paper we present an alternative stable numerical recursion scheme for computation of the CreditRisk$^+$ loss distribution, which is based on series expansions of the respective probability-generating function.

The paper is organized as follows. In Section 2 we give a resume of the CreditRisk$^+$ model. The Panjer algorithm is considered in Section 3, along with our notion of the numerical stability that is introduced. The main result – the CreditRisk$^+$ recursion scheme with the proof of its numerical stability – is presented in Section 4. In Section 5 we extend the algorithm described in Section 4 to a CreditRisk$^+$ model with stochastic exposures as proposed by Tasche (2004). Moreover, we show along the lines of Section 4 that the extended algorithm is numerically stable as well. In Section 6 we conclude with a brief comparison of the computational effort required by the algorithm presented in this paper and the fast Fourier transform inversion method (Merino and Nyfeler, 2002; Reiß, 2003, 2004).

2 The elements of CreditRisk$^+$

We assume some familiarity with the basic principles of CreditRisk$^+$ and therefore restrict ourselves to a concise description. For a more detailed presentation readers are referred to one of the following articles: the original CreditRisk$^+$ document (Credit Suisse First Boston, 1997), Lehrbass, Boland and Thierbach (2002) or Bluhm, Overbeck and Wagner (2003).

In CreditRisk$^+$ the exposure of each obligor is represented as an integer multiple of a basic loss unit L_0, and so the aggregate portfolio loss $\tilde{X}$ (in terms of the loss unit L_0) can be represented as

$$X := \sum_{i=1}^{N} \nu_i Y_i \tag{1}$$

with ν_i denoting the multiplicity of L_0 corresponding to the ith obligor and Y_i being Poisson-distributed random variables with *stochastic* intensities

$$\mathcal{R}_i = p_i \left(w_{0,i} + \sum_{k=1}^{K} w_{k,i} S_k \right), \qquad k = 1, \ldots, K; \quad i = 1, \ldots, N \tag{2}$$

conditional on independent Gamma-distributed random variables

$$S = (S_1, \dots, S_K)$$

with parameters $\mathbf{E}[S_k] = 1$ and $\sigma_k^2 := \mathbf{var}(S_k)$, $(k = 1, \dots, K)$. In (2) p_i, $i = 1, \dots, N$, are the one-year default probabilities of the obligors and the weights $w_{k,i}$ can be considered as affiliations to the sectors S_k, where $S_0 := 1$ is regarded as an idiosyncratic component. The sector variables S_k model the default behavior with respect to a number of meaningfully chosen sectors corresponding to industry branches. Note that

$$\mathbf{E}[Y_i] = \mathbf{E}[\mathcal{R}_i] = p_i \qquad \text{for } i = 1, \dots, N$$

The probability-generating function (PGF) of the CreditRisk+ model $G_{\tilde{X}}(z) = \mathbf{E}[z^{\tilde{X}}]$ can thus be expressed in closed analytical form:

$$G(z) = \exp\left(\sum_{i=1}^{N} w_{0,i}\, p_i (z^{\nu_i} - 1) - \sum_{k=1}^{K} \frac{1}{\sigma_k^2} \ln\left[1 - \sigma_k^2 \sum_{i=1}^{N} w_{k,i}\, p_i (z^{\nu_i} - 1) \right] \right) \tag{3}$$

with $G := G_{\tilde{X}}$ and z a formal variable. On the other hand, by the definition of the PGF of a discrete integer-valued random variable, G may also be represented as

$$G(z) = \sum_{n=0}^{\infty} \mathrm{P}[x = n]\, z^n \tag{4}$$

The efficient and numerically stable computation of the probabilities $\mathrm{P}[\tilde{X} = n]$ in (4) from (3) is the central problem in this paper.

3 Panjer recursion and issues of numerical stability

The Panjer recursion algorithm advocated in the original CreditRisk+ document for the purpose of obtaining the probabilities $p_n := \mathrm{P}[\tilde{X} = n]$ in (4) is derived by using the fact that the log-derivative of G can be written as a rational function of the form $\mathrm{A}(z)/\mathrm{B}(z)$, with polynomials A and B. However, it is known that this algorithm is numerically unstable (see, for example, Gordy (2002) for a theoretical analysis and Giese (2003) for an illustration revealing the problems involved in the computation of the loss distribution by the Panjer method for a real-life loan portfolio). In fact, the numerical instability of the Panjer algorithm arises from an accumulation of roundoff errors, which is nicely explained in Gordy (2002) and has to do with the summation of numbers of similar magnitude but opposite sign, as both the polynomials A and B contain coefficients of both signs. Let us explain this issue in some more detail.

Recall that the relative error, ε_{x+y}, of the addition operation is given by

$$\varepsilon_{x+y} = \frac{x}{x+y}\,\varepsilon_x + \frac{y}{x+y}\,\varepsilon_y \qquad \text{if } x + y \neq 0 \tag{5}$$

in terms of the relative errors ε_x and ε_y of their arguments x and y, respectively. If the summands x and y are of the same sign, we have that $|\varepsilon_{x+y}| \le \max\{|\varepsilon_x|, |\varepsilon_y|\}$. If the arguments of the addition are of opposite sign, at least one of the terms $|x/(x+y)|$, $|y/(x+y)|$ is greater than 1, and hence at least one of the relative errors ε_x or ε_y gets amplified. This amplification becomes particularly big if $x \approx y$ and, hence, a cancellation in the denominator term $x + y$ occurs, leading to an explosion of the relative error ε_{x+y}. Therefore we conclude that the error propagation of the addition of two numbers of equal sign can be considered as harmless even under repeated application, leading to no amplification of the original error terms. On the other hand, if under repeated summation (eg, in a recursive algorithm) there happens to be a constellation where the summands are of similar magnitude but opposite sign, cancellation effects may occur leading at least to spurious results, if not to a complete termination of the algorithm. For a multiplication $x \cdot y$, the relative error is approximately given by

$$\varepsilon_{x \cdot y} \approx \varepsilon_x + \varepsilon_y \tag{6}$$

ie, the relative errors of the arguments simply add up.

We conclude that a recursive algorithm, which relies exclusively on summation and multiplication of numbers of the same sign, can be considered numerically stable. We refer to standard text books on numerical analysis, eg, Stoer and Bulirsch (2002) for more details on the subject.

As alternatives to the Panjer recursion several methods have been proposed in the CreditRisk$^+$ literature. Among others we mention Fourier methods (eg, Merino and Nyfeler, 2002; Reiß, 2003, 2004) and saddlepoint methods for the loss distribution with respect to a particular quantile (Gordy, 2002; Martin, Thompson, Browne, 2001).

4 Numerically stable expansion of the PGF

We introduce the portfolio polynomial of the kth sector to be

$$\mathcal{P}_k(z) := \sum_{i=1}^{N} w_{k,i}\, p_i\, z^{\nu_i}, \qquad k \in \{0, \ldots, K\} \tag{7}$$

For the analysis that follows, it is important to note that the coefficients of $\mathcal{P}_k$ are all non-negative. In terms of $\mathcal{P}_k$, G can be re-expressed as

$$\begin{aligned} G(z) &= \exp\left[-\mathcal{P}_0(1) + \mathcal{P}_0(z)\right] \prod_{k=1}^{K} \left[1 - \sigma_k^2 \left(\mathcal{P}_k(z) - \mathcal{P}_k(1)\right)\right]^{-1/\sigma_k^2} \\ &= \exp\left[-\mathcal{P}_0(1) + \mathcal{P}_0(z) - \sum_{k=1}^{K} \frac{1}{\sigma_k^2} \ln\left(1 + \sigma_k^2 \mathcal{P}_k(1) - \sigma_k^2 \mathcal{P}_k(z)\right)\right] \end{aligned} \tag{8}$$

Observe that (4) can be interpreted as the power series representation of the analytical representation of G around $z = 0$, having a radius of convergence,[1] R, strictly greater than 1. Therefore, it is natural to calculate the coefficients, ie, the probabilities p_n, directly, by applying standard algorithms for the logarithm and exponential of power series, which can be found in the analysis and mathematical physics literature (see, for example, Brent and Kung (1978) and the references therein). We systematically derive a method for calculating the coefficients of the power series expansion of (8) and present a two-step recursive scheme, where the sign structures of the coefficients involved are such that numerical stability of the two steps is ensured by two lemmas. For the convenience of the reader we provide detailed proofs of both lemmas. In fact, a basically equivalent recursion algorithm in this spirit was previously suggested by Giese (2003). However, in Giese (2003) the numerical stability is not analyzed.

Thus, we first look at the power series expansion of the logarithm of a power series. Second, having gained information on the sign structure of the coefficients of the resulting series, we investigate in a further step the power series expansion of its exponential.

We will show that, using this method, the coefficients of the power series of $G(z)$ can be computed in a numerically stable way. In particular, by Lemma 1 and Lemma 2 it will be shown that the stability follows from the particular sign structure of the polynomials under consideration. In fact, in the crucial operations of the recursion scheme only non-negative terms are added up. The numerical stability of such summations is explained in the previous section.

LEMMA 1 EXPANSION OF THE LOGARITHM *Consider a sequence* $(a_k)_{k\geq 0}$ *with* $a_0 > 0$, $a_k \geq 0$ *for all* $k \geq 1$ *and the function* $g(z) := -\ln(a_0 - f(z))$, *where* $f(z) := \sum_{k=1}^{\infty} a_k z^k$. *Let us assume that f has a positive convergence radius, so that g is analytic in a disc* $\{z : |z| < R\}$ *for some* $R > 0$ *and thus can be expanded as* $g(z) =: \sum_{k=0}^{\infty} b_k z^k$ *on this disc. Then, for the coefficients of g we have* $b_k \geq 0$ *for* $k \geq 1$ *and their computation by means of the following recursively defined sequence*[2]

$$b_0 = -\ln(a_0)$$

$$b_k = \frac{1}{a_0}\left[a_k + \frac{1}{k}\sum_{q=1}^{k-1} q\, b_q\, a_{k-q}\right] \qquad \text{for } k \geq 1 \tag{9}$$

is numerically stable.

PROOF Note that $g'(z) = f'(z)/(a_0 - f(z))$; hence

$$\left(a_0 - \sum_{k=1}^{\infty} a_k z^k\right)\sum_{k=0}^{\infty}(k+1)\,b_{k+1} z^k = \sum_{k=0}^{\infty}(k+1)\,a_{k+1} z^k$$

[1] See Haaf and Tasche (2002) for a more precise bound.

[2] As usual, an empty sum, if $k = 1$, is defined to be zero.

Performing the Cauchy product of the power series on the left-hand side of the preceding equation and comparing coefficients, it follows that $(b_k)_{k\geq 0}$ is given by (9) for $k \geq 1$. Substituting $z = 0$ gives $g(0) = -\ln(a_0)$.

From the assumptions on the sequence (a_k) it follows by (9) that $b_k \geq 0$ for $k \geq 1$. So the recursive computation of $(b_k)_{k\geq 0}$ by (9) is numerically stable, as only sums of non-negative terms are involved. □

LEMMA 2 THE EXPONENTIAL OF A POWER SERIES *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^k$ *and* $g(z) := \exp(f(z)) = \sum_{n=0}^{\infty} b_n z^n$ *in a disc* $\{z : |z| < R\}$ *for some* $R > 0$. *Then*

$$b_0 = \exp(a_0)$$

$$b_n = \sum_{k=1}^{n} \frac{k}{n} b_{n-k} a_k \qquad \text{for } n \geq 1 \tag{10}$$

Moreover, the recursion (10) is numerically stable if the coefficients of f satisfy $a_k \geq 0$ for $k \geq 1$.

PROOF The relation $b_0 = \exp(a_0)$ follows by substituting $z = 0$. For the jth derivative we have

$$f^{(j)}(0) = j!\, a_j \qquad \text{and} \qquad g^{(j)}(0) = j!\, b_j \tag{11}$$

On the other hand, for $n \geq 1$ one obtains

$$g^{(n)}(z) = \frac{\mathrm{d}^n}{\mathrm{d}z^n} \exp(f(z)) = \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{n-1} [g(z) \cdot f'(z)]$$

Hence, by Leibniz's rule for the higher derivative of a product,

$$g^{(n)} z = \sum_{k=0}^{n-1} \binom{n-1}{k} f^{(k+1)}(z)\, g^{(n-(k+1))}(z) \tag{12}$$

holds. Then (10) follows straightforwardly by substituting $z = 0$ in (12) and using (11). Finally, the stability assertion is clear, since from $a_k \geq 0$ for $k \geq 1$ and $b_0 > 0$ it follows that $b_n \geq 0$, and so in (10) only positive terms are involved. □

REMARK In fact, the results in Lemma 1 and Lemma 2 may be derived from one another. However, in order to clearly reveal the sign structures of the power series involved and their impact on numerical stability, we have chosen to treat them separately.

4.1 Algorithm 1

Setting

$$a_0^{(k)} := 1 + \sigma_k^2 \mathcal{P}_k(1)$$

$$a_j^{(k)} := \sigma_k^2 \sum_{i=1}^{N} w_{k,i}\, p_i\, \mathbf{1}_{\{\nu_i = j\}} \qquad j = 1, \ldots, M$$

for $k = 1, \ldots, K$, we compute with the procedure defined in Lemma 1 up to a pre-specified order[3] M, the Mth order expansion of

$$-\ln\left(1 + \sigma_k^2 \mathcal{P}_k(1) - \sigma_k^2 \mathcal{P}_k(z)\right)$$

to obtain

$$\ln G(z) = \sum_{j=0}^{M} \beta_j z^j + \mathcal{O}\left(z^{M+1}\right)$$

Note, that Lemma 1 guarantees that $\beta_j \geq 0$ for $j \geq 1$.

In the next step we recursively compute the coefficients γ_n, $n = 0, \ldots, M$, in the expansion

$$G(z) = \sum_{n=0}^{M} \gamma_n z^n + \mathcal{O}\left(z^{M+1}\right)$$

from β_j, $j = 0, \ldots, M$, by applying Lemma 2.

The numerical stability of Algorithm 1 follows from Lemma 1 and Lemma 2 due to the sign structure of the coefficients $a_j^{(k)}$ and β_j, respectively. Note that the coefficients γ_n correspond exactly to the loss probabilities, $\mathrm{P}[\tilde{X} = n]$, of the CreditRisk+ model for $0 \leq n \leq M$.

5 Incorporation of stochastic exposures

We consider stochastic exposures in the CreditRisk+ model introduced by Tasche (2004). We will show that for Tasche's extension the recursion algorithm presented here works as well. In this model the loss variable X as a multiple of the basic loss unit L_0 is represented as

$$\tilde{X} = \sum_{i=1}^{N} \sum_{j=1}^{Y_i} \varepsilon_{i,j} \tag{13}$$

[3] A conservative upper bound for M, in the absence of multiple defaults, would be $\sum_{i=1}^{N} \nu_i$, corresponding to the case that each loan in the entire portfolio defaults. For practical purposes $\mathrm{M} = \mathcal{O}(N)$ is a more meaningful choice.

where the non-negative, integer-valued random variables $\varepsilon_{i,j}$ are independent copies of ε_i, which are also independent of Y_i and of the sector variables S_k.

Conditioned on $S = (S_1, \ldots, S_K)$, $Y_1, \ldots, Y_N$ are mutually independent and the conditional distribution given S is Poisson with intensity $\mathcal{R}_i$ as defined in (2). The PGF $H_i(z) := \mathbf{E}[z^{\varepsilon_i}]$ can be written in the form

$$H_i(z) = \sum_{\ell=0}^{\infty} \pi_{i\ell}\, z^{\ell} \tag{14}$$

where $\pi_{i\ell} := \mathrm{P}[\varepsilon_i = \ell]$, hence $\pi_{i\ell} \geq 0$. Note that for constant exposures $\varepsilon_i \equiv \nu_i$ we obtain the classical CreditRisk$^+$ model again, with the PGF (14) being trivially given by the monomials $H_i(z) = z^{\nu_i}$ for $i = 1, \ldots, N$.

Following Tasche (2004), the portfolio polynomials $\mathcal{P}_k(z)$ naturally generalize to

$$\tilde{\mathcal{P}}_k(z) = \sum_{i=1}^{N} w_{k,i}\, p_i \sum_{\ell=0}^{\infty} \pi_{i\ell}\, z^{\ell} \tag{15}$$

and due to the special structure of $\tilde{X}$ in (13), whose terms correspond to (*conditional*) *compound Poisson sums*, the PGF $\tilde{G}(z)$ of $\tilde{X}$ is given by (8), with $\mathcal{P}_k(z)$ replaced by $\tilde{\mathcal{P}}_k(z)$.

For a meaningful choice of ε we refer to Tasche (2004), where it is suggested that one write $\varepsilon := 1 + E$, with E *negatively binomially distributed* and ε *overdispersed.*[4]

The power series expansion of $\tilde{G}(z)$ can be computed by an algorithm similar to Algorithm 1. Moreover, by writing (8) as

$$\tilde{G}(z) = \exp\Bigg[-\tilde{\mathcal{P}}_0(1) + \tilde{\mathcal{P}}_0(0) + \left(\tilde{\mathcal{P}}_0(z) - \tilde{\mathcal{P}}_0(0)\right) - \sum_{k=1}^{K} \frac{1}{\sigma_k^2} \ln\left(1 + \sigma_k^2\left(\tilde{\mathcal{P}}_k(1) + \tilde{\mathcal{P}}_k(0)\right) - \sigma_k^2\left(\tilde{\mathcal{P}}_k(z) - \tilde{\mathcal{P}}_k(0)\right)\right)\Bigg] \tag{16}$$

and observing that for $k = 0, \ldots, K$, $\tilde{\mathcal{P}}_k(1) - \tilde{\mathcal{P}}_k(0) \geq 0$ and $\tilde{\mathcal{P}}_k(z) - \tilde{\mathcal{P}}_k(0)$ are polynomials of $O(z)$ with non-negative coefficients, it is immediately clear that this computation is numerically stable because of Lemma 1 and Lemma 2.

6 Conclusion

Finally, we conclude that the calculation of the coefficients of the power series representation of G in (8) (and also $\tilde{G}$ in (16)) gives rise to a numerically stable algorithm. For the standard CreditRisk$^+$ model the computational complexity is

[4] ie, $\mathbf{var}[\varepsilon] > \mathbf{E}[\varepsilon]$.

obtained straightforwardly by counting the number of elementary operations to be

$$(K+1)M^2\text{op}_\times + \frac{1}{2}(K+1)M^2\text{op}_+ + \mathcal{O}(KN+KM)\max\left(\text{op}_+, \text{op}_\times\right)$$

where op_+ denotes the cost of an addition and $\text{op}_\times$ the cost of a multiplication.[5] It is known that the Panjer recursion scheme has a computational burden of the order $\mathcal{O}(K^2M^2)$, and thus the suggested algorithm is K times faster. As a consequence, the loss distribution of CreditRisk$^+$ in the standard setting can be determined fast and reliably. Therefore, the method we have presented is very suitable for practical purposes to determine the exact CreditRisk$^+$ loss distribution even for huge loan portfolios.

The recursion algorithm can be applied in an obvious way to determine the risk contributions according to the standard risk measures – namely, quantile and expected shortfall contributions of the CreditRisk$^+$ loss distribution as presented in Haaf and Tasche (2002) for constant exposures and in Tasche (2004) for stochastic ones. This task essentially boils down to the computation of the original portfolio loss distribution and K further distributions, being slightly shifted in terms of the parameters specifying the sector variables.

For generalizations of CreditRisk$^+$-type models we refer to the works of Reiß (2003, 2004), in which Fourier inversion techniques are systematically applied, allowing more freedom in the modeling. In addition, there is no longer a need to introduce a basic loss unit L_0 any more. In fact, for practical purposes using fast Fourier transformation (FFT) techniques, we essentially obtain the loss distribution on a continuous scale.

Of course, the Fourier inversion algorithm can also be applied to the standard CreditRisk$^+$ model. The computational effort of the Fourier inversion algorithm with given, preassigned numerical accuracy (ie, in terms of the fineness of the discretization) and a fixed number of sectors, is of order $\mathcal{O}(N)$. On the other hand, the computational effort of the algorithm presented here is of order $\mathcal{O}(N^2)$, since M ought to be chosen of order $\mathcal{O}(N)$. Hence the Fourier method is faster for very large portfolios. Nevertheless, due to the fixed base expense of the FFT, our series expansion of the PGF is computationally more advantageous for smaller portfolios.

REFERENCES

Bluhm, C., Overbeck, L., and Wagner, C. (2003). *An introduction to credit risk modeling,* Second edition. Chapman & Hall/CRC Financial Mathematics Series.

Brent, R. P., and Kung, H. T. (1978). Fast algorithms for manipulating formal power series. *Journal of the ACM* **25**, 581–95.

[5] Note that on a modern PC the cost of a multiplication is roughly comparable to that of an addition.

Credit Suisse First Boston (1997). CreditRisk$^+$: A credit risk management framework. Available at http://www.csfb.com/creditrisk.

Giese, G. (2003). Enhancing CreditRisk$^+$. *Risk* **16**(4), 73–7.

Gordy, M. B. (2002). Saddlepoint approximation of CreditRisk$^+$. *Journal of Banking and Finance* **26**(7), 1337–55.

Haaf, H., and Tasche, D. (2002). Credit portfolio measurements. *GARP Risk Review*, Issue 7 (July/August), 43–7.

Lehrbass, F., Boland, I., and Thierbach R. (2002). Versicherungsmathematische Risikomessung für ein Kreditportfolio. *Blätter der Deutschen Gesellschaft für Versicherungsmathematik* **XXV**, 285–308.

Martin, R. J., Thompson, K. E., and Browne, C. J. (2001). Taking to the saddle. *Risk* **14**(6), 91–4.

Merino, S., and Nyfeler, M. (2002). Calculating portfolio loss, *Risk* **15**(8), 82–6.

Panjer, H. (1981). Recursive evaluation of a family of compound distributions. *ASTIN Bulletin* **12**, 22–6.

Reiß, O. (2003). Fourier inversion algorithms for generalized CreditRisk$^+$ models and an extension to incorporate market risk. WIAS preprint no. 817, Weierstrass Institute Berlin.

Reiß, O. (2004). Fourier inversion techniques for CreditRisk$^+$. In F. Lehrbass and M. Gundlach (eds), *CreditRisk$^+$ in the banking industry*, pp. 111–28. Springer-Verlag.

Stoer, J., and Bulirsch, R. (2002). *Introduction to numerical analysis.* Springer-Verlag.

Tasche, D. (2004). Capital allocation with CreditRisk$^+$. In F. Lehrbass and M. Gundlach (eds), *CreditRisk$^+$ in the banking industry*, pp. 25–43. Springer-Verlag.