Robust option replication for a Black–Scholes model extended with nondeterministic trends *

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Statistical analysis on various stocks reveals long range dependence behaviour of the stock prices that is not consistent with the classical Black and Scholes model. This memory or non deterministic trend behaviour is oftenly seen as a reflection of market sentiments and causes that the historical volatility estimator becomes unreliable in practice. We propose an extension of the Black and Scholes model by adding a term to the original Wiener term involving a *smoother* process which accounts for these effects. The problem of arbitrage will be discussed. Using a generalized stochastic integration theory [8], we show that it is possible to construct a self financing replicating portfolio for a European option without any further knowledge of the extension and that, as a consequence, the classical concept of volatility needs to be re–interpretated.

Keywords: Black and Scholes option price theory, long-range dependence, stochastic analysis of square zero variation processes, portfolios, arbitrage.

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1 Introduction

The Black and Scholes model for the price S of a stock, given by the stochastic differential equation

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t, \tag{1}$$

where μ is the return rate, σ the volatility and W a Wiener process, is widely accepted as a tool for the valuation of contingent claims (options) on the underlying stock [1]. The presence of long range dependence in the structure of data of stock prices from financial markets suggests, however, that the Black and Scholes model is not entirely realistic [6] and has led to proposals some years ago that the Wiener process in (1) should be replaced by a fractional Brownian motion [4, 8]. Fractional Brownian motion (fBm) B^h with Hurst index $\frac{1}{2} < h < 1$, introduced by Mandelbrot and Van Ness [9] to model long range dependence, is a zero mean Gaussian process with covariance function

$$\Gamma^h(t,s) := \frac{1}{2}\left(t^{2h} + s^{2h} - |t-s|^{2h}\right). \tag{2}$$

The fBm version of the Black and Scholes model is then

$$dS_t = \mu S_t \, dt + \sigma_B S_t \, dB_t^h. \tag{3}$$

However, as we will see later on, fractional Brownian motion is *not* a semi–martingale and there is no equivalent martingale measure, so by general results this implies *almost* that there must be arbitrage. In fact, Rogers [10] has shown that the model (3) admits arbitrage opportunities by constructing an arbitrage explicitly using the specific nature of the fBm. Unfortunately the arbitrage strategy in [10] is quite technical and not easy to carry out in practice. In the present paper we will argue along other lines that the model (3) cannot stand and that *both* a Wiener process *and* an additional process Z to model long range dependence behaviour are required for an appropriate stock price model. In particular, we propose the model

$$dS_t = \mu S_t dt + \sigma S_t \, dW_t + S_t \, dZ_t, \tag{4}$$

where W is a standard Wiener process and Z is a continuous process of square variation zero, which are not necessarily independent. For technical reasons[1], however, we assume that Z can be split up by two continuous, square zero variation processes $Z^{(1)}$ and $Z^{(2)}$ as

$$Z = Z^{(1)} + Z^{(2)},$$

[1]These reasons became clear after helpful comments from Prof. M. Zaehle

such that $Z^{(1)}$ is adapted to W and $Z^{(2)}$ is independent of W. As such Z is a smoother process than the Wiener process, but the distribution of Z is considered to be completely unknown. From the additional assumptions it follows that W is a martingale with respect to the filtration generated by $\{W, Z^{(2)}\}$ and that Z is adapted to this filtration.

In Section (3) we discuss the question of arbitrage, but we will set aside this problem for a moment and consider as an example for Z the process $Z = Z^{(2)} = \sigma_B B^h$ where σ_B is an additional parameter depending on the intensity of the long range effects and B^h is fBm with Hurst index $h \in]\frac{1}{2}, 1[$. It is known that B^h for $h \in]\frac{1}{2}, 1[$ is a process of unbounded variation and square variation zero. See [8, 10]. From this it follows that B^h is not a semi–martingale and the use of fractional Brownian motion B^h or a more general process Z with zero quadratic variation in a stochastic differential equation requires a different concept of stochastic integral since stochastic calculus based on semi–martingale integrators is not applicable. In this respect we could use non–probabilistic path–wise integration methods of Föllmer [5] and Bick,Willinger [3]. Also S.J. Lin [8] defined a stochastic integral with respect to a continuous process Z with zero quadratic variation for integrands of the form $\phi(Y_t, Z_t)$, where $\phi : \mathbb{R}^2 \to \mathbb{R}$ is a smooth enough function, Y is an arbitrary continuous semi–martingale on a filtered probability space and Z is adapted to this filtration. We will reformulate Lin's definition in Section (2) and give some extensions of his ideas on stochastic differential equations, including an Itô–like formula for solutions of these equations. In Section (4) we will construct a self financing replicating portfolio for a European option claim and will show that the initial value of this portfolio can be valuated in a way similar to the Black and Scholes theory and depends only on the coefficient σ of the Wiener process in (4) and, surprisingly, not on the specific nature of the process Z. However, in Section (5) we will see that the volatility σ can now no longer be regarded as the deviation of the stock return $\log \frac{S_{t+1}}{S_t}$, but rather that σ^2 is merely the rate of the second variation of the process $\log S_t$.

2 Stochastic integrals for integrands with zero quadratic variation

We start by recapitulating Lin's definition of a stochastic integral with respect to a continuous process with zero quadratic variation, such as a fractional Brownian motion. After adding some measurability requirements and changing the notation in [8] slightly, we have

Definition 1 *Let Y be a continuous semi–martingale on a filtered probability space and let Z be a continuous process with zero quadratic variation adapted to this filtration. Given $\phi : \mathbb{R}^2 \to \mathbb{R}$, if there exists a $\Phi \in C^2(\mathbb{R}^2 \to \mathbb{R})$ such that $\Phi_z(y,z) = \phi(y,z)$ we define*

$$\int_0^T \phi(Y_t, Z_t)\, dZ_t := \Phi(Y_T, Z_T) - \Phi(Y_0, Z_0) - \int_0^T \Phi_y(Y_t, Z_t)\, dY_t - \frac{1}{2}\int_0^T \Phi_{yy}(Y_t, Z_t)\, d[Y,Y]_t,$$

where $[Y,Y]$ is the quadratic variation process of Y.

Lin showed that the stochastic integral defined in this way can be regarded as a limit in probability of Riemann sums,

$$\int_0^T \phi(Y_t, Z_t)\, dZ_t = \lim_{\delta \to 0} \sum_{i=1}^{N} \phi(Y_{t_{i-1}}, Z_{t_{i-1}}) \left(Z_{t_i} - Z_{t_{i-1}} \right),$$

where $0 = t_0 < t_1 < \cdots < t_N = T$ and $\delta := \max\{t_i - t_{i-t} :\ i = 0, \ldots, N\}$.

Following [8], we will see that under suitable conditions the stochastic differential equation (SDE)

$$dX_t = a(X_t)dt + b(X_t)\, dW_t + c(X_t)\, dZ_t \quad \text{with } X_0 = x_0, \tag{5}$$

where W is a Wiener process and Z is a square zero variation process which satisfies the same conditions as in (4), has a solution X of the form $X_t = \psi(U_t, Z_t)$ for a certain semi–martingale U which is to be determined and a function ψ such that the composite function $c \circ \psi$ is like the function ϕ in Definition (1).

Proposition 1 *Suppose that $c \in C^1(\mathbb{R} \to \mathbb{R})$ is strictly positive or strictly negative and that the functions a and b are locally Lipschitz continuous on $\mathbb{R}$. Further, suppose that the function g satisfies*

$$\int_0^{g(\tau)} \frac{ds}{c(s)} = \tau$$

and for some $p_0 \in \mathbb{R}$ define the functions ψ, ρ, η by

$$\begin{aligned}
\psi(y,z) &:= g\left(y + z + p_0\right), \\
\rho(y,z) &:= \left(\frac{ac - \frac{1}{2} b^2 c'}{c} \right) \circ g\left(y + z + p_0\right), \\
\eta(y,z) &:= \left(\frac{b}{c} \right) \circ g\left(y + z + p_0\right).
\end{aligned}$$

Then the Itô SDE with random coefficients

$$dU_t = \rho(U_t, Z_t)\, dt + \eta((U_t, Z_t)\, dW_t, \qquad U_0 = 0,$$

has a unique strong solution U and the solution X of the SDE (5) is given by

$$X_t = \psi\left(U_t, Z_t\right) = g\left(U_t + Z_t + p_0\right) \tag{6}$$

where $g(p_0) = x_0$.

Proof The conditions are such that the differentials in (5) are properly defined, so all we have to do is to replace Φ by ψ in Definition (1) and everything works out straightforwardly.

Corollary 1 *If $a(x) = \mu x$, $b(x) = \sigma x$ and $c(x) = x$, then the SDE (5) has the explicit solution*

$$X_t = x_0 \, \exp\left\{ \left(\mu - \frac{1}{2}\sigma^2\right) \, t + \sigma W_t + Z_t \right\}. \tag{7}$$

Now that we have defined a stochastic differential equation driven by a Wiener process and a continuous zero quadratic variation process, we can derive a transformation formula similar to the Itô formula for Itô diffusions.

Proposition 2 *If X is a solution of the SDE (5) as in Proposition (1) and $f \in C^{1,2}(\mathbb{R} \times \mathbb{R} \to \mathbb{R})$, then*

$$\begin{aligned} df(t, X_t) \quad &= \quad \left(f_t(t, X_t) + f_x(t, X_t) a(X_t) + \frac{1}{2} f_{xx}(t, X_t) b^2(X_t) \right) \, dt \\ &\qquad + f_x(t, X_t) b(X_t) \, dW_t + f_x(t, X_t) c(X_t) \, dZ_t, \end{aligned}$$

or in condensed form

$$df = f_t \, dt + f_x \, dX_t + \frac{1}{2} f_{xx} \, d[X, X]_t. \tag{8}$$

Proof It is obvious how to generalize Definition (1) to integrands of the form $\phi(t, Y_t, Z_t)$. We then use the representation for X in Proposition (1) and insert $f(t, \psi(U_t, Z_t))$ for $\Phi(t, U_t, Z_t)$ into this generalization of Definition (1).

3 Arbitrage free models, mathematical arbitrage versus practical "bubble" arbitrage

When dealing with a stock price model such as (4) a delicate problem which has to be considered is the possibility of arbitrage opportunities. As a general result it is known that that an arbitrage free stock price model admits an equivalent martingale measure and thus needs to be a semi–martingale at least. For instance, if $\sigma = 0$ and if Z is known to be equal to $\sigma_B B^h$ it follows that there is arbitrage and by Rogers [10] an arbitrage strategy is constructed. However, as shown by Rogers [10] and independently Anh et al. [2], it is possible to modify the fBm slightly while keeping long range dependence behaviour of fractional Brownian motion, such that the nodified process is a semi–martingale and arbitrage is avoided. For instance, Rogers suggested replacing the fBm in this case by a semi–martingale process of the form $W+A$, where A is a process of finite variation (even differentiable) and adapted to W. It is clear that this situation can be considered as a special case of the model (4), where Z is proportional to A. At this point it is an interesting question whether the arbitrage construction in [10], or maybe a slight modification of it, can serve as an arbitrage for the model (4), for instance, in in the case $\sigma \neq 0$ and $Z = \sigma_B B^h$, independent of W. However, in the next section we will show that

it is always possible to replicate, or hedge, a European option by a self financing portfolio without having any further knowledge of the process Z!

We consider this as an important fact for the following reasons.

i) If Z is such that the model (4) is mathematically arbitrage free, then the value of this portfolio at any time point before maturity is equal to the value of the European option at that time point in the usual "no arbitrage theory".

ii) There is lot of practical evidence that markets are not always in equilibrium and alow for arbitrage opportunities for a very short time due to the fact that these opportunities can not be seen immediately. See [11]. In this situation the stock price model (4) may not be arbitrage free in the strict mathematical sense, but still will be in practice because market participants need time to discover an arbitrage opportunity due to the unknown distribution of Z, at least at the beginning. Once dealers get hold of the distribution of Z and an arbitrage strategy is seen, they will try to carry it out, but, then this will influence the stock price evolution in such a way that the possibility of arbitrage disappears again. In the model (4) this change will be reflected by a change in the distribution of Z after the discovery of the arbitrage. As time goes on there may arise a new arbitrage opportunity which will, however, disappear again after its discovery, and so on.

In this more general situation which allows for "bubble arbitrage opportunities" we will see that the European option can still be replicated almost surely by the same self financing strategy which thus can be regarded as a robuust strategy with respect to unknown smoother perturbations of the standard Black and Scholes model.

4 Replicating a European option

Here we will show how the pay-off of a European contingent claim (option) can be replicated by a self financing portfolio when a stock price follows an SDE (4) and where the process Z may reflect long range dependence. It is somewhat suprising that we do not need to know anything more about the specific nature of Z.

Proposition 3 *Suppose a stock price S_t follows the SDE (4) and let $g(S_T)$ be a contingent claim with exercise date T.*

(i) *If $\sigma > 0$, then there exists a function $C_g(\cdot, \cdot)$ and a self financing portfolio with value $C_g(t, S_t)$ at a time instant $t < T$ prior to T and terminal value $C_g(T, S_T) = g(S_T)$ at maturity time T such that the function C_g is completely determined by the risk free interest rate r, the volatility coefficient σ of the Wiener process and the maturity time T. In particular, $C_g^{r,\sigma,T}(t, S_t)$ is given by the standard Black and Scholes formula*

$$C_g^{r,\sigma,T}(t, S_t) = e^{-r(T-t)}\, \mathbb{E}\, g\left(\hat{S}_T^{t,S_t}\right), \tag{9}$$

where the process $\hat{S}^{t,s}$ is the solution of the SDE

$$d\hat{S}_\tau = r\hat{S}_\tau\, d\tau + \sigma \hat{S}_\tau\, dW_\tau, \quad \hat{S}_t = s.$$

(ii) *If $\sigma = 0$, then the formula (9) in* (i) *collapses to*

$$C_g^{r,0,T}(t, S_t) = e^{-r(T-t)}\, g\left(S_t e^{r(T-t)}\right). \tag{10}$$

Remarks

1) It is *not true in general,* here that there exists an equivalent measure $\mathbb{P}^*$ such that the process $e^{-rt}S_t$ is a martingale with respect to $\mathbb{P}^*$ and

$$C_g^{r,\sigma,T}(t, S_t) = e^{-r(T-t)}\mathbb{E}^*\, g\left(S_T^{t,S_t}\right),$$

as in the standard theory for option pricing. This is due to the fact that it is not possible to change a process which is not a semi–martingale into a semi–martingale by an equivalent measure transformation.

2) If the process Z has finite variation and if there exists an equivalent measure $\mathbb{P}^*$ such that the $\mathbb{P}^*$ distribution of

$$W_t + \frac{\mu - r}{\sigma} t + \frac{1}{\sigma} Z_t$$

is equal to the distribution of the Wiener process W, then it is easy to see that $e^{-rt} S_t$ is a martingale with respect to $\mathbb{P}^*$ and that the present result also follows from the standard theory for option pricing. From Girsanov's theorem it can be seen that, if Z is a to W adapted process with almost sure continuous differentiable sample paths, such a $\mathbb{P}^*$ exists. Moreover, the semi–martingale with long range dependence proposed in Rogers [10] is covered by this situation.

Proof Suppose that $C(t,x)$ satisfies the Black and Scholes parabolic partial differential equation

$$C_t + \frac{1}{2}\sigma^2 x^2 C_{xx} + rxC_x - rC = 0$$

with final value $C(T,x) = g(x)$ and $t_0 \le t \le T$. Consider at time t a portfolio consisting of $C_x(t,S_t)$ shares of stock and an amount of money equal to $C(t,S_t) - C_x(t,S_t)S_t$ invested against the risk free interest r. If V is the total value of the portfolio, we have $V(t,S_t) = C(t,S_t)$ for $t_0 \le t \le T$ with $V(T,S_T) = g(S_T)$. We will show that this portfolio is *self financing.* From Proposition (2) we observe that

$$dV(t,S_t) = C_t(t,S_t)\,dt + C_x(t,S_t)\,dS_t + \frac{1}{2} C_{xx}(t,S_t)\,d[S,S]_t.$$

Since Z is a zero quadratic variation process we have $d[S,S]_t = \sigma^2 S_t^2 dt$, just as in the ordinary Black and Scholes model. Using this and the partial differential equation for C it follows that

$$dV(t,S_t) = C_x(t,S_t)\,dS_t + r\left(C(t,S_t) - S_t C_x(t,S_t)\right)\,dt.$$

The first term here is just the infinitesimal return of the stock, while the second term is the infinitesimal return of the risk free investment. From these considerations we see that the portfolio V is self financing and replicates the pay–off value of the contingent claim with probability 1.

5 Conclusions, a different interpretation of volatility

It is remarkable that the price of any contingent claim depends only on σ, the coefficient of the Wiener term in the stock price model (4), and not on the specific nature of the process Z. Consequently, a Wiener component in the model (4) is of crucial importance, because, if we could take σ equal to zero, then according to Proposition (3) the option prices on the market would only depend on the present stock price, the risk free interest rate and the time to maturity of the option, regardless of the nature of the underlying stock. This is not consistent with what actual happens in financial markets.

We note that the Wiener volatility σ is characterised by

$$[\log S, \log S]_t = [\sigma W, \sigma W]_t = \sigma^2 t. \tag{11}$$

Assuming a frictionless market, we may regard the market prices of options as being correct within small margins and from these prices we can derive the so-called implied volatilities by inverting the Black and Scholes formula. From our new stock price model (4) it follows that the squared implied volatility of a stock, which must be in accordance with (11), is substantially different from the variance of $\log \frac{S_{t+1}}{S_t}$, which in turn can be estimated from a sample of the stationary, in general *dependent*, sequence $\log \frac{S_{t_{n+1}}}{S_{t_n}}$, $n = 0,\ 1,\ 2,\ \ldots$, of identical distributed Gaussian random variables, where the time points t_n are supposed to be equally spaced. Indeed, this discrepancy is observed from actual data of stock prices, see for example [7]. Our generalized Black-Scholes model (4) provides an explanation, at least partially. In order to detect abnormalities in the stock market we need to compare the squared implied volatility of a particular stock with the rate of the second variation of the process $\log S_t$ of the stock. Thus we have to observe a particular stock during a not necessarily very long time interval $[t, t+T]$ on a very detailed time scale $t = t_0 < \cdots < t_N = t + T$ and compare the squared implied volatility with the estimator for the second variation

$$\hat{\sigma}^2 := \frac{1}{T} \sum_{n=1}^{N} \left(\log S_{t_n} - \log S_{t_{n-1}} \right)^2, \tag{12}$$

which is asymptotically consistent with σ^2 as the mesh size δ of the partition $\{t_0, \cdots, t_N\}$ tends to zero.

Finally, we note that there are several extensions of the Black and Scholes model studied in the literature, for instance models where the risk free interest rate is time dependent or where the volatility depends on S_t and t explicitly [7]. It is not difficult to show that one can also extend several of these models by including a smoother process Z to account for long range dependence behaviour and that similar conclusions can be made concerning the pricing of European options and the concept of volatility.

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