Optimal dual martingales, their analysis and application to new algorithms for Bermudan products[1,2]

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Abstract

In this paper we introduce and study the concept of optimal and surely optimal dual martingales in the context of dual valuation of Bermudan options, and outline the development of new algorithms in this context. We provide a characterization theorem, a theorem which gives conditions for a martingale to be surely optimal, and a stability theorem concerning martingales which are near to be surely optimal in a sense. Guided by these results we develop a framework of backward algorithms for constructing such a martingale which can be utilized for computing an upper bound of the Bermudan product. The methodology is purely dual in the sense that it doesn't require certain input approximations to the Snell envelope. In an Itô-Lévy environment we outline a particular regression based backward algorithm which allows for computing dual upper bounds without nested Monte Carlo simulation. Moreover, as a by-product this algorithm also provides approximations to the continuation values of the product, which in turn determine a stopping policy. We hence obtain lower bounds at the same time. We finally supplement our presentation with numerical experiments.

Keywords: Bermudan options, duality, Monte Carlo simulation, linear regression, surely optimal martingales, backward algorithm.

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1 Introduction

It is well-known that the evaluation of Bermudan callable derivatives comes down to solving an optimal stopping problem. For many callable exotic prod-

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ucts, e.g. interest products, the underlying state space is high-dimensional however. As such these products are usually computationally expensive to solve with deterministic (PDE) methods and therefore simulation based (Monte Carlo) methods are called for. The first developments in this respect concentrated on the construction of a “good” exercise policy. We mention, among others, regression based methods by Carriere (1996), Longstaff and Schwartz (2001), and Tsistsiklis and Van Roy (2001), the stochastic mesh method of Broadie and Glasserman (2004), and quantization algorithms by Bally and Pages (2003). Especially for very high dimensions, Kolodko and Schoenmakers (2004) developed a policy improvement approach which can be effectively combined with Longstaff and Schwartz (2001) for example (see Bender et al. (2008) and Bender et al. (2006)).

As a common feature, the aforementioned simulation methods provide lower biased estimates for the Bermudan product under consideration. As a new breakthrough, Rogers (2002), and Haugh and Kogan (2004) introduced a dual approach, which comes down to minimizing over a set of martingales rather than maximizing over a family of stopping times. By its very nature the dual approach gives upper biased estimates for the Bermudan product and after its discovery several numerical algorithms for computing dual upper bounds have been proposed. Probably the most popular one is the method of Andersen and Broadie (2004), although this method requires nested Monte Carlo simulation (see also Kolodko and Schoenmakers (2004) and Schoenmakers (2005)). In a Wiener environment, Belomestny et al. (2009) provides a fast generic method for computing dual upper bounds which avoids nested simulations. Further Brown et al. (2009) consider dual optimization via enlarging the information were an exercise decision may depend on. In this setting they also provide an example were a tight dual upper bound can be obtained by non-nested simulation.

The algorithms for computing dual upper bounds so far have in common that they start with some given “good enough” approximation of the Snell envelope and then construct the Doob martingale due to this approximation. In a recent paper by Rogers (2010), points out how to construct a particular ’good’ martingale via a sequence of martingales which are constant on an even bigger time interval. In this construction no input approximation to the Snell envelope is used. The methods proposed in this paper have some flavor of the method of Rogers (2010), in the sense that no approximation to the Snell envelope is involved either. In a recent paper Desai et al. (2010) treat the dual problem by methods from convex optimization theory.

In applications of the algorithm of Andersen and Broadie (2004) one generally observes that the lower the variance of the upper bound estimator, i.e. the closer the corresponding martingale is to a surely optimal one, the sharper is the corresponding dual upper bound. Actually this observation was not well studied from a mathematical point of view so far. In this paper, we study this phenomenon as one of our main contributions and give an explanation of it.

The structure of this paper is as follows. Starting with a short resume of well-known facts on Bermudan derivatives in Section 2, we analyze in Section 3 the almost sure property of the dual representation in detail. There we introduce the concept of a surely optimal martingale, which is loosely speaking, a martingale that minimizes the dual representation with a particular almost sure property. In this respect we will point out that a martingale which minimizes the dual representation is not necessarily surely optimal, and on the other hand, a surely optimal martingale is generally not unique.

In Section 4 we present, as one of the main contributions of this paper, a characterization theorem for surely optimal martingales (Theorem 6). Moreover, we provide another result that guarantees that a martingale is surely optimal if it satisfies a certain measurability criterion (Theorem 11).

Guided by the new theoretical insights we develop in Section 6 algorithms for constructing dual martingales that are based on minimization of the variance (respectively expected conditional variance) of corresponding dual representations and estimators. Also we compare (sample) variance minimization with direct minimization of the (sample) expectation (as in Desai et al. (2010)), and illustrate by two stylized examples that variance minimization may allow for smaller sample sizes. As for computational complexity, this is of particular benefit. Moreover, we present in an Itô-Lévy environment a regression based backward procedure that constructs a dual martingale via minimizing backwardly in time the expected (conditional) variances of the dual estimators corresponding to the Snell envelope. We so obtain a martingale that allows for computing upper bounds without nested Monte Carlo (like in Belomestny et al. (2009)). Furthermore we obtain, as a by-product, estimations of continuation values. Thus, as a result, we end up with a procedure that computes upper bounds as well as lower bounds simultaneously via a non-nested simulation procedure. The procedure effectively boils down to linear Monte Carlo based regression, thus is straightforward to implement and may be considered as a valuable alternative to the non-nested method of Belomestny et al. (2009), where a dual martingale is obtained by constructing a discretized Clark-Ocone derivative of some (input) approximation to the Snell envelope via regression. In particular, our new procedure only requires regression at each exercise date, in contrast to the procedure of Belomestny et al. (2009) that requires regression at each time point of a sufficient refinement of the exercise grid.

In Section 7, we present a numerical study of our algorithm. We illustrate at several multi-dimensional benchmark products a backward regression algorithm that, regarding accuracy and computational effort, produces fast lower and upper bounds. In particular, we price a Bermudan basket put which has been considered in Bender et al. (2006) and a Bermudan max-call which has been considered in Andersen and Broadie (2004). The lower and upper bounds are in the same range as their benchmark counterparts at a significant less computational effort however. We also test our method at a Bermudan max-call option with an up-and-out feature, a product which is more sensitive with regard to delta-hedging.

2 Bermudan derivatives and optimal stopping

Let $(Z_i : i = 0, 1, \ldots, T)$[1] be a non-negative stochastic process in discrete time on a filtered probability space $(\Omega, \mathcal{F}, P)$, adapted to a filtration $\mathbb{F} := (\mathcal{F}_i : 0 \le i \le T)$ which satisfies $E|Z_i| < \infty$, for $0 \le i \le T$. The measure P may be considered as a pricing measure and the process Z may be seen as a (discounted) cash-flow which an investor may exercise once in the time set $\{0, ..., T\}$. Hence, she is faced with a Bermudan product. A well-known fact is that a fair price of such a derivative is given by the Snell envelope

$$Y_i^* = \sup_{\tau \in \{i, \ldots, T\},} E_i Z_\tau, \quad 0 \le i \le T, \tag{1}$$

at time $i = 0$. In (1), τ denotes a stopping time, $E_i := E_{\mathcal{F}_i}$ denotes the conditional expectation with respect to the σ-algebra $\mathcal{F}_i$, and sup (inf) is to be understood as *essential supremum* (*essential infimum*) if it ranges over an uncountable family of random variables. Let us recall some well-known facts (e.g. see Neveu (1975)).

1. The Snell envelope Y^* of Z is the smallest super-martingale that dominates Z.
2. A family of optimal stopping times is given by

   $$\tau_i^* = \inf\{j : j \ge i, \quad Z_j \ge Y_j^*\}, \quad 0 \le i \le T.$$

   In particular,

   $$Y_i^* = E_i Z_{\tau_i^*}, \quad 0 \le i \le T,$$

   and the above family is the family of first optimal stopping times if several optimal stopping families exist.

The optimal stopping problem (1) has a natural interpretation from the point of view of the option holder: she seeks for an optimal exercise strategy which optimizes her expected payoff. On the other hand, the seller of the option rather seeks for the minimal cash amount (smallest supermartingale) he has to have at hand in any case the holder of the option exercises.

3 Duality and surely optimal martingales

We briefly recall the dual approach proposed by Rogers (2002) and, independently, Haugh and Kogan (2004). The dual approach is based on the following observation: for any martingale (M_j) with $M_0 = 0$ we have

$$Y_0^* = \sup_{\tau \in \{0, \ldots, T\}} E_0 Z_\tau = \sup_{\tau \in \{0, \ldots, T\}} E_0 (Z_\tau - M_\tau) \le E_0 \max_{0 \le j \le T} (Z_j - M_j), \tag{2}$$

[1] For notational convenience we have chosen for this stylized time set. The reader may reformulate all statements and results in this paper for a general discrete time set $\{T_0, T_1, \ldots, T_J\}$ in a trivial way.

hence the right-hand side provides an upper bound for Y_0^*. Rogers (2002) and Haugh and Kogan (2004) showed that (2) holds with equality for the martingale part of the Doob decomposition of Y^*, i.e. $Y_j^* = Y_0^* + M_j^* - A_j^*$, where M^* is a martingale with $M_0^* = 0$, and A^* is predictable with $A_0^* = 0$. More precisely we have

$$M_j^* = \sum_{l=1}^{j} \left(Y_l^* - E_{l-1} Y_l^* \right), \quad A_j^* = \sum_{l=1}^{j} \left(Y_{l-1}^* - E_{l-1} Y_l^* \right), \tag{3}$$

from which we see A^* is non-decreasing due to Y^* being a supermartingale. In addition, they showed that

$$Y_0^* = \max_{0 \le j \le T} \left(Z_j - M_j^* \right) \quad \text{a.s.} \tag{4}$$

The next lemma, by Kolodko and Schoenmakers (2006), provides a somewhat more general class of supermartingales, which turns relation (2) into an equality such that moreover (4) holds.

Lemma 1 *Let S be a supermartingale with $S_0 = 0$. Assume that $Z_j - Y_0^* \le S_j$, $1 \le j \le T$. It then holds that*

$$Y_0^* = \max_{0 \le j \le T} (Z_j - S_j) \quad a.s. \tag{5}$$

For the proof see Kolodko and Schoenmakers (2006).

Example 2 *Obviously, by taking for S the Doob martingale as constructed in (3), Lemma 1 applies. However, the Doob martingale is not the only one. For example, in the case $Z > 0$ a.s. we may also take*

$$S_j = (N_j^* - 1) Y_0^*,$$

where N^* is the multiplicative Doob part of the Snell envelope. More precisely, $Y_j^* = Y_0^* N_j^* B_j^*$ for a martingale N^* with $N_0^* = 1$ and predictable B^* with $B_0^* = 1$. Hence

$$N_j^* = \prod_{l=1}^{j} \frac{Y_l^*}{E_{l-1} Y_l^*}, \quad B_j^* = \prod_{l=1}^{j} \frac{E_{l-1} Y_l^*}{Y_{l-1}^*}. \tag{6}$$

Indeed, since B^* is non-increasing due to Y^* being a supermartingale, we have

$$S_j = Y_0^* \left(\frac{Y_j^*}{Y_0^* B_j^*} - 1 \right) \ge Y_0^* \left(\frac{Y_j^*}{Y_0^*} - 1 \right) = Y_j^* - Y_0^* \ge Z_j - Y_0^*,$$

thus, Lemma 1 applies again.

The multiplicative Doob decomposition in (6) is used by Jamshidian (2007) for constructing a multiplicative dual representation. In a comparative study, Chen and Glasserman (2007) pointed out however, that from a numerical point of view additive dual algorithms perform better due to the nice almost sure property (4).

Remark 3 *It is* ***not*** *true that for any martingale M which turns (2) into equality the almost sure statement (4) holds. As a simple counterexample, consider $T=1$, $Z_0=0$, $Z_1=2$, $M_0=0$, and $M_1=\pm 1$ each with probability $1/2$. Indeed, we see that $Y_0^*=2=E_0(2-M_1)=E_0\max(0,2-M_1)$, but, $Y_0^*\neq\max(0,2-M_1)$ a.s.*

In order to have a unified dual representation for the Snell envelope Y_i^* at any i, it is convenient to drop the assumption that martingales start at zero. We then may restate the dual theorem as

$$Y_i^* = \inf_{M\in\mathcal{M}} E_i \max_{i\leq j\leq T}(Z_j - M_j + M_i) \tag{7}$$

$$= \max_{i\leq j\leq T}\left(Z_j - M_j^* + M_i^*\right) \quad \text{a.s.}, \tag{8}$$

for any i, $0\leq i\leq T$, where $\mathcal{M}$ is the set of all martingales and M^* is the Doob martingale part of Y^*.

In view of Remark 3 and Examples 2, a martingale for which the infimum (7) is attained must not necessarily satisfy an almost sure property such as (8), and, martingales which do satisfy such almost sure property are generally not unique. We hence propose the following concept of *surely optimal* martingales.

Definition 4 *We say that a martingale M is* ***surely optimal*** *for the Snell envelope Y^* at a time i, $0\leq i\leq T$, if it holds*

$$Y_i^* = \max_{i\leq j\leq T}(Z_j - M_j + M_i) \quad a.s. \tag{9}$$

Remark 5 *Obviously, the Doob martingale of Y^* is surely optimal at each i, $0\leq i\leq T$, and any martingale M is trivially surely optimal at $i=T$. However, it is* ***not*** *true that sure optimality for some i with $i<T$ implies sure optimality at $i+1$. As a counterexample let us consider $T=2$, and $Z_0=4$, $Z_1=0$, $Z_2=2$. Take as martingale $M_0=0$, $M_1=\pm 1$, each with probability $1/2$, and $M_2=M_1\pm 1$, each with probability $1/2$ conditional on M_1. Then $\max_{0\leq j\leq 2}(Z_j-M_j+M_0)=4$ a.s. Since we have trivially $Y_0^*=4$, M is surely optimal at $i=0$. But, $\max_{1\leq j\leq 2}(Z_j-M_j+M_1)=2-M_2+M_1\notin\mathcal{F}_1$, so M is not surely optimal for Y^* at $i=1$.*

4 Characterization of surely optimal martingales

In this section we give a characterization of martingales that are surely optimal for all $i=0,\ldots,T$.

Theorem 6 *A martingale M with $M_0=0$ is surely optimal for $i=0,\ldots,T$, if and only if there exists a sequence of adapted random variables $(\zeta_i)_{0\leq i\leq T}$, such that $E_{i-1}\zeta_i=1$, and $\zeta_i\geq 0$ for all $0<i\leq T$, and*

$$M_i = M_i^* - A_i^* + \sum_{l=1}^{i}\left(A_l^* - A_{l-1}^*\right)\zeta_i, \tag{10}$$

where, respectively, M^* is the Doob martingale and A_i^* the predictable process of the Snell envelope Y^* as given in (3).

Proof. i) Let us assume that M is surely optimal as stated. Then by (9) it holds for any $0 < i \leq T$,

$$\begin{aligned} Y_{i-1}^* &= \max_{i-1 \leq j \leq T} (Z_j - M_j + M_{i-1}) \\ &= \max(Z_{i-1}, M_{i-1} - M_i + \max_{i \leq j \leq T} (Z_j - M_j + M_i)) \\ &= \max(Z_{i-1}, M_{i-1} - M_i + Y_i^*). \end{aligned} \tag{11}$$

Since $Z_{i-1} \leq Y_{i-1}^*$, and since $Z_{i-1} < Y_{i-1}^*$ implies $A_{i-1}^* = A_i^*$, we obtain from (11) and the Doob decomposition $Y_i^* = Y_0^* + M_i^* - A_i^*$

$$\begin{aligned} Y_{i-1}^* - Z_{i-1} &= (M_{i-1} - M_i + Y_i^* - Z_{i-1})^+ \\ &= \left(M_{i-1} - M_i + M_i^* - M_{i-1}^* - A_i^* + A_{i-1}^* + Y_{i-1}^* - Z_{i-1}\right)^+ \\ &= 1_{Z_{i-1} < Y_{i-1}^*} \left(M_{i-1} - M_i + M_i^* - M_{i-1}^* + Y_{i-1}^* - Z_{i-1}\right)^+ \\ &+ 1_{Z_{i-1} = Y_{i-1}^*} \left(M_{i-1} - M_i + M_i^* - M_{i-1}^* - A_i^* + A_{i-1}^*\right)^+. \end{aligned}$$

So we must have

$$\begin{gathered} 1_{Z_{i-1} < Y_{i-1}^*} \left(Y_{i-1}^* - Z_{i-1}\right) = \\ 1_{Z_{i-1} < Y_{i-1}^*} \left(M_{i-1} - M_i + M_i^* - M_{i-1}^* + Y_{i-1}^* - Z_{i-1}\right), \quad \text{and} \\ 1_{Z_{i-1} = Y_{i-1}^*} \left(M_{i-1} - M_i + M_i^* - M_{i-1}^* - A_i^* + A_{i-1}^*\right)^+ = 0, \end{gathered}$$

respectively. Hence we get

$$1_{Z_{i-1} < Y_{i-1}^*} \left(M_{i-1} - M_i + M_i^* - M_{i-1}^*\right) = 0, \quad \text{and} \tag{12}$$

$$1_{Z_{i-1} = Y_{i-1}^*} \left(M_{i-1} - M_i + M_i^* - M_{i-1}^* - A_i^* + A_{i-1}^*\right) = -1_{Z_{i-1} = Y_{i-1}^*} \mu_i, \tag{13}$$

for some non-negative $\mathcal{F}_i$-measurable random variable μ_i. W.l.o.g. we assume that $\mu_i \equiv 0$ on the set $\{Z_{i-1} < Y_{i-1}^*\}$. By taking $\mathcal{F}_{i-1}$ conditional expectations on both sides of (13), and using the martingale property of both M and M^*, and the predictability of A^*, it then follows that

$$E_{i-1}\mu_i = 1_{Z_{i-1} = Y_{i-1}^*} E_{i-1}\mu_i = 1_{Z_{i-1} = Y_{i-1}^*} \left(A_i^* - A_{i-1}^*\right). \tag{14}$$

In particular, since $\mu_i \geq 0$ almost surely, it follows from (14) that $\mu_i = 0$ on the set $\{A_i^* = A_{i-1}^*\}$ (in which $\{Z_{i-1} < Y_{i-1}^*\}$ is contained as a subset). We next define

$$\zeta_i := \begin{cases} \left(A_i^* - A_{i-1}^*\right)^{-1} \mu_i, & \text{if } A_i^* > A_{i-1}^*, \\ 1, & \text{else,} \end{cases} \tag{15}$$

and we see that we have a.s. $\mu_i = \left(A_i^* - A_{i-1}^*\right) \zeta_i$. By (14) we have (using the convention $0 \cdot \infty = 0$)

$$\begin{aligned} E_{i-1}\zeta_i &= 1_{A_i^* > A_{i-1}^*} E_{i-1} \left(A_i^* - A_{i-1}^*\right)^{-1} \mu_i + 1_{A_i^* = A_{i-1}^*} \\ &= 1_{A_i^* > A_{i-1}^*} 1_{Z_{i-1} = Y_{i-1}^*} + 1_{A_i^* > A_{i-1}^*} 1_{Z_{i-1} < Y_{i-1}^*} + 1_{A_i^* = A_{i-1}^*} = 1, \end{aligned}$$

since the middle term is trivially zero. We thus obtain from (12) and (13)

$$M_{i-1} - M_i + M_i^* - M_{i-1}^* - A_i^* + A_{i-1}^* = -\left(A_i^* - A_{i-1}^*\right)\zeta_i,$$

from which (10) follows.

ii) Conversely, if a martingale M satisfies (10), we have for any $0 \le i \le T$,

$$\begin{aligned}\max_{i\le j\le T}\left(Z_j - M_j + M_i\right) &= \max_{i\le j\le T}\left(Z_j - M_j^* + A_j^* - \sum_{l=1}^{j}\left(A_l^* - A_{l-1}^*\right)\zeta_l\right. \\ &\qquad \left. + M_i^* - A_i^* + \sum_{l=1}^{i}\left(A_l^* - A_{l-1}^*\right)\zeta_l\right) \\ &= Y_i^* + \max_{i\le j\le T}\left(Z_j - Y_j^* - \sum_{l=i+1}^{j}\left(A_l^* - A_{l-1}^*\right)\zeta_l\right) \le Y_i^*,\end{aligned}$$

and then by (7) the almost sure optimality follows. ■

By Theorem 6 we have immediately the following alternative characterization of almost sure martingales. It basically says that a martingale is surely optimal if the Snell envelope can be representated in a way that resembles the Doob decomposition but where the predictable process is replaced by a process which is in general only adapted.

Corollary 7 *A martingale M with $M_0 = 0$ is surely optimal for $i = 0, \ldots, T$, if and only if there exists an non-decreasing adapted process N with $N_0 = 0$ such that*[2]

$$Y_i^* = Y_0^* + M_i - N_i.$$

Proof. If M is surely optimal as stated, we have by the "if" part of Theorem 6 (see (10)),

$$Y_i^* - Y_0^* - M_i = -\sum_{l=1}^{i}\left(A_l^* - A_{l-1}^*\right)\zeta_i = -N_i, \tag{16}$$

with N being adapted, non-decreasing and $N_0 = 0$. Conversely, if

$$Y_i^* = Y_0^* + M_i - N_i$$

for some martingale M, $M_0 = 0$, and non-decreasing adapted N, $N_0 = 0$, we consider for each i, $0 \le i \le T$,

$$\max_{i\le j\le T}\left(Z_j - M_j + M_i\right) = \max_{i\le j\le T}\left(Z_j - Y_j^* - N_j + Y_i^* + N_i\right) \le Y_i,^*$$

and then apply (7) again. ■

We have the following remark.

[2]Note that N is not assumed to be predictable.

Remark 8 *Let the martingale M with $M_0 = 0$ be surely optimal for $i = 0, \ldots, T$. For the non-decreasing process N defined by (16) it holds that*

$$Y_i^* - M_i + M_{i-1} - Z_{i-1} = Y_{i-1}^* - N_i + N_{i-1} - Z_{i-1} =: U_i,$$

and since by (16), $N_i - N_{i-1} = \left(A_i^* - A_{i-1}^*\right)\zeta_i$, we obtain from (11)

$$(U_i)^+ = Y_{i-1}^* - Z_{i-1} \quad a.s.$$

*So, in particular we have that $(U_i)^+$ is $\mathcal{F}_{i-1}$-measurable while U_i itself is generally **not**, except for the case where $M = M^*$. A similar observation will be encountered later on in* (25).

Remark 9 *In a complete market environment, Corollary 7 implies that $Y_0^* + M$ may be considered as a discounted dynamic hedge portfolio whenever M is a martingale that is surely optimal at $\{0, ..., T\}$.*

From Theorem 6 it is clear that there exist infinitely many martingales which are surely optimal for all $i = 0, \ldots, T$. In the following example we construct a one-parametric family of such martingales which includes the Doob martingale of the Snell envelope.

Example 10 *Let us assume $Z > 0$ a.s. (if Z is strictly bounded from below by a constant $-K$, we may consider the equivalent stopping problem due to $Z + K$). Then $Y^* > 0$ a.s., and for any α, $0 \le \alpha \le 1$, we consider*

$$\zeta_i := 1 - \alpha + \alpha \frac{Y_i^*}{E_{i-1} Y_i^*} = 1 - \alpha + \alpha \frac{N_l^*}{N_{l-1}^*},$$

where N^* is the martingale part of the multiplicative decomposition $Y_i^* = Y_0^* N_i^* B_i^*$ of the Snell envelope (see Example 2). Obviously, it holds $E_{i-1}\zeta_i = 1$ and $\zeta_i \ge 0$, and hence, by Theorem 6 we obtain for every $0 \le \alpha \le 1$ a martingale

$$\begin{aligned} M_i &= M_i^* - A_i^* + \sum_{l=1}^{i} \left(A_l^* - A_{l-1}^*\right) \left(1 - \alpha + \alpha \frac{N_l^*}{N_{l-1}^*}\right) \\ &= M_i^* - \alpha A_i^* + \alpha \sum_{l=1}^{i} \left(A_l^* - A_{l-1}^*\right) \frac{N_l^*}{N_{l-1}^*}, \end{aligned}$$

which is surely optimal for $i = 0, ..., T$. Thus, for $\alpha = 0$ (i.e. $\zeta_i \equiv 1$) we retrieve *the standard Doob martingale of the Snell envelope, and for $\alpha = 1$ we obtain*

$$\begin{aligned} M_i &= Y_i^* - Y_0^* + \sum_{l=1}^{i} \left(A_l^* - A_{l-1}^*\right) \frac{N_l^*}{N_{l-1}^*} \\ &= \sum_{l=1}^{i} \left(Y_l^* - Y_{l-1}^* + Y_{l-1}^* \left(1 - \frac{B_l^*}{B_{l-1}^*}\right) \frac{N_l^*}{N_{l-1}^*} \right) \\ &= Y_0^* \sum_{l=1}^{i} \left(N_l^* B_l^* - N_{l-1}^* B_{l-1}^* + B_{l-1}^* \left(1 - \frac{B_l^*}{B_{l-1}^*}\right) N_l^* \right) \\ &= Y_0^* \sum_{l=1}^{i} B_{l-1}^* \left(N_l^* - N_{l-1}^*\right). \end{aligned} \tag{17}$$

Note that this martingale differs from the martingale $Y_0^* \left(N_i^* - 1\right)$ from Example 2 (they would coincide after dropping the factors B_{l-1}^*). It is easy to show (using Theorem 6 again) that the latter martingale is in general only optimal at $i = 0$, while the martingale (17) is surely optimal for all $i = 0, ..., T$, by construction.

The next theorem provides a key criterion for identifying surely optimal martingales.

Theorem 11 *Let Y^* be the Snell envelope of the cash-flow Z and let M be any martingale. Then, for any $i \in \{0, ..., T\}$ it holds*

$$\max_{i \le j \le T} \left(Z_j - M_j + M_i\right) \in \mathcal{F}_i \quad \Longrightarrow \quad \max_{i \le j \le T} \left(Z_j - M_j + M_i\right) = Y_i^*.$$

Proof. Let us suppose $\vartheta_i := \max_{i \le j \le T} \left(Z_j - M_j + M_i\right) \in \mathcal{F}_i$ and define the stopping time

$$\tau_i = \inf \left\{ j \ge i : Z_j - M_j + M_i \ge \vartheta_i \right\}.$$

By the definition of ϑ_i we have $i \le \tau_i \le T$ almost surely. We thus have

$$Y_i^* \ge E_i\, Z_{\tau_i} \ge E_i \left(M_{\tau_i} - M_i + \vartheta_i\right) = \vartheta_i,$$

by Doob's optional sampling theorem and the fact that $\vartheta_i \in \mathcal{F}_i$. On the other hand we have $\vartheta_i = E_i\, \vartheta_i \ge Y_i^*$ due to (7). ■

Remark 12 *While in this paper we work in a discrete time setting, Theorem 11 can be proved in a similar way for continuous time exercise as well.*

5 Stability of surely optimal martingales

In equivalent terms, Theorem 11 states that, if a martingale M is such that the conditional variance of

$$\vartheta_i(M) := \max_{i \le j \le T} \left(Z_j - M_j + M_i\right) \tag{18}$$

is zero for some $0 \le i \le T$, i.e.

$$\mathrm{Var}_i\, \vartheta_i(M) := E_i \left(\vartheta_i(M) - E_i\vartheta_i(M)\right)^2 = 0, \text{ a.s.},$$

then $\vartheta_i(M) = Y_i^*$. Hence the martingale M is surely optimal at i. In this section we present a stability result for martingales M which are, loosely speaking, close to be surely optimal at some i, in the sense that $\mathrm{Var}_i\, \vartheta_i(M)$ is small. More specifically, we provide mild conditions on a sequence of martingales $(M^{(n)})_{n\ge1}$ which guarantee that the corresponding upper bounds converge to the Snell envelope in a sense, although the sequence of martingales $(M^{(n)})$ does not necessarily converge. We have the following result.

Theorem 13 *Let $i \in \{0, ..., T\}$. If $\mathrm{Var}_i\, \vartheta_i^{(n)} \overset{P}{\to} 0$ for $n \to \infty$, where $\vartheta_i^{(n)} := \vartheta_i(M^{(n)})$, and if in addition the sequence of martingales $\left(M_i^{(n)}\right)_{n\ge1}$ is uniformly integrable, then it holds*

$$E_i\, \vartheta_i^{(n)} \overset{L_1}{\to} Y_i^*.$$

Proof. Fix an $i \in \{0, ..., T\}$ and suppose that the assumptions of the theorem are satisfied. Now take an $\epsilon > 0$. By introducing an auxiliary time $S > T$ and setting $Z_S = 0$ we next define the stopping time

$$\tau_i^{(n)} = \inf\left\{j \ge i : Z_j - M_j^{(n)} + M_i^{(n)} \ge E_i\vartheta_i^{(n)} - \epsilon\right\} \wedge S.$$

We thus have with $M_S^{(n)} := M_T^{(n)}$, $n \ge 1$,

$$\begin{aligned}
Y_i^* &\ge E_i\, Z_{\tau_i^{(n)}} = E_i\, Z_{\tau_i^{(n)}} 1_{\tau_i^{(n)}<S} \ge E_i\, \left(M_{\tau_i^{(n)}}^{(n)} - M_i^{(n)} + E_i\vartheta_i^{(n)} - \epsilon\right) 1_{\{\tau_i^{(n)}<S\}} \\
&= E_i \left(M_{\tau_i^{(n)}}^{(n)} - M_i^{(n)} + E_i\vartheta^{(n)} - \epsilon\right) - E_i \left(M_T^{(n)} - M_i^{(n)} + E_i\vartheta_i^{(n)} - \epsilon\right) 1_{\{\tau_i^{(n)}=S\}} \\
&= E_i\vartheta_i^{(n)} - \epsilon - E_i \left(M_T^{(n)} - M_i^{(n)} + E_i\vartheta_i^{(n)} - \epsilon\right) 1_{\{\tau_i^{(n)}=S\}} \quad \text{a.s.,}
\end{aligned}$$

hence

$$\begin{aligned}
E_i\vartheta_i^{(n)} &\le Y_i^* + \epsilon + E_i \left|M_T^{(n)} - M_i^{(n)} + E_i\vartheta_i^{(n)} - \epsilon\right| 1_{\tau_i^{(n)}=S} \\
&=: Y_i^* + \epsilon + E_i\, U_i^{(n)} 1_{\tau_i^{(n)}=S} \quad \text{a.s.} \qquad (19)
\end{aligned}$$

Now it is easy to see that the family of random variables $\left(U_i^{(n)}\right)_{n\ge1}$ is uniformly integrable too. We so may take $K_\epsilon > 0$ such that

$$\sup_{n\ge0} E\, U_i^{(n)} 1_{U_i^{(n)}>K_\epsilon} \le \epsilon.$$

Further observe that due to a conditional version of Chebyshev's inequality,

$$0 \le E_i\, 1_{\left\{\tau_i^{(n)}=S\right\}} = E_i 1_{\left\{\vartheta_i^{(n)}<E_i\vartheta_i^{(n)}-\epsilon\right\}} \le \frac{\mathrm{Var}_i\, \vartheta_i^{(n)}}{\epsilon^2} \overset{P}{\to} 0.$$

Since the family $\left(E_i\, 1_{\left\{\tau_i^{(n)}=S\right\}}\right)_{n\geq 0}$ is bounded by 1, it is uniformly integrable. Hence, it follows that

$$E_i 1_{\left\{\tau_i^{(n)}=S\right\}} \overset{L_1}{\to} 0. \tag{20}$$

We thus have

$$\begin{aligned} E\, U_i^{(n)} 1_{\tau_i^{(n)}=S} &= E\, U_i^{(n)} 1_{U_i^{(n)}>K_\epsilon} 1_{\tau_i^{(n)}=S} + E\, U_i^{(n)} 1_{U_i^{(n)}\leq K_\epsilon} 1_{\tau_i^{(n)}=S} \\ &\leq \epsilon + K_\epsilon E\, 1_{U_i^{(n)}\leq K_\epsilon} 1_{\tau_i^{(n)}=S} \leq \epsilon + K_\epsilon E\, E_i 1_{\tau_i^{(n)}=S} < 2\epsilon \end{aligned}$$

for $n > N_{\epsilon,K_\epsilon}$ by (20). So for $n > N_{\epsilon,K_\epsilon}$, we derive from (19)

$$E\vartheta_i^{(n)} \leq EY_i + \epsilon + E\, U_i^{(n)} 1_{\tau_i^{(n)}=S} \leq EY_i^* + 3\epsilon.$$

Thus,

$$\overline{\lim}_{n\to\infty}\; E\vartheta_i^{(n)} \leq EY_i^* + 3\epsilon,$$

where $\overline{\lim}$ denotes lim sup. Since $\epsilon > 0$ was arbitrary,

$$\overline{\lim}_{n\to\infty}\; E\vartheta_i^{(n)} \leq EY_i^*.$$

On the other hand, due to (7) we have $E_i\vartheta_i^{(n)} \geq Y_i^*$ a.s. for all n, so

$$0 \leq \overline{\lim}_{n\to\infty}\, E\left|E_i\vartheta^{(n)} - Y_i^*\right| = \overline{\lim}_{n\to\infty}\left(E\vartheta_i^{(n)} - EY_i^*\right) \leq 0,$$

which finally proves $E_i\vartheta_i^{(n)} \overset{L_1}{\to} Y_i^*$. ■

Remark 14 *Like Theorem 11, Theorem 13 can be formulated in a continuous time setting as well with (almost) literally the same proof.*

The following simple example illustrates that Theorem 13 would not be true when the uniform integrability condition is dropped.

Example 15 *Take* $T = 1$, $Z_0 = Z_1 = 0$, $M_0^{(n)} = 0$, $M_1^{(n)} =: -\xi_n$ *with* $E_0\xi_n = 0$, $n = 1, 2, \ldots$ *Then obviously* $Y_0^* = 0$, *and we have*

$$\vartheta_0^{(n)} = \max(Z_0 - M_0^{(n)}, Z_1 - M_1^{(n)}) = \max(0, \xi^{(n)}) = \xi_+^{(n)}.$$

Now take

$$\xi^{(n)} = \begin{cases} 1 \text{ with Prob. } \frac{n-1}{n} \\ 1-n \text{ with Prob. } \frac{1}{n} \end{cases}$$

(hence $E_0\xi^{(n)} = 0$*). Then, for* $n \to \infty$ *we have* $\mathrm{Var}_0\,\vartheta_0^{(n)} = E_0\,(\xi_+^{(n)})^2 - \left(E_0\xi_+^{(n)}\right)^2 = \frac{n-1}{n} - \left(\frac{n-1}{n}\right)^2 = \frac{n-1}{n^2} \to 0$, *whereas* $E_0\vartheta_0^{(n)} = E_0\xi_n^+ = \frac{n-1}{n} \to 1$. *Clearly, for each* $K > 1$, $E_0\left|M_1^{(n)}\right| 1_{\left\{\left|M_1^{(n)}\right|>K\right\}} \geq \frac{n-1}{n} 1_{\{n-1>K\}} \to 1$ *as* $n \to \infty$, *hence the* $\left(M_1^{(n)}\right)$ *are not uniformly integrable.*

In view of the next Corollary, Theorem 13 may be considered as a stability theorem related to Theorem 11.

Corollary 16 *Let $\mathcal{M}^{UI}$ be a set of uniformly integrable martingales. Then for any $i \in \{0, \dots, T\}$ it holds: For every $\epsilon > 0$ there exist a $\delta > 0$ such that*

$$\left[M \in \mathcal{M}^{UI} \quad \text{and} \quad E \operatorname{Var}_i \vartheta_i(M) < \delta\right] \quad \Longrightarrow \quad 0 \leq E\, \vartheta_i(M) - Y_i^* < \epsilon.$$

Proof. Suppose the statement is not true for some i. Then there exists an $\epsilon_0 > 0$ such that for all $n \in \mathbb{N}$ there exists a martingale $M^{(n)} \in \mathcal{M}^{UI}$, for which $E \operatorname{Var}_i \vartheta_i(M^{(n)}) < 1/n$ and $E\, \vartheta_i(M^{(n)}) - Y_i^* \geq \epsilon_0$. Since convergence in L_1 implies convergence in probability along a subsequence (indexed again by n) we thus have $\operatorname{Var}_i \vartheta_i(M^{(n)}) \xrightarrow{P} 0$, and $E \left|\vartheta_i(M^{(n)}) - Y_i^*\right| \geq \epsilon_0$ along this subsequence. This contradicts Theorem 13. ■

Remark 17 *Theorem 13 is important in practical situations, for instance, for (possibly high dimensional) underlyings of jump-diffusion type in a Lévy-Itô setup. In this environment we may consider the following class of uniformly integrable martingales.*

Let W be an m-dimensional Brownian motion and let N denote a Poisson random measure, independent of W, with (deterministic) compensator measure $\nu(s, du)ds$ such that

$$\int_0^t \int_{\mathbb{R}^q} (u^2 \wedge 1)\nu(s, du)ds < \infty, \ 0 \leq t \leq T.$$

Let $(\mathcal{F}_t)_{0 \leq t \leq T}$ be the filtration generated by W and N, augmented by null sets. Now let X be a D-dimensional Markov process, adapted to $(\mathcal{F}_t)$, and consider the mappings $c : [0, T] \times \mathbb{R}^D \to \mathbb{R}_{\geq 0}$ and $d : [0, T] \times \mathbb{R}^D \times \mathbb{R}^q \to \mathbb{R}_{\geq 0}$ satisfying

$$E \int_0^T |c(s, X_s)|^2 ds < \infty, \quad E \int_0^T \int_{\mathbb{R}^q} |d(s, X_s, u)|^2 \nu(s, du)ds < \infty. \tag{21}$$

We define the class of uniformly integrable martingales, $\mathcal{M}^{UI}$, as the set of all martingales M satisfying

$$\begin{aligned} M_t &= M_0 + M_t^c + M_t^d \\ &= M_0 + \int_0^t \varphi^c(s, X_s)dW_s + \int_0^t \int_{\mathbb{R}^q} \varphi^d(s, X_s, u)\tilde{N}(ds, du), \end{aligned}$$

where φ^c and φ^d satisfy

$$|\varphi^c| \leq c, \quad |\varphi^d| \leq d,$$

and $\tilde{N} = N - \nu$ is the compensated Poisson measure. Note that M is indeed a martingale and that the expected quadratic variation of M is given by

$$\begin{aligned} E\left[M, M\right]_t &= E \int_0^t |\varphi^c(s, X_s)|^2 ds + E \int_0^t \int_{\mathbb{R}^q} |\varphi^d(s, X_s, u)|^2 \nu(s, du)ds \\ &\leq E \int_0^t |c(s, X_s)|^2 ds + E \int_0^t \int_{\mathbb{R}^q} |d(s, X_s, u)|^2 \nu(s, du)ds. \end{aligned}$$

We then have for every $t \in [0,T]$,

$$\sup_{M \in \mathcal{M}^{UI}} E|M_t|^2 \leq \sup_{M \in \mathcal{M}^{UI}} E \sup_{0 \leq t \leq T} |M_t|^2 \leq \sup_{M \in \mathcal{M}^{UI}} CE\big[M,M\big]_T < \infty,$$

where the second estimation results from the Burkholder-Davis-Gundy inequality and the third estimation follows from (21). *Finally, an application of the de la Vallée Poussin criterion yields that $\mathcal{M}^{UI}$ is indeed a family of uniformly integrable martingales.*

6 New dual algorithms for pricing of Bermudan derivatives

In this section we consider the design of new dual algorithms for solving multiple stopping problems, hence pricing Bermudan products, which are based on the theoretical insights from Theorem 11, Theorem 13, and Corollary 16.

6.1 Dual variance minimization

The trust of Corollary 16 is that, loosely speaking, if one sorts out within a suitable class of martingales one for which the variance of (18) (for $i = 0$) is sufficiently small, then this martingale gives rise to a tight upper bound. In fact the results of Section 5 provide a necessary theoretical platform for dual algorithms that aim at minimizing the variance of (18) at $i = 0$, or the expected conditional variance (18) for $i > 0$ over a suitable parametric set of martingales.

Of course, such algorithms only can involve empirical (Monte Carlo based) estimations of these variances rather than the true ones. Therefore, in the remainder of this work we aim at the development of fast dual algorithms via minimizing suitable empirical variance estimators that in any case provide true dual martingales. Hence, the upper bound estimates are upper biased, regardless the distance of the estimated variances to the true ones and the "richness" of the family of martingales.

Naturally, the question about 'how close' the martingale obtained by empirical minimization is to the true minimizing one, arises. However, such a study would involve the application of deep results of the theory of empirical processes (uniform concentration inequalities in connection with specific Vapnik-Chervonenkis classes, see e.g. Dudley (1999)) and is considered therefore beyond the scope of this article. But, at least on an intuitive level it is clear that, typically, if a parametric family of martingales contains one for which the (expected conditional) variance of (18) is very small, only a relatively small sample size will be needed to detect a martingale that is at least "close" to this one, by minimization of the sample variance (cf. Corollary 16). This we consider a main appealing feature of variance minimizing dual algorithms. Let us note that, in contrast, minimization of an (accurate enough) sample version of the expectation of (18) directly, would typically require a relatively large sample size. In the following two paragraphs, we explore the merits of variance minimization and minimization of the expectation in a stylized framework where the family of martingales contains a surely optimal one.

Variance minimization in the presence of a surely optimal martingale

Let Q be some index set and $\mathcal{M} = \{M^q : q \in Q\}$ be a set of uniformly integrable martingales such that $\mathcal{M}$ contains a martingale M^{q^*} which is surely optimal at $i = 0$. Suppose that for any $q \in Q$ we have N samples of $\vartheta_0(M^{q,(n)})$, $n = 1, ..., N$. Based on these samples we may estimate $\mathrm{Var}_0\, \vartheta_0(M^q) = \mathrm{Var}\, \vartheta_0(M^q)$ as usual by

$$\mathrm{Var}^{(N)}\, \vartheta_0(M^q) := \frac{1}{N-1} \sum_{n=1}^{N} \left(\vartheta_0(M^{q,(n)}) - \overline{\vartheta_0(M^q)_N} \right)^2, \quad \text{with}$$
$$\overline{\vartheta_0(M^q)_N} := \frac{1}{N} \sum_{n=1}^{N} \vartheta_0(M^{q,(n)}). \tag{22}$$

So, in principle, only two realizations ($N = 2$) would be enough to identify a q^* such that

$$0 = \mathrm{Var}\, \vartheta_0(M^{q^*}) = \mathrm{Var}^{(2)}\, \vartheta_0(M^{q^*}) = \min_{q \in Q} \sum_{n=1}^{2} \left(\vartheta_0(M^{q,(n)}) - \overline{\vartheta_0(M^q)_N} \right)^2,$$

and then obtain $Y_0^* = \vartheta_0(M^{q^*}) = \vartheta_0(M^{q^*,(1)})$. Due to this stylized argumentation we may expect that in a case where although the set $\mathcal{M}$ doesn't contain a martingale that is surely optimal at $i = 0$ but at least one martingale M^q such that $\mathrm{Var}\, \vartheta_0(M^q)$ is "small enough", we only need a relatively small sample size N to estimate this martingale, leading to a tight upper bound $Y_0^{up} := E\vartheta_0(M^q)$.

Minimization of the expectation in the presence of a surely optimal martingale

In a stylized example where

$$T = 2, \quad Z_0 = 0, \quad Z_1 = \xi, \quad Z_2 = 1, \quad \text{with} \quad P(\xi \in dx) = \frac{1}{2} 1_{[0,2]}(x) dx, \tag{23}$$

we study minimizing the (estimated) dual expectation of (18) at $i = 0$ (cf. (7)) over a parametric family of martingales that contains an almost surely optimal one. More specifically, we consider multiples of the Doob martingale of the Snell envelope. This study reveals some surprising insights. In particular, it shows that in general it may happen that, though the martingale family contains an almost surely optimal one, the minimization of the (estimated) expectation yields a martingale that is only asymptotically optimal in the sense of (7) (for sample sizes going to infinity), but not surely optimal in the sense of (8)! The full details are presented in Appendix 8. The bottom line is that minimization of (estimated) dual expectations may require larger sample sizes, both for the minimization procedure as well as for the second simulation in order to determine the (upper biased) upper bound. Conferred with the previous stylized example, variance minimization would here identify the almost sure one with only two samples again.

The arguments above suggest to minimize the estimated variance of (18) at $i = 0$ over a parametric set of martingales using a relatively small sample size N. However, since the parametric set of martingales $\mathcal{M}$ needs to be "rich enough", in practice there may be many parameters involved, which in turn may lead to a non-convex minimization problem with many local minima. As a remedy to this problem, rather than directly minimizing the variance of the dual estimator at time zero, we propose to *minimize backwardly the expected conditional variances* $E\,\mathrm{Var}_i\,\vartheta_i(M^q)$ over $q \in Q$, starting from $i = T$ (where the conditional variance is trivially zero) down to $i = 0$, using a simple but effective recursive relationship between $\vartheta_i(M^q)$ and $\vartheta_{i+1}(M^q)$. This is the subject of the next subsection. For this *backward minimization* procedure the arguments above apply as well and moreover, as we will see, it opens the door to utilizing linear regression, hence to fast numerical implementations.

6.2 Backward dual variance minimization

Motivated by Section 6.1 we now develop a backward recursive simulation based algorithm for the construction of a dual martingale M that yields tight upper bounds. In view of a such a Monte Carlo approach, we assume a Markov setting generated by some underlying Markov process $X := (X_t)_{0 \le t \le T}$, and a cash-flow of the form $Z_j := Z_j(X_j) := Z(j, X_j)$. First we describe the algorithm in a pseudo language which involves terms such as conditional expectations and conditional variances. Then, we spell out an implementable Monte Carlo algorithm where these expressions are replaced by their empirical counterparts.

To start out on a pseudo algorithmic level we construct a martingale M backwardly in a recursive way by establishing that from $i = T$ down to $i = 0$ the expected conditional variances $E\,\mathrm{Var}_i\vartheta_i(M)$ are "as small as possible" in a sense that we will describe. The martingale M is such that for $j > i$, any increment

$$M_j - M_i \text{ is measurable with respect to } \Delta\mathcal{F}_{i,j} := \sigma\{X_s : i \le s \le j\}. \tag{24}$$

It is easy to see that the Doob martingale of the Snell envelope meets this measurability property, however, in general Theorem 6 yields that there may exist many other surely optimal martingales satisfying this property.

A corner stone of the whole procedure is the following recursion that holds for any martingale M and any $i < T$,

$$\begin{aligned}\vartheta_i(M) &= \max\left(Z_i, \max_{i+1\le j\le T}(Z_j - M_j + M_i)\right)\\ &= \max\left(Z_i, \vartheta_{i+1}(M) + M_i - M_{i+1}\right)\\ &= Z_i + \left(\vartheta_{i+1}(M) + M_i - M_{i+1} - Z_i\right)^+ . \qquad (25)\end{aligned}$$

Obviously, at every $i = 0, ..., T$, $\vartheta_i(M)$ only depends on $(M_j - M_i)_{i\le j\le T}$, and at the starting time $i = T$ we initially have $\vartheta_T(M) = Z_T$ which trivially satisfies $E\,\mathrm{Var}_T(\vartheta_T(M)) = 0$. Note that if M were already surely optimal, i.e. $\vartheta(M)$ were already equal to Y^*, then Remark 8 would imply that for $U_i := \vartheta_{i+1}(M) + M_i - M_{i+1} - Z_i$, $(U_i)^+ = \vartheta_i(M) - Z_i = Y_i^* - Z_i$ is already $\mathcal{F}_i$-measurable.

Now the essential idea is comprised in the following backward induction: Assume that for $i+1 \le T$ we have constructed the increments $(M_j - M_{i+1})_{i+1\le j\le T}$ and $\vartheta_{i+1}(M)$. Now the task is to find a random variable ξ_{i+1} such that

$$\xi_{i+1} \text{ is } \Delta\mathcal{F}_{i,i+1}\text{-measurable}, \qquad E_i\xi_{i+1} = 0, \qquad (26)$$

that solves the following minimization problem

$$\begin{aligned}\xi_{i+1} \quad &:= \quad \operatorname*{arg\,min}_{\xi\in\Delta\mathcal{F}_{i,i+1},\ E_i\xi=0} E\,\mathrm{Var}_i\vartheta_i(M(\xi))\\ &= \quad \operatorname*{arg\,min}_{\xi\in\Delta\mathcal{F}_{i,i+1},\ E_i\xi=0} E\,\mathrm{Var}_i\left(\vartheta_{i+1}(M) - \xi - Z_i\right)^+ . \qquad (27)\end{aligned}$$

Intuitively, ξ_{i+1} represents the optimal martingale increment and thus, we put $M_j(\xi_{i+1}) - M_i(\xi_{i+1}) := M_j - M_{i+1} + \xi_{i+1}$ for $j \ge i+1$. By construction, the random variable ξ_{i+1} satisfies (26), therefore, we obtain a set of martingale increments $(M_j(\xi_{i+1}) - M_i(\xi_{i+1}))_{i\le j\le T}$, which has now been extended from $j = i+1$ to $j = i$ and which satisfies for $j \ge i+1$,

$$\begin{aligned}M_j(\xi_{i+1}) - M_{i+1}(\xi_{i+1}) &= M_j(\xi_{i+1}) - M_i(\xi_{i+1}) + M_i(\xi_{i+1}) - M_{i+1}(\xi_{i+1})\\ &= M_j - M_{i+1}\end{aligned}$$

and by construction also the measurability requirement (24). Now we extend the increments $(M_j - M_{i+1})_{i+1\le j\le T}$ from $j = i+1$ to $j = i$ by setting

$$(M_j - M_i)_{i\le j\le T} = (M_j(\xi_{i+1}) - M_i(\xi_{i+1}))_{i\le j\le T}$$

Finally we put

$$\vartheta_i(M) = Z_i + \left(\vartheta_{i+1}(M) - \xi_{i+1} - Z_i\right)^+ .$$

After carrying out these steps backwardly from $i = T$ down to $i = 0$ we end up with a family of martingale increments $(M_j - M_0)_{0\le j\le T}$, hence a martingale $(M_j)_{0\le j\le T}$, as $M_0 = 0$ without loss of generality. This martingale will be subsequently used to compute a dual upper bound for Y_0^* via

$$Y_0^{up} = E \max_{0\le j\le T}\left(Z_j - M_j\right).$$

The key step in the above procedure is to find a solution to minimization problem (27): Suppose that the martingale increments satisfying (26) for some fixed i may be parameterized as $\xi_{i+1}(\beta)$ where β is some generic parameter. Based on a set of simulated trajectories of X one may then estimate for some β (which we specify in more details below) the conditional variance

$$E\,\mathrm{Var}_i\left(\vartheta_{i+1}(M)-\xi_{i+1}(\beta)-Z_i\right)^+ =: E\,\mathrm{Var}_i U_i^+(\beta)$$

by using e.g. kernel estimators (e.g. see Liero (1989)), and next minimize with respect to β. In particular when the dimension of the parameter space is very small (typically one-dimensional) this may lead to a feasible Monte Carlo procedure. However, if the set of martingale increments $\xi_{i+1}(\beta)$ is "rich enough" and is moreover linearly structured in β, that is

$$\xi_{i+1}(\beta)=\sum_{k=1}^{K}\beta_k\mathfrak{m}_{i+1}^{(k)},$$

where $\beta=(\beta_1,\ldots,\beta_K)\in\mathbb{R}^K$ and the random variables $\mathfrak{m}_{i+1}^{(k)}, k=1,\ldots,K,$ satisfy (26) for $K\geq 1$ sufficiently large, we rather solve the dominating problem

$$\begin{aligned}\arg\min_{\beta\in\mathbb{R}^K} E\,\mathrm{Var}_i U_i(\beta) &:= \arg\min_{\beta\in\mathbb{R}^K} E\,\mathrm{Var}_i\left(\vartheta_{i+1}(M)-\xi_{i+1}(\beta)-Z_i\right)\\ &= \arg\min_{\beta\in\mathbb{R}^K} E\,\mathrm{Var}_i\left(\vartheta_{i+1}(M)-\sum_{k=1}^{K}\beta_k\mathfrak{m}_{i+1}^{(k)}\right). \qquad (28)\end{aligned}$$

The reason is twofold. On the one hand, if we succeed to find $\beta^\circ\in\mathbb{R}^K$ such that $E\,\mathrm{Var}_i U_i(\beta^\circ)$ is sufficiently small (if it were zero, we would have arrived at a surely optimal martingale increment), then since

$$\arg\min_{\beta\in\mathbb{R}^K} E\,\mathrm{Var}_{X_{T_i}} U_i^+(\beta)\leq E\,\mathrm{Var}_i U_i^+(\beta^\circ)\leq E\,\mathrm{Var}_i U_i(\beta^\circ),$$

$E\,\mathrm{Var}_i U_i^+(\beta^\circ)$ is generally even closer to zero and so β° can be considered a good approximation to (27) as well. On the other hand, most importantly, problem (28) can be treated as a linear regression problem,

$$[\beta^\circ,\gamma^\circ]=\arg\min_{\beta\in\mathbb{R}^K,\gamma\in\mathbb{R}^{K_1}} E\left|\vartheta_{i+1}(M)-\sum_{k=1}^{K}\beta_k\mathfrak{m}_{i+1}^{(k)}-\sum_{k=1}^{K_1}\gamma_k\psi_k(i,X_i)\right|^2, \qquad (29)$$

which employs an additional set of basis functions $\psi_k(t,x)$, $k=1,...,K_1$. To see this, note that (29) is equivalent to

$$[\beta^\circ,\gamma^\circ] = \underset{\beta\in\mathbb{R}^K,\gamma\in\mathbb{R}^{K_1}}{\arg\min} EE_i \left(\vartheta_{i+1}(M) - \sum_{k=1}^{K}\beta_k \mathfrak{m}_{i+1}^{(k)} - E_i\vartheta_{i+1}(M)\right.$$
$$\left. + E_i\vartheta_{i+1}(M) - \sum_{k=1}^{K_1}\gamma_k\psi_k(i,X_i)\right)^2$$
$$= \underset{\beta\in\mathbb{R}^K,\gamma\in\mathbb{R}^{K_1}}{\arg\min} \left\{E\,\mathrm{Var}_i\left(\vartheta_{i+1}(M) - \sum_{k=1}^{K}\beta_k \mathfrak{m}_{i+1}^{(k)}\right)\right.$$
$$\left. + E\left(E_i\vartheta_{i+1}(M) - \sum_{k=1}^{K_1}\gamma_k\psi_k(i,X_i)\right)^2\right\}.$$

Hence, β° satisfies (28). Moreover for γ° it holds

$$\gamma^\circ = \underset{\gamma\in\mathbb{R}^{K_1}}{\arg\min} E\left(E_i\vartheta_{i+1}(M) - \sum_{k=1}^{K_1}\gamma_k\psi_k(i,X_i)\right)^2. \tag{30}$$

Further, the regression procedure (29) delivers as by-product

$$\mathcal{C}_i(x) := \sum_{k=1}^{K_1}\gamma_k^\circ\psi_k(i,x),$$

an approximate continuation function that may be used afterwards to define a stopping rule and to simulate a corresponding lower biased estimation of Y_0^*.

Remark 18 *(i) In virtually all practical applications we are in a setting as described in Remark 17. In this environment we may model ξ_{i+1} as linear combinations of the form*

$$\xi_{i+1}(\beta) := \sum_{k=1}^{N_1}\beta_k^c\int_{T_i}^{T_{i+1}}\varphi_k^c(s,X_s)dW_s$$
$$+ \sum_{k=1}^{N_2}\beta_k^d\int_{T_i}^{T_{i+1}}\varphi_k^d(s,X_s,u)d\widetilde{N}(ds,du), \tag{31}$$

where $N_1+N_2=K$ and $\varphi_k^c(s,x)$ and $\varphi_k^d(s,x,u)$ are suitable sets of basis functions satisfying the conditions in Remark 17. In this setting, we have

$$\mathfrak{m}_{i+1}^{(k)} = \int_{T_i}^{T_{i+1}}\varphi_k^c(s,X_s)dW_s + \int_{T_i}^{T_{i+1}}\varphi_k^d(s,X_s,u)d\widetilde{N}(ds,du)$$

and $\beta = (\beta_1^c,\ldots,\beta_{N_1}^c,\beta_1^d,\ldots,\beta_{N_2}^d)\in\mathbb{R}^K$.

As an alternative, we may also take

$$\xi_{i+1}(\beta) := \sum_{k=1}^{K} \beta_k \left(B_{i+1}^{(k)} - B_i^{(k)} \right) \tag{32}$$

for an arbitrary given set of discounted tradables $\left(B_j^{(k)}\right)_{0 \le j \le T}$ *where the* $B_j^{(k)} = B^{(k)}(j, X_j)$ *are provided by some specific problem under consideration. For example it may happen that discounted European options are available in closed form. In any case, (31) and (32) satisfy the requirements (26) for any vector parameter* $\beta \in \mathbb{R}^K$.

(ii) Suppose that the system of basis martingale increments and basis functions in the regression based minimization (29) is sufficiently "rich" that there even exist $\beta^{\circ\circ}$ *and* $\gamma^{\circ\circ}$ *such that*

$$\vartheta_{i+1}(M) - \sum_{k=1}^{K} \beta_k^{\circ\circ} \mathfrak{m}_{i+1}^{(k)} - \sum_{k=1}^{K_1} \gamma_k^{\circ\circ} \psi_k(i, X_i) = 0 \quad a.s.$$

Then one would need only one trajectory for X *to identify* $\beta^{\circ\circ}$ *and* $\gamma^{\circ\circ}$ *via (29). This is a similar situation as discussed in Section 6.1: In practice when the system (31) is rich enough, a relatively low sample size will be sufficient to solve (29) effectively. This phenomenon will be confirmed by our numerical experiments in Section 7.*

Description of the Monte Carlo algorithm

Let us now spell out the empirical, implementable counterpart of the procedure described above. Based on a set of trajectories $\left(X_j^{(n)}\right)_{j=0,\dots,T}$, $n = 1, \dots, N$, we carry out the following procedure.

Step 1: At $i = T$ we set on each trajectory $\vartheta_T^{(n)} := \vartheta_T^{(n)}(M) := Z_T(X_T^{(n)})$ and $\left(M_j^{(n)} - M_T^{(n)}\right)_{T \le j \le T} = M_T^{(n)} - M_T^{(n)} = 0$ for $n = 1, \dots, N$.

Step 2: For $n = 1, \dots, N$ let $\left(M_j^{(n)} - M_{i+1}^{(n)}\right)_{i+1 \le j \le T}$ be constructed. For $i = T - 1$ down to $i = 0$, based on the N samples, we solve the regression problem

$$\left[\widehat{\beta}^{(i)}, \widehat{\gamma}^{(i)}\right] := \underset{\beta \in \mathbb{R}^K, \gamma \in \mathbb{R}^{K_1}}{\arg\min} \frac{1}{N} \sum_{n=1}^{N} \left(\vartheta_{i+1}^{(n)}(M) - \sum_{k=1}^{K} \beta_k \mathfrak{m}_{i+1}^{(k,n)} - \sum_{k=1}^{K_1} \gamma_k \psi_k(i, X_i^{(n)}) \right)^2 .$$

We then put

$$\widehat{\xi}_{i+1}^{(n)} := \sum_{k=1}^{K} \widehat{\beta}_k^{(i)} \mathfrak{m}_{i+1}^{(k,n)},$$
$$M_j^{(n)} - M_i^{(n)} := \left(M_j^{(n)} - M_{i+1}^{(n)} \right) + \widehat{\xi}_{i+1}^{(n)},$$

and

$$\vartheta_i^{(n)}(M) := Z_i^{(n)} + \left(\vartheta_{i+1}^{(n)}(M) - \widehat{\xi}_{i+1}^{(n)} - Z_i^{(n)} \right)^+ .$$

Step 3: We simulate $\widetilde{N}$ new independent samples $\left(\widetilde{X}_j^{(n)}\right)_{j=0,\dots,T}$, $n = 1,...,\widetilde{N}$, which give rise to the new martingale samples

$$\widetilde{M}_i^{(n)} = \sum_{j=1}^{i} \sum_{k=1}^{K} \widehat{\beta}_k^{(j)} \widetilde{\mathfrak{m}}_j^{(k,n)}, \quad k = 1,...,K, \quad n = 1,...,\widetilde{N}.$$

Then, an upper biased estimate for the upper bound is given by

$$\widehat{Y}_0^{\mathrm{up}} := \frac{1}{\widetilde{N}} \sum_{n=1}^{\widetilde{N}} \max_{0 \le i \le T} \left(Z_i^{(n)}(\widetilde{X}_i^{(n)}) - \sum_{j=1}^{i} \sum_{k=1}^{K} \widehat{\beta}_k^{(j)} \widetilde{\mathfrak{m}}_j^{(k,n)} \right). \tag{33}$$

Step 4: Based on the stopping rule

$$\tau_0(X_i) := \inf\{i \ge 0 : Z_i(X_i) \ge \sum_{k=1}^{K_1} \widehat{\gamma}_k^{(i)} \psi_k(i, X_i)\} \tag{34}$$

we put

$$\widehat{Y}_0^{\mathrm{low}} := \frac{1}{\widetilde{N}} \sum_{n=1}^{\widetilde{N}} Z^{(n)}_{\tau_0(\widetilde{X}^{(n)})}(\widetilde{X}^{(n)}_{\tau_0(\widetilde{X}^{(n)})}).$$

which yields a lower biased estimate to Y_0^*.

At this point, let us briefly compare our algorithm with the algorithm from Belomestny et al. (2009). The methodology of Belomestny et al. (2009) to compute dual martingales is built upon a procedure to numerically approximate Clark-Ocone derivatives of an approximative Snell envelope Y with respect to a Wiener filtration. The key ingredient there is to approximate the Clark-Ocone derivative on a (fine) grid $\pi = \{t_0, \dots, t_N\}$ which contains the exercise grid $\{0, 1, \dots, T\}$ using the estimator,

$$Z_{t_j}^{\pi} := \frac{1}{\Delta_j^{\pi}} \mathbb{E}_{t_j}[\Delta^{\pi} W_j \; Y_{i+1}], \tag{35}$$

where $\Delta_j^\pi = t_{j+1} - t_j$ and $\Delta^\pi W_j = W_{t_{j+1}} - W_{t_j}$. Due to (35), Belomestny et al. (2009) in fact have to carry out a regression at each $t_j \in \pi$ on the fine grid π. In contrast, our algorithm only needs to carry out regressions on the coarser grid of the possible exercise dates $\{0, \dots, T\} \subset \pi$. Moreover, (35) requires an input approximation Y, which needs to be obtained by another method, such as the method of Longstaff and Schwartz (2001). The tightness of the upper bound obviously also depends on the quality of this approximation. Finally, we underline that obtaining numerically the Clark-Ocone derivative (35) in a non-Wiener filtration (e.g. filtrations generated by Lévy processes) is not so straightforward, while in our framework, the regression procedure (29) may include jump martingales as depicted in Remark 17.

7 Numerical examples

In this section we present the numerical results pertaining to the backward algorithm described in Section 6. The performance and accuracy of our algorithm is illustrated by applying it to the pricing of a Bermudan basket-put on 5 assets (see e.g. Bender et al. (2006a)) and a Bermudan max-call on 2 and 5 assets (see e.g. Andersen and Broadie (2004)). We furthermore study our method in case that the product is less well posed in the sense that the deltas can become highly negative and reach a point where hedging essentially becomes impossible. This has repercussions on the choice of suitable martingales. In our test, we price an up-and-out max-call option, see e.g. Desai et al. (2010). In the examples, the risk-neutral dynamics of each asset are given as

$$dX_t^d = (r - \delta) X_t^d dt + \sigma X_t^d dW_t^d, \quad d = 1, ..., D,$$

where $D \in \mathbb{N}$ is the number of assets, W_t^d, $d = 1, ..., D$, are independent one-dimensional Brownian motions, and r, δ and σ are constant real valued parameters. Exercise opportunities are equally spaced at times $T_j = \frac{jT}{J}$, $j = 0, ..., J$. The discounted payoff from exercise at time t is given by

(i) $Z_t(X_t) = e^{-rt}(K - \frac{X_t^1 + \ldots + X_t^D}{D})^+$ for the Bermudan basket-put,

(ii) $Z_t(X_t) = e^{-rt}(\max(X_t^1, \ldots, X_t^D) - K)^+$ for the Bermudan max-call,

(iii) $Z_t(X_t) = \begin{cases} e^{-rt}(\max(X_t^1, \ldots, X_t^D) - K)^+, & \text{if } \max_{\substack{0 \le u \le t \\ 1 \le d \le D}} X_u^d \le B, \\ 0 & \text{else} \end{cases}$

for the up-and-out Bermudan max-call with knock out barrier $B > 0$.

We denote $X_t = (X_t^1, \ldots, X_t^D)$. For both products, the time interval $[T_j, T_{j+1}]$, $j = 0, \ldots, J - 1$, is partitioned into L equally spaced subintervals of width $\Delta t = \frac{T}{N}$ with $N = J \times L$. The implementation can be outlined as follows. We first simulate M independent samples of Brownian increments

$$\Delta W_i = (\Delta W_i^{1,(m)}, \ldots, \Delta W_i^{D,(m)}), \quad i = 1, \ldots, N, \ m = 1, \ldots, M.$$

Then the trajectories of $X_i^{(m)} = (X_i^{1,(m)}, \ldots, X_i^{D,(m)})$, $i = 1, \ldots, N$, $m = 1, \ldots, M$, are given by

$$X_i^{d,(m)} = X_{i-1}^{d,(m)} \exp\left((r - \delta - \frac{1}{2}\sigma^2)\Delta t + \sigma \Delta W_i^{d,(m)} \right), \tag{36}$$

for $d = 1, \ldots, D$ and initial data $X_0 = (X_0^1, \ldots, X_0^D)$.

We now carry out the backward Monte Carlo regression algorithm as described in Section 6. In this Wiener setting, we recall Remark 18 (i) and choose as the spanning family of surely optimal martingales the Wiener integrals $\mathfrak{m}_{i+1}^{(k)} = \int_{T_i}^{T_{i+1}} \varphi_k^c(s, X_s) dW_s$. More precisely, we solve in a first step the regression problem backward in time

$$(\widehat{\beta}^{(i)}, \widehat{\gamma}^{(i)}) := \arg\min_{(\beta,\gamma)} \frac{1}{M} \sum_{m=1}^{M} \left[\vartheta_{i+1}^{(m)} - \sum_{k=1}^{K} \beta_k \int_i^{i+1} \varphi_k(u, X_u^{(m)}) dW_u^{(m)} \right. \\ \left. - \sum_{k=1}^{K_1} \gamma_k \psi_k(i, X_i^{(m)}) \right]^2, \quad i = T-1, \ldots, 0, \tag{37}$$

for two families of basis functions $(\varphi_k) = \left(\varphi_k^{(d)}\right)$ with $\varphi_k^{(d)} = \varphi_k^{(1)}$, and (ψ_k), chosen as explained below. In (37) the Wiener integrals are approximated by the standard Euler scheme, using the same Brownian increments as in (36). Finally, a new independent simulation is launched and we estimate an upper bound $\widehat{Y}_0^{up}$ and a lower bound $\widehat{Y}_0^{low}$ by means of (33) and (34).

As one may expect, the choice of basis functions is crucial to obtain tight upper and lower bounds. In this respect, special information on the pricing problem may help us finding suitable basis functions. One way of retrieving additional information is to employ martingales representations and Itô's formula to obtain more specific insights into the structure of the pricing dynamics. We illustrate this by considering the following stylized setting: By the Markov property of X, we have that $E_t(Z_T(X_T)) = f(t, X_t)$ for some measurable function $f(t,x)$ and $0 \le t \le T = T_J$. Let us assume that $f(t,x)$ is differentiable in x. Then, by Itô's formula and the fact that $E_t(Z_T)$ is a martingale we have

$$Z_T(X_T) - E_{T_{J-1}}(Z_T(X_T)) = \sum_{d=1}^{D} \sigma \int_{T_{J-1}}^{T} f_{x^d}(t, X_t) X_t^d dW_t^d.$$

Recall that $\widehat{\vartheta}_T = Z_T$ and $E_{T_{J-1}}(Z_T(X_T))$ can be expressed in the following form

$$E_{T_{J-1}}(Z_T(X_T)) = e^{-rT_{J-1}} EP(T_{J-1}, X_{T_{J-1}}; T),$$

where $EP(t, x; T)$ is the price of the corresponding European option with maturity T at time t. Thus, it is natural to choose from time T to time T_{J-1} European option values for the basis $(\psi_k(t,x))$ and the corresponding European deltas multiplied by the value of the underlying asset for the basis $(\varphi_k(t,x))$. Although for the following steps $(t < T_{J-1})$ there is no easy way to predict optimal choices of (ψ_k) and (φ_k), the above analysis suggests to always include the still-alive European options into the basis (ψ_k) and include the information on the European deltas into the basis (φ_k). In fact, based on similar arguments, this choice of basis functions was already proposed in Belomestny et al. (2009).

7.1 Bermudan basket-put

In this example, we take the following parameter values,

$$r = 0.05, \quad \delta = 0, \quad \sigma = 0.2, \quad D = 5, \quad T = 3,$$

and

$$X_0^1 = \ldots = X_0^D = x_0, \quad K = 100.$$

We perform the simulation of the underlying asset X from (36) with a time step size $\Delta t = 0.01$. For $T_j \le t < T_{j+1}$, $j = 0, \ldots, J-1$, we choose the set

$$\Big\{1, Pol_3(X_t), Pol_3(EP(t, X_t; T_{j+1})), Pol_3(EP(t, X_t; T_J))\Big\}$$

as basis functions (ψ_k), where $Pol_n(y)$ denotes the set of monomials of degree up to n in the components of a vector y and $EP(t, X; T)$ denotes the (approximated) value of a European basket-put with maturity T at time t. Recall that the family (ψ_k) serves as the regression basis for the continuation value. Further we choose

$$\left\{1, \left(X_t^d \frac{\partial EP(t, X_t; T_{j+1})}{\partial X_t^d}\right)_{1\le d\le D}, \left(X_t^d \frac{\partial EP(t, X_t; T_J)}{\partial X_t^d}\right)_{1\le d\le D}\right\}$$

as a regression basis (φ_k) spanning the family of the surely optimal martingales. Since there is no closed-form formula for the still-alive European basket-put, we use the moment-matching method to approximate their values (see e.g. Brigo et al. (2004), and Lord (2006)). To this end, Let $S_t = \dfrac{X_t^1 + \ldots + X_t^D}{D}$, and consider another asset G_t whose risk-neutral dynamic follows

$$dG_t = rG_t dt + \tilde{\sigma} G_t dW_t^1,$$

where $\tilde{\sigma}$ is a constant. The value of the European put on this asset can be easily computed by the well-known Black-Scholes formula, that is,

$$E[e^{-rT}(K - G_T)^+] = BS(G_0, r, \tilde{\sigma}, K, T). \tag{38}$$

If S_T and G_T have the same moments up to two, then the Black-Scholes price in (38) can be regarded as a good approximation for the value of the European basket-put $E\left(e^{-rT}(K-S_T)^+\right)$, for details see Lord (2006). Since

$$E(S_T) = \frac{1}{D}\sum_{d=1}^{D} X_0^d e^{rT},$$

$$E(S_T^2) = \frac{1}{D^2}e^{2rT}\left(\sum_{i,j=1}^{D} X_0^i X_0^j \exp(1_{i=j}\sigma^2 T)\right)$$

and

$$E(G_T) = G_0 e^{rT}, \qquad E(G_T^2) = G_0^2 e^{2rT+\tilde{\sigma}^2 T},$$

we can simply set

$$G_0 = \frac{1}{D}\sum_{d=1}^{D} X_0^d$$

and

$$\tilde{\sigma}^2 = \frac{1}{T}\ln\left(\frac{1}{(\sum_{d=1}^{D} X_0^d)^2}\sum_{i,j=1}^{D} X_0^i X_0^j \exp(1_{i=j}\sigma^2 T)\right).$$

The European deltas can be approximated by

$$\frac{\partial BS}{\partial G_0}\frac{\partial G_0}{\partial X_0^d} = -\mathcal{N}(-d_1)\frac{1}{D}, \quad d = 1,\ldots,D,$$

where $d_1 = \dfrac{\ln(\frac{G_0}{K}) + (r+\frac{\tilde{\sigma}^2}{2})T}{\tilde{\sigma}\sqrt{T}}$ and $\mathcal{N}$ denotes the cumulative standard normal distribution function. These formulas are straightforwardly extended to the pricing at times $t > 0$.

Table 1: Lower and upper bounds for Bermudan basket-put on 5 assets with parameters $r = 0.05$, $\delta = 0$, $\sigma = 0.2$, $K = 100$, $T = 3$ and different J and x_0

J	x_0	Low (SE)	Up (SE)	BKS Price Interval
	90	10.000 (0.000)	10.000 (0.000)	[10.000, 10.004]
3	100	2.164 (0.007)	2.168 (0.005)	[2.154, 2.164]
	110	0.539 (0.004)	0.555 (0.003)	[0.535, 0.540]
	90	10.000 (0.000)	10.000 (0.000)	[10.000, 10.000]
6	100	2.407 (0.006)	2.432 (0.005)	[2.359, 2.412]
	110	0.573 (0.003)	0.608 (0.003)	[0.569, 0.580]
	90	10.000 (0.0000)	10.000 (0.000)	[10.000, 10.005]
9	100	2.475 (0.0063)	2.539 (0.006)	[2.385, 2.502]
	110	0.5915 (0.0034)	0.635 (0.003)	[0.577, 0.600]

The numerical results are shown in Table 1. We use 1000 paths for estimating a surely optimal martingale and the continuation function via the regression procedure. Another 300000 paths are used to compute the lower bound and 5000 paths are used to compute the upper bound. Note that we have chosen a relatively small sample size (1000) for estimating the martingale in the regression procedure and a small size (5000) for the simulation of the upper bound. This confirms the general idea behind our algorithm, that for a rich enough family of integrable martingales, only a small number of samples are required for identifying a good approximation to a surely optimal martingale. Moreover, as the objective of our method is the minimization of variance, for the subsequent calculation of a tight (upper biased) upper bound, a relatively small sample size is sufficient again. We compare our results to the price intervals obtained in Bender et al. (2006a) which are displayed in the last column of Table 1. Typically, the bulk of the computation time is occupied with the simulation of the lower bound which amounts to about 80% of the total computation time whereas the simulation of the upper bound amounts to about 8% and the regression procedure to 12%. In this regard, producing good upper bounds in order of a few minutes is a drastic computational improvement compared to Bender et al. (2006a) whose upper bounds are computed with nested Monte Carlo simulation which has a complexity typically of the order

$$D \times N_{\text{outer}} \times N_{\text{inner}} \times J^2.$$

Our method computes the upper bound with a complexity of

$$D \times N \times J,$$

where N denotes the sample size for calculating the upper bound. Moreover, the strength of our method in this example is that N can be chosen to be small whereas in Bender et al. (2006a), the choice was $N_{\text{outer}} = 2000$ and $N_{\text{inner}} = 1000$, leading to a complexity 400 times as high as in our case.

7.2 Bermudan max-call

We use the same parameter values as in Section 7.1 except $\delta = 0.1$ and $D = 2$ or 5. As in the previous example we use European (call) options in the basis (ψ_k) and the corresponding deltas in the basis (φ_k). The value of the European max-call option is computed by the following formula (Johnson (1987)),

$$\begin{aligned} C_{max} = &\sum_{l=1}^{D} X_0^l \frac{e^{-\delta T}}{\sqrt{2\pi}} \int_{(-\infty, d_+^l]} \exp[-\frac{1}{2}z^2] \prod_{\substack{l'=1 \\ l' \neq l}}^{D} \mathcal{N}\left(\frac{\ln \frac{X_0^l}{X_0^{l'}}}{\sigma\sqrt{T}} - z + \sigma\sqrt{T} \right) dz \\ &- Ke^{-rT} + Ke^{-rT} \prod_{l=1}^{D} \left(1 - \mathcal{N}\left(d_-^l\right)\right), \end{aligned} \tag{39}$$

where

$$d_-^l := \frac{\ln \frac{X_0^l}{K} + (r - \delta - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}, \quad d_+^l = d_-^l + \sigma\sqrt{T}.$$

Moreover, straightforward computations reveal that the deltas are given by

$$\frac{\partial C_{max}}{\partial X_0^l} = \frac{e^{-\delta T}}{\sqrt{2\pi}} \int_{(-\infty, d_+^l]} \exp[-\frac{1}{2}z^2] \prod_{\substack{l'=1 \\ l' \neq l}}^{D} \mathcal{N}\left(\frac{\ln \frac{X_0^l}{X_0^{l'}}}{\sigma\sqrt{T}} - z + \sigma\sqrt{T} \right) dz, \tag{40}$$

and that C_{max} satisfies the linear homogeneity[3]

$$C_{max} = \sum_{l=1}^{D} X_0^l \frac{\partial C_{max}}{\partial X_0^l} + K \frac{\partial C_{max}}{\partial K}. \tag{41}$$

Table 2: Lower and upper bounds for Bermudan max-call with parameters $r = 0.05$, $\delta = 0.1$, $\sigma = 0.2$, $K = 100$, $T = 3$ and different D and x_0.

D	x_0	Low (SE)	Up (SE)	A&B price interval
2	90	8.0556 (0.021)	8.15284 (0.014)	[8.053, 8.082]
	100	13.8850 (0.027)	14.0145 (0.019)	[13.892, 13.934]
	110	21.3671 (0.0319)	21.5187 (0.022)	[21.316, 21.359]
5	90	16.5973 (0.0296)	16.7718 (0.027)	[16.602, 16.655]
	100	26.1325 (0.0356)	26.3440 (0.031)	[26.109, 26.292]
	110	36.7348 (0.0403)	37.0431 (0.039)	[36.704, 36.832]

The numerical results are shown in Table 2. They are based on 1000 paths for the regression procedure, 300000 paths for computing the lower bound and 5000 paths for computing the upper bound. As before, we have chosen a relatively small number of samples (1000) for estimating the martingale in the regression procedure. To achieve upper bounds with standard errors of the same magnitude as the lower bounds, again only a small number of simulation paths is needed. In this example, 5000 paths were sufficient. The integrals in (39) and (40) are numerically evaluated using a simple adaptive Gauss-Kronrod procedure with 31 points. The price intervals in the last column are quoted from Andersen and Broadie (2004). The worst case complexity of Andersen and Broadie (2004) is

$$D \times N_{\text{outer}} \times N_{\text{inner}} \times J^2,$$

where D denotes the dimension, N_{outer} the number of the outer paths and N_{inner} the number of the inner paths and J the number of exercise time points, see eqn. (18) in Andersen and Broadie (2004). In this example, Andersen and Broadie (2004) chose $N_{\text{outer}} = 1500$ and $N_{\text{inner}} = 10000$ paths. The complexity of our algorithm is

$$D \times N \times J$$

where N is the number of simulation paths for the upper bound. Thus, it should be stressed that the complexity of our algorithm grows linearly and by

[3]Compare also with (Johnson, 1987, eq. (9)).

the device of having the minimization of the variance as its objective, typically we only need few simulation paths to compute upper bounds with low standard errors. The reduction of computation time by a factor of the ratio of the two complexities has been observed in experiments with several one-dimensional products. Typically, of the total computation time, the regression procedure accounts for 5%, the calculation of the lower bound for 65% and the calculation of the upper bound for 30%.

7.3 Bermudan up-and-out max-call

Let us now carry out the analysis for the same product as in Section 7.2 with the additional feature of an upper knock out barrier. In the case of standard one-dimensional European up-and-out call options, it is well known that the delta becomes negative when the spot approaches the barrier and once the barrier is crossed, the delta collapses to zero as the option itself has expired. This singular behaviour of the delta makes delta-hedging an up-and-out call virtually impossible when the spot is very close to the barrier. This feature makes guessing a good class of approximating martingales challenging because both singularities of collapsing to zero at crossing the barrier and approaching negative values when the spot is close to the barrier have to be account for. For (ψ_k), we use the following basis: denoting $Y_t := \max_{\substack{0\leq u\leq t \\ 1\leq d\leq D}} X_u^d$, we employ

$$\left\{1 \times \eta_t \times 1_{Y_t \leq B}, Pol_3(X_t) \times \eta_t \times 1_{Y_t \leq B}\right\}$$

as a regression basis where $\eta_t = 1 - e^{-\alpha \frac{B-Y_t}{B}}$ serves as a damping factor which forces the basis elements to drop smoothly to zero when the dynamics of Y_t get close to the barrier. The constant α serves as a weighting factor on the relative distance to the barrier and in our numerical experiments, we tested several values for the factor α. This way, we account for the bahaviour that up-and-out call options drop smoothly to zero when the barrier is approached. The indicator takes care of the fact once the barrier is hit, the option has expired, thus its value remains zero. In a similar fashion, we employ as martingale integrand (φ_k) the deltas of the European max-call expiring at the hitting time $\tau := \inf\{t \geq 0 : Y_t > B\}$,

$$\left\{\widetilde{\eta}_t \times \left(X_t^d \frac{\partial C_{max}(t, X_t; \tau)}{\partial X_t^d}\right)_{1\leq d\leq D} \times 1_{\{Y_t \leq B\}}\right\},$$

where $\frac{\partial C_{max}}{\partial x^d}$ is from (40) and where the damping factor $\widetilde{\eta}_t = 1 - 2e^{-\tilde{\alpha} \frac{B-Y_t}{B}}$ caters for the decay of the deltas of up-and-out call options whose spot values are close to the barrier. As before, the indicator accounts for the fact once the barrier is crossed, the option has expired and thus all deltas also have expired.

The results of the numerical experiments are displayed in Table 3. We have chosen a sample size of 1000 paths for the regression procedure, 300000 samples

Table 3: Lower and upper bounds for up-and-out Bermudan max-call with parameters $r = 0.05$, $\delta = 0.1$, $\sigma = 0.2$, $K = 100$, $T = 3$ and different D and x_0.

Barrier	D	x_0	Low (SE)	Up (SE)
170	2	90	7.998 (0.018)	8.092 (0.024)
		100	13.645 (0.024)	13.758 (0.027)
		110	20.784 (0.026)	20.883 (0.030)
170	5	90	16.198 (0.022)	16.426 (0.033)
		100	24.808 (0.029)	25.248 (0.039)
		110	33.222 (0.030)	34.458 (0.054)

in the simulation of the lower bound and 5000 samples on the simulation of the upper bound. The figures for dimension $D = 2$ are based on damping factors with the weight coefficients $\alpha = 20$ and $\tilde{\alpha} = 16$ which seem to produce a best overall gap between lower and upper bound. The figures for dimension $D = 5$ are based on damping factors with the weight coefficients $\alpha = 20$ and $\tilde{\alpha} = 16$. In this example, our method shows a deficiency which becomes clear for the case of $B = 150$. Moving closer to the barrier and thus increasing the probability that the option knocks out almost surely before maturity leads to a more complex behaviour in the deltas. The increasing gap between lower and upper bounds is indicating that the choice of approximating martingales is not good enough. Testing several different bases (ψ_k) and (φ_k) reveals that the lower bound is very robust concerning its regression basis whereas the upper bound is sensitive in terms of the choice of its regression basis. The conlcusion we draw from this example is that the class of martingales spanned by (φ_k) does not provide a rich enough class which allows for calculating a tight upper bound to the Snell envelope. Hence, for products with complex behaviour in the deltas which complicates the choice of good martingales, our method has problems to produce tight upper bounds. In such cases, standard inner simulation based methods as in Andersen and Broadie (2004) or in Desai et al. (2010) produce better upper bounds at the expense of a drastic increase of complexity however.

8 Appendix: example from Section 6.1

For the cash-flow process from (23), i.e.

$$T = 2, \quad Z_0 = 0, \quad Z_1 = \xi, \quad Z_2 = 1, \quad \text{with} \quad \mathbb{P}(\xi \in dx) = \frac{1}{2} 1_{[0,2]}(x) dx,$$

we obviously have that

$$\begin{aligned} Y_2^* &= 1, \\ Y_1^* &= \max(\xi, 1) = 1 + (\xi - 1)^+ \\ Y_0^* &= E_0 Y_1^* = 1 + \frac{1}{2} \int (x-1)^+ 1_{[0,2]}(x) dx = \frac{5}{4}, \end{aligned}$$

and so the corresponding (almost sure) Doob martingale has increments given by

$$M_2^* - M_1^* = Y_2^* - E_1 Y_2^* = 0$$
$$M_1^* - M_0^* = M_1^* = Y_1^* - E_0 Y_1^* = (\xi - 1)^+ - \frac{1}{4}.$$

Now consider for $\alpha \in \mathbb{R}$ the family of martingales $\{\alpha M^* : \alpha \in \mathbb{R}\}$. Let us note that we have

$$\begin{aligned}\mathcal{Z}(\alpha, \xi) &:= \max\left(Z_0 - \alpha M_0^*, Z_1 - \alpha M_1^*, Z_2 - \alpha M_2^*\right) \\ &= 1_{\xi \le 1}\left(1 + \frac{1}{4}\alpha\right)^+ + 1_{\xi > 1}\left(\xi + \alpha\left(\frac{5}{4} - \xi\right)\right)^+ \\ &=: 1_{\xi \le 1}\mathcal{Z}_-(\alpha) + 1_{\xi > 1}\mathcal{Z}_+(\alpha, \xi).\end{aligned}$$

For the present case study we will later recall on the following facts which follow by straightforward computations.

Proposition 19 *It holds that*

$$E\mathcal{Z}(\alpha, \xi) = \begin{cases} \frac{1}{64}(\alpha + 4)\frac{9\alpha - 4}{\alpha - 1} & \text{if} \quad \alpha > \frac{8}{3}, \\ \frac{1}{64}\frac{(3\alpha - 8)^2}{1 - \alpha} & \text{if} \quad \alpha \le -4, \end{cases},$$

$$\underset{\alpha \ge \frac{8}{3}}{\arg\min} E\mathcal{Z}(\alpha, \xi) = \frac{8}{3} \text{ with } E\mathcal{Z}\left(\frac{8}{3}, \xi\right) = \frac{5}{4} = Y_0^*, \text{ and } Var\,\mathcal{Z}\left(\frac{8}{3}, \xi\right) = \frac{125}{432},$$

$$\underset{\alpha \le -4}{\arg\min} E\mathcal{Z}(\alpha, \xi) = -4 \text{ with } E\mathcal{Z}(-4, \xi) = \frac{5}{4} = Y_0^*, \text{ and } Var\,\mathcal{Z}(-4, \xi) = \frac{125}{48}.$$

Now consider a sample of i.i.d. realizations $(\xi^{(n)})$, $n = 1, ..., N$, and for each $\alpha \in \mathbb{R}$ the estimator

$$\begin{aligned}\overline{\mathcal{Z}_N(\alpha)} &:= \frac{1}{N}\sum_{n=1}^{N}\mathcal{Z}\left(\alpha, \xi^{(n)}\right) \qquad (42) \\ &= \frac{1}{N}\sum_{n=1}^{N}\left(1_{\xi^{(n)} \le 1}\mathcal{Z}_-(\alpha) + 1_{\xi^{(n)} > 1}\mathcal{Z}_+\left(\alpha, \xi^{(n)}\right)\right) \\ &= \pi_N \mathcal{Z}_-(\alpha) + \frac{1}{N}\sum_{n=1}^{N} 1_{\xi^{(n)} > 1}\mathcal{Z}_+^{(n)}(\alpha),\end{aligned}$$

where $\pi_N := \#\left\{n : \xi^{(n)} \le 1,\ 1 \le n \le N\right\}/N$, as an estimate for the expected value

$$E_0 \max\left(Z_0 - \alpha M_0^*, Z_1 - \alpha M_1^*, Z_2 - \alpha M_2^*\right).$$

Minimization of the (unbiased) sample estimator (42) comes down to solving for

$$\arg\min_{\alpha} \overline{\mathcal{Z}_N(\alpha)} = \arg\min_{\alpha} \left\{ \pi_N \mathcal{Z}_-(\alpha) + \frac{1}{N}\sum_{n=1}^{N} 1_{\xi^{(n)}>1} \mathcal{Z}_+^{(n)}(\alpha) \right\} \tag{43}$$

$$= \arg\min_{\alpha} \left\{ \pi_N \left(1+\frac{1}{4}\alpha\right)^+ + \frac{1}{N}\sum_{n=1}^{N} 1_{\xi^{(n)}>1} \left(\xi^{(n)} + \alpha\left(\frac{5}{4} - \xi^{(n)}\right)\right)^+ \right\}.$$

W.l.o.g. we may assume that $\xi^{(n)} \neq 5/4$, $\xi^{(n)} \neq 1$, and $\xi^{(n)} \neq 2$ for all n as the event that ξ attains a constant $\kappa \in [0,2]$ has zero probability.

We now distinguish and discuss the following possible cases.

Case I: $N = 1$

(a) If $\xi^{(1)} < 1$ then for any $\alpha \leq -4$ we get $\overline{\mathcal{Z}_1(\alpha)} = 0$, leading to $\alpha_{\inf} \in (-\infty, -4]$ and

$$\mathcal{Z}(\alpha_{\inf}, \xi) = 1_{\xi>1}\left(\xi + \alpha_{\inf}\left(\frac{5}{4} - \xi\right)\right)^+, \quad \alpha_{\inf} \leq -4.$$

So in view of Proposition 19 the best choice is $\alpha_{\inf} = -4$, after which an independent new simulation $\widetilde{\xi}^{(1)}, ..., \widetilde{\xi}^{(\widetilde{N})}$ would yield the estimate

$$Y^{\widetilde{N},-4} := \frac{1}{\widetilde{N}}\sum_{n=1}^{\widetilde{N}} \mathcal{Z}\left(-4, \widetilde{\xi}^{(n)}\right) \quad \text{with} \quad E\,Y^{\widetilde{N},-4} = Y_0^*, \quad \text{and} \quad \operatorname{Var} Y^{\widetilde{N},-4} = \frac{125}{48\widetilde{N}}, \tag{44}$$

due to the optimal but not surely optimal martingale $-4M^*$.

(b) If $1 < \xi^{(1)} < 5/4$ then $\pi_1 = 0$ and we have $\overline{\mathcal{Z}_1(\alpha_{\inf})} = 0$ for $\alpha_{\inf} = \xi^{(1)}/(\xi^{(1)} - 5/4) < -4$. By Proposition 19 we then have due to an independent new sample of size $\widetilde{N}$,

$$Y^{\widetilde{N},\alpha_{\inf}} := \frac{1}{\widetilde{N}}\sum_{n=1}^{\widetilde{N}} \mathcal{Z}\left(\alpha_{\inf}, \widetilde{\xi}^{(n)}\right) \quad \text{with} \quad E\,Y^{\widetilde{N},\alpha_{\inf}} = \frac{1}{64}\frac{(3\alpha_{\inf} - 8)^2}{1-\alpha_{\inf}} > Y_0^*,$$

and a significant non-zero variance decay proportional to $\widetilde{N}^{-1}$.

(c) If $1 < \xi^{(1)} < 5/4$ then $\pi_1 = 0$ and we get $\overline{\mathcal{Z}_1(\alpha_{\inf})} = 0$ for $\alpha_{\inf} = \xi^{(1)}/(\xi^{(1)} - 5/4) > 8/3$. By Proposition 19 we then have due to an independent new sample of size $\widetilde{N}$,

$$Y^{\widetilde{N},\alpha_{\inf}} := \frac{1}{\widetilde{N}}\sum_{n=1}^{\widetilde{N}} \mathcal{Z}\left(\alpha_{\inf}, \widetilde{\xi}^{(n)}\right) \quad \text{with} \quad E\,Y^{\widetilde{N},\alpha_{\inf}} = \frac{1}{64}(\alpha_{\inf}+4)\frac{9\alpha_{\inf} - 4}{\alpha_{\inf} - 1} > Y_0^*,$$

and a significant non-zero variance decay proportional to $\widetilde{N}^{-1}$.

Case II: $N > 1$ and $\xi^{(n)} < 5/4$ for all n (probability $(5/8)^N$)

(a) If there is at least one n with $1 < \xi^{(n)} < 5/4$ we find for any $\alpha_{\inf} \le -A < -4$ with large enough $A > 0$, $\overline{\mathcal{Z}_N(\alpha_{\inf})} = 0$. We thus obtain an upper bound estimator $Y^{\tilde{N},\alpha_{\inf}}$ similar to the one in Case I-b.

(b) If for all n, $\xi^{(n)} < 1$ (probability $(1/2)^N$) one may "luckily" find $\alpha_{\inf} = -4$ and then apply the estimator from (44), see Case I-a.

Case III: $N > 1$ and $\xi^{(n)} > 5/4$ for all n (probability $(3/8)^N$)

By taking $\alpha_{\inf} \ge A > 8/3$ for large enough $A > 0$, we obtain $\overline{\mathcal{Z}_N(\alpha_{\inf})} = 0$. Thus, the corresponding upper bound estimator has properties similar to Case I-c (depending on $\alpha_{\inf}$).

Case IV: $N > 1$ and $\xi^{(n)} > 5/4$ and $\xi^{(n')} < 5/4$ for some pair $1 \le n, n' \le N$, (probability $1 - (5/8)^N - (3/8)^N \to 1$ as $N \to \infty$)

This is the most interesting case and also the most probable one for large N. We now have that $\overline{\mathcal{Z}_N(\alpha)} \to +\infty$ for $\alpha \to \pm\infty$. Further we observe that the function

$$\alpha \mapsto \overline{\mathcal{Z}_N(\alpha)}$$

is a non-negative convex piecewise linear map. As a consequence, this map attains a global (non-negative) minimum at some possibly non-unique $-\infty < \alpha_{\inf} < \infty$. Let us define the set of global minima

$$\mathcal{A}_{\inf} := \left\{ \alpha_{\inf} \in \mathbb{R} : \overline{\mathcal{Z}_N(\alpha)} \ge \overline{\mathcal{Z}_N(\alpha_{\inf})} \ \text{ for all } \ \alpha \in \mathbb{R} \right\}$$

and consider the collection of "kink points"

$$\alpha^{(0)} = -4, \quad \alpha^{(n)} = \frac{\xi^{(n)}}{\xi^{(n)} - \frac{5}{4}} \in (-\infty, 4) \cup \left(\frac{8}{3}, \infty \right), \quad \text{for all } n \text{ with } 1 < \xi^{(n)} < 2,$$

denoted by $\mathcal{K}$. Then we claim that

$$\mathcal{A}_{\inf} \subset (-\infty, -4] \cup (8/3, \infty).$$

If not, there must be some $\beta \in \mathcal{A}_{\inf}$ with $-4 < \beta \le 8/3$. Since $\beta \notin \mathcal{K}$, there must exist $\alpha_{\inf}, \alpha'_{\inf} \in \mathcal{A}_{\inf} \cap \mathcal{K}$ such that $\alpha_{\inf} \le -4 < \beta \le 8/3 < \alpha'_{\inf}$ and $[-4, 8/3] \subset [\alpha_{\inf}, \alpha'_{\inf}] \subset \mathcal{A}_{\inf}$. Thus, in particular $1 \in \mathcal{A}_{\inf}$. However, this is impossible. Indeed, $1 \in \mathcal{A}_{\inf}$ would imply that $\overline{\mathcal{Z}_N(\alpha)}$ attains a global (and thus also local) minimum at $\alpha = 1$. Now for small enough δ we have that (note that the sample is fixed)

$$\begin{aligned}
\overline{\mathcal{Z}_N(1+\delta)} &= \pi_N \left(1 + \frac{1}{4}(1+\delta) \right)^+ + \frac{1}{N} \sum_{n=1}^{N} 1_{\xi^{(n)}>1} \left(\xi^{(n)} + (1+\delta)\left(\frac{5}{4} - \xi^{(n)} \right) \right)^+ \\
&= \frac{5}{4} + \delta \left(\frac{1}{4}\pi_N + \frac{1}{N} \sum_{n=1}^{N} 1_{\xi^{(n)}>1} \left(\frac{5}{4} - \xi^{(n)} \right) \right) =: \frac{5}{4} + \delta \cdot (*)
\end{aligned}$$

which never attains a local minimum at $\delta = 0$, except if $(*) = 0$. However, it is easy to check that the latter case corresponds to

$$\frac{1}{N}\sum_{n=1}^{N} 1_{\xi^{(n)}>1}\left(\xi^{(n)} - 1\right) = \frac{1}{4}$$

which is an event of probability zero.

The important conclusion is that whatever $\alpha_{\inf}$ the minimization procedure for (43) delivers, it must hold that either

$$\alpha_{\inf} \leq -4 \quad \text{or} \quad \alpha_{\inf} > 8/3 \quad \text{with probability 1,} \tag{45}$$

and thus one never finds $\alpha = 1$ (which corresponds to the almost surely optimal martingale!). Further observe that due to Proposition 19, $-4M^*$ and $\frac{8}{3}M^*$ are the only optimal martingales corresponding to (45), but, they are not surely optimal (M^* is the only surely optimal one in the class $(\alpha M^*)_{\alpha\in\mathbb{R}}$). Now let us consider a general result in Desai et al. (2010) that implies that we must have

$$E\mathcal{Z}\left(\alpha_{\inf,N}, \xi\right) \to Y_0^* \quad \text{if} \quad N \to \infty.$$

For the present example that means that the limit set of the sequence $(\alpha_{\inf,N})$ is just $\{-4, 8/3\}$ (attracting set!), and that in the limit for $N \to \infty$ one ends up with either the martingale $-4M^*$ or $\frac{8}{3}M^*$, which both are optimal but not surely optimal.

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