Systematic generation of parametric correlation structures for the LIBOR market model *

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Abstract

We present a conceptual approach of deriving parsimonious correlation structures suitable for implementation in the LIBOR market model. By imposing additional constraints on a known ratio correlation structure, motivated by economically sensible assumptions concerning forward LIBOR correlations, we yield a semi-parametric framework of non-degenerate correlation structures with realistic properties. Within this framework we derive systematically low parametric structures with, in principal, any desired number of parameters. As illustrated, such structures may be used for smoothing a matrix of historically estimated LIBOR return correlations. In combination with a suitably parametrized deterministic LIBOR volatility norm we so obtain a parsimonious multi-factor market model which allows for joint calibration to caps and swaptions. See Schoenmakers [2002] for a stable full implied calibration procedure based on the correlation structures developed in this paper.

1 Introduction

This paper is devoted to a systematic construction of parsimonious non-degenerate correlation structures with economically sensible properties suitable for implementation in a BGM/Jamshidian LIBOR market model (Brace, Gatarek, Musiela [1997], Jamshidian [1997]). In principal, a multi-factor market model involving a thus obtained parsimonious correlation structure and some parsimoniously structured volatility norm may be calibrated to liquid market prices of caps and swaptions. In Section 4 we give a brief sketch of such a calibration procedure which is based on approximative swaption pricing, see Hull & White [2000], Jäckel & Rebonato [2000], Schoenmakers [2002], and historical correlations of forward LIBOR returns smoothed by a particular correlation structure developed in this paper. In this respect, Schoenmakers [2002] has shown that full implied standard least squares calibration may suffer from stability problems and proposes an implied calibration procedure which is stabilised by an additional market concept, the Market Swaption Formula. Our motivation for studying parsimonious multi-factor LIBOR models is basically twofold. (i) Parameter stability and protection against over fitting in a calibration procedure. This may be important, for instance, in situations where the given system of cap and swaption prices is not completely liquid. (ii) As illustrated in section 2.1 below, in Schoenmakers, Coffey [1998], and somewhat differently in Rebonato [1996], LIBOR models with only two or three factors tend to imply unrealistic correlation structures between increments of different forward LIBOR rates and, as a consequence, pricing correlation sensitive products with such models may be unreliable in practice.

In Jamshidians framework (Jamshidian [1997]), briefly recalled in Section 2, a d-factor LIBOR market model for a tenor structure $T_1, \ldots, T_n$ is determined by a deterministic volatility structure $t \to \gamma(t) \in \mathbb{R}^{n-1} \times \mathbb{R}^d$. With $\gamma_i(t) := [\gamma_{i1}, \ldots, \gamma_{id}](t)$ being the volatility vector of the i-th forward LIBOR, the instantaneous covariance between the i-th and the j-th forward is given by the scalar product $\gamma_i \cdot \gamma_j$, hence the instantaneous correlation by $\rho_{ij}(t) := \frac{\gamma_i \cdot \gamma_j}{|\gamma_i||\gamma_j|}(t)$. A parsimonious market model is so identified by a parsimonious parametrisation of the volatility matrix $\gamma(t)$ and as a first step to parsimony we basically follow Rebonato [1999] by assuming a certain simple functional form of the volatility norms $|\gamma_i|(t) =: c_i g(T_i - t)$, where g is typically some hump shaped function involving a few parameters. However, our main contribution is the next step: In a systematic way we provide various suitable parametrisations of the correlation matrix $[\rho_{ij}]$ which obey sensible economical principles, see Section 3. In fact, we introduce and motivate a semi-parametric full rank correlation structure for the matrix $[\rho_{ij}]$ which is obtained by subjecting a ratio correlation structure (e.g. Curnow, Dunnett [1962]) to additional constraints motivated by economically relevant assumptions. This structure may be regarded as *semi-parametric* in the sense that its parameter dimension is is $\mathcal{O}(n)$ rather than $\mathcal{O}(n^2)$, where in the latter case we would speak of a *non*-parametric structure. Although for typical n, for example $n = 40$ or 80, a parameter dimension of $\mathcal{O}(n)$ is still relatively large, the extra constraints allow for both stable and realistic behaviour of implied model correlations. In fact, the proposed ratio correlation structure is essentially suggested earlier in Schoenmakers, Coffey [1998] and Kurbanmuradov, Sabelfeld, Schoenmakers [1999], but is motivated in more detail and analysed further in this paper.

As shown in Schoenmakers [2002], for our parsimonious models the attainable overall fit to the (ATM) cap/swaption market is about 0.5% -4% relative, depending on the set of swaptions included in the calibration. This accuracy may be considered as good enough, since usually the ultimate purpose is pricing of exotic LIBOR instruments with respect to the (relatively liquid) cap/swaption market and bid-ask spreads of exotic products are in general considerably larger. Of course, by using a general non-parametric volatility structure it will be possible to achieve a more exact fit, but, such a fit may be generally instable in the sense that a slight change in the market data likely results into a completely different set of model input parameters, the typical problem of over fitting. Finally, we note that by choosing a desired number of principal components in a calibrated multi-factor model, and then re-calibrating the volatility norms, the multi-factor model may be converted into a low factor model which is thus calibrated in a stable way, see for details Schoenmakers [2002].

2 The LIBOR market model

We consider a Jamshidian LIBOR market model (Jamshidian [1997]) for the forward LIBOR processes L_i with respect to a given tenor structure $0 < T_1 < T_2 < \ldots < T_n$ in the terminal bond measure $\mathbb{P}_n$,

$$dL_i = -\sum_{j=i+1}^{n-1} \frac{\delta_j L_i L_j \, \gamma_i \cdot \gamma_j}{1 + \delta_j L_j} \, dt + L_i \, \gamma_i \cdot dW^{(n)}, \qquad (1)$$

where for $i = 1, \ldots, n-1$ the L_i are defined in the intervals $[t_0, T_i]$, $\delta_i = T_{i+1} - T_i$ are the day count fractions and $\gamma_i = (\gamma_{i,1}, \ldots, \gamma_{i,d})$ are given deterministic functions, called factor loadings, defined in $[t_0, T_i]$. In (1), $(W^{(n)}(t) \mid t_0 \leq t \leq T_{n-1})$ is a standard d-dimensional Wiener process under $\mathbb{P}_n$, where d, $d \leq n-1$, is the number of driving factors.

2.1 Problems with low factor models

It is known that low factor models have intrinsic problems to match (instantaneous) correlations between forward LIBORs realistically, see e.g. Rebonato [1996], Schoenmakers, Coffey [1998]. Let us illustrate this fact here again and consider a two factor version of the LIBOR model (1), where

$$\gamma_i(t) = g_i(t) e_i, \quad e_i \in \mathbb{R}^2, \quad i = 1, \ldots, n-1 \tag{2}$$

and e_i are unit vectors specifying the instantaneous correlations which we assume to be time independent. In general, the volatility norms $g_i = |\gamma_i|$ may be taken to be time dependent. For instance, a plausible assumption would be

$$g_i(t) = c_i g(T_i - t), \tag{3}$$

where the i-independent function g takes care of the typical "hump shaped" volatility behaviour as function of time to LIBOR maturity. However, as instantaneous correlations are completely determined by the choice of e_i, the choice of g_i does not effect the problem sketched below. Indeed, we have

$$\frac{\gamma_i(t) \cdot \gamma_j(t)}{|\gamma_i(t)||\gamma_j(t)|} = e_i \cdot e_j = \cos(\phi_i - \phi_j), \tag{4}$$

for a set of angles $\phi_1, \ldots, \phi_{n-1}$ with $\phi_1 := 0$ and it is clear that any two factor market model with constant instantaneous correlations can be represented in this way.

We now consider an artificial example. Take $n = 20$ and suppose that the market tells us the correlations $\rho_{1,j}$ behave like $\rho_{1j} = 18/(17+j)$, thus falling down from 1 to 0.5. Then, if we calibrate this two-factor model, i.e. the ϕ_i, to these correlations it is easily seen that, as an immediate consequence, the correlations $\rho_{j,19}$ have to be $\rho_{j,19} = \frac{9}{17+j} + \frac{\sqrt{3}}{2}\sqrt{1 - \left(\frac{17}{18} + \frac{j}{18}\right)^{-2}}$, see figure (1). However, the behaviour of the correlations $\rho_{j,19}$ in figure (1) is clearly *not* consistent with their real behaviour in the market which should look more or less the same as $\rho_{1,j}$, mirrored at $j = 10$. Of course the situation will be better when we increase the number of factors but a two factor example reveals this problem most clearly. Here we should note that in Brigo, Mercurio [2001] calibration of the cosine structure (4) is extensively tested and their conclusions concerning the behaviour of implied correlations more or less confirm the difficulties sketched above.

As a solution for the intrinsic low factor problem we propose a multi-factor market model by the identification of a natural correlation structure which matches correlation behaviour observed in practice with a relatively small number of parameters, in fact, essentially of the same order as in a two factor market model.

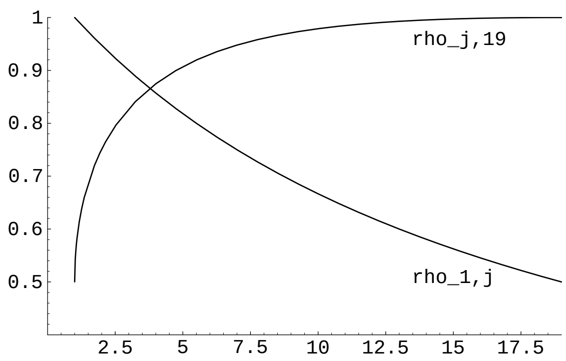

Figure 1:

3 Market model with semi-parametric correlation structure

Assumption 3.0.1 (semi-parametric correlation structure) *We propose a LIBOR market model (1) with $d = n - 1$ and a deterministic volatility of the following structure*

$$\gamma_i(t) = g_i(t)e_i, \quad e_i \in \mathbb{R}^{n-1}, \tag{5}$$

where $g_i = |\gamma_i|$ are in general suitable humped shaped real functions and e_i are time independent unit vectors determining an instantaneous correlation structure of the form

$$\rho_{ij} := e_i \cdot e_j = \frac{\min(b_i, b_j)}{\max(b_i, b_j)}, \tag{6}$$

where the sequence $b = (b_1, \ldots, b_{n-1})$ is required to be positive, strictly increasing, and, such that

$$i \longrightarrow b_i/b_{i+1} \textit{ is strictly increasing.} \tag{7}$$

Without further restriction we may assume in (6) that $b_1 = 1$.

As the calibration of correlation structure (6) involves the identification of $\mathcal{O}(n)$ parameters whereas the calibration of a general or *non-parametric* correlation matrix would require the identification $\mathcal{O}(n^2)$ entries we call (6) a *semi-parametric* correlation structure. In the next section we will motivate (6) and show in particular that (6) defines a correlation structure indeed.

3.1 Motivation of the semi-parametric correlation structure (6)

Let b_i, $i = 1, \ldots, m$, be an arbitrary positive increasing sequence with $b_1 = 1$ and let $a_1 = b_1 = 1$, $a_i := \sqrt{b_i^2 - b_{i-1}^2}$, for $i = 2, \ldots, m$. Let further Z_i, $i = 1, \ldots, m$, be standard normally distributed independent real random variables and consider the random variables

$$Y_i := \sum_{k=1}^{i} a_k Z_k. \tag{8}$$

Then, for $i \leq j$ the covariance between Y_i and Y_j is given by

$$Cov(Y_i, Y_j) = \sum_{k=1}^{i} a_k^2 = b_i^2$$

and so the correlation is given by $\rho_{Y_i,Y_j} = b_i/b_j$. Hence, it follows that the correlation structure of Y is given by (6), where $m = n - 1$ and, in particular, that (6) defines a correlation structure indeed.

Remark 3.1.1 We note that (8) is not the only Gaussian vector with correlation structure (6): It is not difficult to prove that for an $m \times m$ matrix B and $Z := (Z_1, \ldots, Z_m)^\top$, the centered Gaussian vector $U := BZ$ has correlation structure (6) if and only if there exists a positive diagonal matrix Λ and orthogonal matrix Q such that $B = \Lambda AQ$, where $A_{ik} := a_k 1_{k \leq i}$.

We may assume without restriction in (5) that the time independent $(n-1) \times (n-1)$ matrix E defined by $E_{ik} = e_{i,k}$ is lower triangular, otherwise, we may apply an orthogonal transformation to the Brownian vector $W^{(n)}$. So, we have

$$\begin{aligned} \Delta L_i &= \ldots \Delta t + L_i \gamma_i \cdot \Delta W^{(n)} \\ &= \ldots \Delta t + L_i g_i \sum_{k=1}^{i} e_{i,k} \Delta W_k^{(n)}. \end{aligned}$$

By the imposed correlation structure (6) and remark (3.1.1) it then follows that

$$\begin{aligned} e_{i,k} &= \frac{1}{b_i} a_k 1_{k \leq i}, \quad \text{and so} \\ \Delta L_i &= \ldots \Delta t + L_i g_i \sum_{k=1}^{i} \frac{a_k}{\sqrt{\sum_{l=1}^{i} a_l^2}} \Delta W_k^{(n)} = \ldots \Delta t + L_i g_i \sum_{k=1}^{i} \frac{\sqrt{b_k^2 - b_{k-1}^2}}{b_i} \Delta W_k^{(n)} \\ &= \ldots \Delta t + L_i g_i \sqrt{1 - \frac{b_{i-1}^2}{b_i^2}}\, \Delta W_i^{(n)} + \frac{L_i g_i}{L_{i-1} g_{i-1}} \frac{b_{i-1}}{b_i} \Delta L_{i-1} \qquad (9) \end{aligned}$$

where $b_0 := 0$. The interpretation of (9) is clear: The risky part of a forward LIBOR increment ΔL_i at time t, $t < T_{i-1}$ is a linear combination of the forward increment ΔL_{i-1} and an independent random shock $\Delta W_i^{(n)}$, with coefficients determined by $L_{i-1}(t)$, $L_i(t)$, $g_{i-1}(t)$, $g_i(t)$, b_{i-1} and b_i, respectively. We emphasize that decomposition (9) is due to the special structure of $e_{i,k}$ hence the correlation structure (6) and does not hold for a general correlation structure. Finally, the additional assumption (7) forces that for fixed p

$$i \longrightarrow \rho_{i,i+p} = Cor(\Delta L_i, \Delta L_{i+p}) \quad \textit{is increasing.} \qquad (10)$$

This is a very important realistic feature of the model which states, for instance, that the correlation between a seven and a nine year forward is higher than the correlation between a three and a five year forward. For more interesting features around the correlation structure (6), we refer to Curnow, Dunnett [1962].

3.2 Alternative characterization of the correlation structure (6) and the generation of (6)-consistent low parametric structures

By the next theorem it is possible to transform the non-linear constraints on the sequence $(b_1, \ldots, b_{n-1})$ in (6) to the region $\mathbb{R}_+^{n-2}$.

Theorem 3.2.1 *Every correlation structure of type (6) can be represented by*

$$b_i = \exp\left[\sum_{l=2}^{m} \min(l-1, i-1)\Delta_l\right], \tag{11}$$

$$\tag{12}$$

for a sequence of nonnegative numbers $\Delta_i, \Delta_i \geq 0$, $i = 2, \ldots, m := n-1$.

Conversely, (11) satisfies (6) for any sequence Δ_l, $\Delta_l \geq 0$ $l = 2, \ldots, m$.

Proof. For a sequence (b_i) satisfying (6) let us define $\xi_i := \ln b_i$, $1 \leq i \leq m := n-1$. Then, $\xi_1 = 0$ since $b_1 = 1$ and for ξ_i we have the following constraints:

$$\begin{aligned} \xi_i &\leq \xi_{i+1} \quad 1 \leq i \leq m-1 \\ \xi_{i-1} + \xi_{i+1} &\leq 2\xi_i \quad 2 \leq i \leq m-1. \end{aligned} \tag{13}$$

We introduce the new variables,

$$\begin{aligned} \Delta_i &:= 2\xi_i - \xi_{i-1} - \xi_{i+1} = \xi_i - \xi_{i-1} - (\xi_{i+1} - \xi_i) \geq 0, \quad 2 \leq i \leq m-1, \\ \Delta_m &:= \xi_2 - \sum_{l=2}^{m-1} \Delta_l \end{aligned} \tag{14}$$

and so we have for $2 \leq i \leq m$,

$$\begin{aligned} \xi_i = \xi_i - \xi_1 &= \sum_{k=2}^{i} \xi_k - \xi_{k-1} \\ &= \sum_{k=2}^{i} \{\xi_k - \xi_{k-1} - (\xi_2 - \xi_1) + \xi_2 - \xi_1\} \\ &= (i-1)\xi_2 + \sum_{k=3}^{i} \sum_{l=2}^{k-1} \{\xi_{l+1} - \xi_l - (\xi_l - \xi_{l-1})\} \\ &= (i-1)\xi_2 - \sum_{k=3}^{i} \sum_{l=2}^{k-1} \Delta_l \\ &= (i-1)\xi_2 - \sum_{l=2}^{i-1} \sum_{k=l+1}^{i} \Delta_l \\ &= (i-1)\xi_2 - \sum_{l=2}^{i-1} (i-l)\Delta_l \end{aligned} \tag{15}$$

where an empty sum is defined to be zero. It follows that

$$\xi_{i+1} - \xi_i = \xi_2 - \sum_{l=2}^{i} \Delta_l$$

and the constraints (13) transform into

$$\Delta_i \geq 0, \qquad 2 \leq i \leq m. \tag{16}$$

Then by (14) and (15) we may express ξ_i, resp. b_i in the new coordinates Δ_i via

$$\begin{aligned} \xi_i &= (i-1)\sum_{l=2}^{m} \Delta_l - \sum_{l=2}^{i-1}(i-l)\Delta_l \\ &= (i-1)\sum_{l=i}^{m} \Delta_l + \sum_{l=2}^{i-1}(l-1)\Delta_l \\ &= \sum_{l=2}^{m} \min(l-1, i-1)\Delta_l, \\ b_i &= \exp(\xi_i). \end{aligned} \tag{17}$$

The converse follows straightforwardly by checking (6) for the sequence (b_i) defined by (11).

For a correlation structure (6) representation by (11) it now follows that

$$\rho_{ij} = \exp[-\sum_{l=i+1}^{m} \min(l-i, j-i)\Delta_l\,] \qquad i < j. \tag{18}$$

From representation (11) in theorem (3.2.1) or, equivalently, from (18) we may derive conveniently various low parametric structures consistent with (6). Below we give some examples.

Example 3.2.2 Let us take $\Delta_2 = \ldots = \Delta_{m-1} =: \alpha \geq 0$ and $\Delta_m =: \beta \geq 0$. Then, (18) yields the correlation structure

$$\rho_{ij} = e^{-|i-j|(\beta+\alpha(m-\frac{i+j+1}{2}))}, \quad i,j = 1,\ldots,m. \tag{19}$$

Note that for $\alpha = 0$ we get $\rho_{ij} = e^{-\beta|i-j|}$, a simple correlation structure frequently used in practice in spite of the, in fact, unrealistic consequence that $i \to Cor(\Delta L_i, \Delta L_{i+p})$ is *constant* rather than increasing for fixed p. Let us introduce new parameters $\rho_\infty := \rho_{1m}$ and $\eta := \alpha(m-1)(m-2)/2$, hence

$$\beta = -\frac{\alpha}{2}(m-2) - \frac{\ln \rho_\infty}{m-1} = \frac{-\eta - \ln \rho_\infty}{m-1}$$

and then (19) becomes

$$\boxed{\rho_{ij} = e^{-\frac{|i-j|}{m-1}(-\ln \rho_\infty + \eta\frac{m-i-j+1}{m-2})}, \quad 0 \leq \eta \leq -\ln \rho_\infty, \quad i,j = 1,\ldots,m.} \tag{20}$$

While the structures (19) and (20) are essentially the same, by the re-parametrization of (19) into (20) the parameter stability is improved: Relatively small movements in the $b-$sequence connected with (20), thus the (market) correlations, causes relatively small movements in the parameters ρ_∞ and η. In fact, this can be seen also by analytical comparison of the parameter sensitivities (derivatives) in (19) and (20).

The following three parametric structure is a refinement of (19):

Example 3.2.3 Suppose $m > 2$ and let Δ_i be linear dependent of i, for $2 \leq i \leq m-1$, with

$\Delta_2 = \alpha_1 \geq 0$, $\Delta_{m-1} = \alpha_2 \geq 0$, and $\Delta_m = \beta \geq 0$. Hence for $i = 2, \ldots, m-1$,

$$\Delta_i = \alpha_1 \frac{m-i-1}{m-3} + \alpha_2 \frac{i-2}{m-3}.$$

Then, from (18) we get by rather tedious but elementary algebra the correlation structure

$$\rho_{ij} = \exp\left[-|j-i|\left(\beta - \frac{\alpha_2}{6m-18}(i^2+j^2+ij-6i-6j-3m^2+15m-7)\right.\right.$$
$$\left.\left.+\frac{\alpha_1}{6m-18}(i^2+j^2+ij-3mi-3mj+3i+3j+3m^2-6m+2)\right)\right]. \qquad (21)$$

Note that (21) collapses too (19) for $\alpha_1 = \alpha_2 = \alpha$. As in (19) we now re-parameterize (21) by $\rho_\infty = \rho_{1m}$ which yields

$$\beta = \frac{-\ln\rho_\infty}{m-1} - \frac{\alpha_1}{6}(m-2) - \frac{\alpha_2}{3}(m-2).$$

In order to gain parameter stability we set, as in (20),

$$\alpha_1 = \frac{6\eta_1 - 2\eta_2}{(m-1)(m-2)}, \quad \alpha_2 = \frac{4\eta_2}{(m-1)(m-2)}$$

and then (21) becomes

$$\rho_{ij} = \exp\left[-\frac{|j-i|}{m-1}\left(-\ln\rho_\infty + \eta_1 \frac{i^2+j^2+ij-3mi-3mj+3i+3j+2m^2-m-4}{(m-2)(m-3)} + \right.\right.$$
$$\left.\left.-\eta_2 \frac{i^2+j^2+ij-mi-mj-3i-3j+3m+2}{(m-2)(m-3)}\right)\right], \qquad (22)$$
$$i,j = 1,\ldots,m, \quad 3\eta_1 \geq \eta_2 \geq 0,\ 0 \leq \eta_1 + \eta_2 \leq -\ln\rho_\infty.$$

Obviously, for $\eta_1 = \eta_2 = \eta/2$, (22) yields (20) again.

Remark 3.2.4 (A realistic two-parametric correlation structure) Correlation structure (22) is used in the implied calibration procedure of Schoenmakers [2002] and turns out to work well in practice. Moreover, implied calibration usually returns $\eta_2 \approx 0$, and, since calibrating a three parametric structure takes longer than a two-parametric one, we suggest another two parameter structure which is obtained from (22) by taking $\eta_2 = 0$ (equivalently, $\alpha_2 = 0$ in (21)). For this correlation structure the magnitude of concavity of sequence $\ln b_i$ in (6) is decreasing to zero rather than being constant like in (19). We thus advocate the structure

$$\begin{aligned} \rho_{ij} &= \exp\left[-\frac{|j-i|}{m-1}\left(-\ln\rho_\infty + \eta\frac{i^2+j^2+ij-3mi-3mj+3i+3j+2m^2-m-4}{(m-2)(m-3)}\right)\right], \\ & i,j = 1,\ldots,m, \quad \eta > 0,\ 0 < \eta < -\ln\rho_\infty, \end{aligned} \tag{23}$$

obtained from (22) by setting $\eta_2 := 0$ and $\eta := \eta_1$ as being, in a sense, a realistic two-parametric structure for the LIBOR market model.

Example 3.2.5 It is easily checked that the sequence b defined by

$$b_i = e^{\beta(i-1)^\alpha}, \quad 1 \le i \le m = n-1,\ \beta > 0,\ 0 < \alpha < 1, \tag{24}$$

satisfies the requirements in assumption (3.0.1) and thus yields a correlation structure by (6), which was proposed in Schoenmakers, Coffey [1998] and used in Kurbanmuradov, Sabelfeld, Schoenmakers [1999] for simulation experiments. By theorem (3.2.1) the structure (24) has a representation (11) with

$$\begin{aligned} \Delta_i &= 2\beta(i-1)^\alpha - \beta(i-2)^\alpha - \beta i^\alpha, \quad 2 \le i \le m-1, \\ \Delta_m &= \beta - \sum_{l=2}^{m-1}\Delta_l = \beta(m-1)^\alpha - \beta(m-2)^\alpha, \end{aligned} \tag{25}$$

where indeed $\Delta_i > 0$ for $1 \le i \le m$ and Δ_i is decreasing for $2 \le i \le m-1$. By introducing $\rho_\infty := 1/b_m$ we get from (24),

$$\rho_{ij} = e^{\ln\rho_\infty\left|\left(\frac{i-1}{m-1}\right)^\alpha - \left(\frac{j-1}{m-1}\right)^\alpha\right|}, \quad i,j = 1,\ldots,m,\ 0 < \alpha < 1,\ \beta > 0. \tag{26}$$

We note that the structure (26) has, in principal, similar properties as (23), however, (23) shows to have somewhat better parameter stability in practice.

Remark 3.2.6 Let us now consider the parametrization of market correlations used by Rebonato [1999a], which has for a an equidistant tenor structure the following form

$$\rho_{ij} = \rho_\infty + (1-\rho_\infty)\exp\left[-|j-i|(\beta - \alpha\max(i,j))\right]. \tag{27}$$

The structure (27) can be seen as a perturbation of the correlation structure

$$\hat\rho_{ij} = \rho_\infty + (1-\rho_\infty)\exp\left[-|j-i|\beta\right] \tag{28}$$

and has the desirable property that $i \to \rho_{i,i+p} = \rho_\infty + (1-\rho_\infty)\exp\left[-p(\beta - \alpha(i+p))\right]$ is *increasing* for $\alpha > 0$ and thus may produce for a given tenor structure realistic market correlations for properly chosen ρ_∞, $\beta > 0$ and (small) $\alpha > 0$, see Rebonato [1999a]. However, while (27) may produce realistic correlations, it should be noted that its $(\alpha, \beta, \rho_\infty)$ domain of positivity is not explicitly specified. Hence, for a particular choice of parameters it is not directly guaranteed that (27) defines a correlation structure indeed. In particular. it can be verified easily that (27) does not fit in the framework of (6). It is clear that all the correlation structures consistent with (6), in particular (19), (21), (24) and their equivalent representations do not suffer for this problem as, from the way they are constructed, they are endogenously positive definite and incorporate the economically realistic properties $j \to \rho_{ij}$ is *decreasing* for $j > i$ *and* $i \to \rho_{i,i+p}$ is *increasing* for fixed p, for any choice of parameters *in their well specified domain.* This property makes the correlation structures consistent with (6) presented in this section particularly suitable for calibration purposes as search routines are thus prevented for searching in parameter regions where the matrix ρ_{ij} fails to be a correlation structure in fact.

4 Smoothing historical LIBOR correlations with parametric structures, calibration to caps and swaptions (sketch)

We assume a fixed time tenor structure $T_0 < T_1 < T_2 < ... < T_n$ with equi-distant tenors, i.e. $\delta = T_{i+1} - T_i$ for all i, and a system of risk-free zero-coupon bonds B_i also called "present values" which mature at T_i, hence $B_i(T_i) = 1$. We further suppose that up to the present calendar date T_0 time series of historical data of the present values $B_i(t)$, $t \le T_0$, $i = 1, \ldots, n$ are available, for instance on a daily base. Note that these values may be obtained from the present value curve $T \to B(t,T)$, $t \le T_0$, $T \ge T_0$, which in turn may be constructed by interpolating the available present values at each historical date $t \le T_0$. For more details on yield curve smoothing methods, see e.g. Fisher et al. [1995], Linton et al. [1998]. So, in principle, time series of forward LIBOR rates are available via

$$L_i(t) = (\frac{B_i(t)}{B_{i+1}(t)} - 1)\delta^{-1}, \quad t \le T_0,\ i = 1, \ldots, n.$$

As a rough approximation of the LIBOR market model, by neglecting the drifts in fact, we assume that returns of the forward LIBORs may be modelled by a multi-dimensional Gaussian distribution. We thus set

$$\frac{\Delta L_i(t)}{L_i(t)} \approx \gamma_i \cdot \Delta Z \sqrt{\Delta t}$$

where $\gamma_i \in \mathbb{R}^{n-1}$ is here assumed to be constant over the time interval where the time series are considered and ΔZ is a $(n-1)$-dimensional standard Gaussian vector, i.e. $\mathbb{E}\,\Delta Z_i \Delta Z_j = \delta_{ij}$. So we have

$$\mathbb{E}\,\frac{\Delta L_i}{L_i}\frac{\Delta L_j}{L_j} = \gamma_i \cdot \gamma_j \Delta t = |\gamma_i||\gamma_j|\rho_{ij}\Delta t. \tag{29}$$

In (29) the $|\gamma_i|$ are obtained by the historical estimates of the left-hand-side for $i = j$, since always $\rho_{ii} = 1$. Then, in (29) we plug in the estimations for $|\gamma_i|$ and one of the parametric correlation structures derived in Section 3, for example (22) or (23). Next we calibrate via least squares the parameters of the correlation structure via (29) for $i \neq j$. Resuming we propose the following program for identification of a smooth LIBOR return correlation structure:

I *Construct daily time series for the different forward LIBORs*

II *Estimate historically the left-hand-side of (29) in the usual way and thus obtain for $i = j$ in particular historical estimations for the $|\gamma_i|$.*

III *Fit the (for instance two or three) parameters of a parametric correlation structure ρ via (29) by standard least squares to the historical estimations obtained for $i \neq j$.*

Application: Calibration to caps and swaptions

One may take the above historically identified instantaneous correlation structure ρ as fixed in the LIBOR market model and then calibrate by standard least squares volatility norms of the form (3) to the prices of caps and swaptions. For instance, by using a parametrization of g consistent with Rebonato [1999],

$$g(s) := g_{a,b,g_\infty}(s) := g_\infty + (1 - g_\infty + as)e^{-bs}, \quad a, b, g_\infty > 0,$$

and approximative swaption pricers which can be found in Jäckel & Rebonato [2000], Schoenmakers [2002]. We note, however, that this calibration method involves historical estimation of LIBOR correlations which, on one hand, may be a rather elaborate job and, on the other hand, is not really desirable in the view of implied calibration enthusiasts who follow the idea that the LIBOR model should be identified by exclusively using price information relevant at the calibration date (T_0). A full implied calibration method is presented in Schoenmakers [2002]. >From this paper it follows that, while implied methods do not require historical data they may suffer from stability problems and therefore require incorporation of an additional concept to get the calibration stable. In this respect Schoenmakers [2002] proposes incorporation of a Market Swaption Formula, an intuitive link among cap and swaption volatilities and correlations of forward LIBOR returns.

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