Noname manuscript No.
(will be inserted by the editor)

A pure martingale dual for multiple stopping

John Schoenmakers

Received: date / Accepted: date

Abstract In this paper we present a dual representation for the multiple stopping problem, hence multiple exercise options. As such it is a natural generalization of the method in Rogers (2002) and Haugh and Kogan (2004) for the standard stopping problem for American options. We term this representation as a 'pure martingale' dual as it is solely expressed in terms of an infimum over martingales rather than an infimum over martingales *and* stopping times as in Meinshausen and Hambly (2004). For the multiple dual representation we propose Monte Carlo simulation methods which require only one degree of nesting.

Keywords Multiple stopping · Dual representations · Multiple callable derivatives

Mathematics Subject Classification (2000) 60G40 · 62L15

JEL Classification C61 · C63

1 Introduction

The key issue in valuation of financial derivatives with several exercise rights is solving a multiple stopping problem. Such derivatives are encountered, for example, in electricity markets (swing options) and interest rate markets (chooser caps). Typically, the dimension of the underlying financial object is rather high, for instance a Libor interest rate model, and therefore Monte Carlo

Work related to project 'Financial derivatives and valuation of risk', supported by the DFG Research Center MATHEON 'Mathematics for key technologies' in Berlin.

J. Schoenmakers
Weierstrass Institute for Applied Analysis and Stochastics, Mohrenstr. 39, D-10117 Berlin, Germany.
E-mail: schoenma@wias-berlin.de

based methods are called for. In this respect the last decades have seen several breakthroughs for standard American (or Bermudan style) derivatives, hence the standard stopping problem. Among the most popular ones are the regression based methods of Longstaff and Schwartz (2001), Tsitsiklis and Van Roy (2001), and alternative approaches by Andersen (1999), Broadie and Glasserman (2004) and others. These methods allow for the computation of a lower approximation of the price of the product under consideration by straightforward (non-nested) Monte Carlo simulation when the underlying dimension is not too high. More recently, Kolodko and Schoenmakers (2006) proposed a policy improvement procedure and it is demonstrated in Bender et al. (2006) and Bender et al. (2008) that this method can be effectively combined with Longstaff and Schwartz (2001) for very high-dimensional products. In Bender and Schoenmakers (2006) this policy iteration method is extended to multiple stopping problems. Evaluation of products with multiple exercise rights (on a low dimensional underlying) is also possible by using trinomial forests (Jaillet et al., 2004). In Carmona and Touzi (2008) a Malliavin calculus based approach for the valuation of swing options is presented.

In Rogers (2002) and Haugh and Kogan (2004) a dual approach is developed (inspired by Davis and Karatzas (1994)) which allows for computing tight upper bounds for American style products. Jamshidian (2007) proposed a multiplicative version of the dual representation, Belomestny and Milstein (2006), and Belomestny et al. (2009b) proposed to compute upper bounds based on the concept of consumption processes. Effective algorithms for dual upper bounds are proposed in Andersen and Broadie (2004), Kolodko and Schoenmakers (2004), and Belomestny et al. (2009a). For products with multiple exercise possibilities Meinshausen and Hambly (2004) found a dual representation for the marginal excess value of the product due to one additional exercise right. In this representation an infimum over a family of stopping times and a family of martingales is involved. Generalizations of this method to multiple exercise products under volume constraints are developed in Bender (2008) and Aleksandrov and Hambly (2008). While the mentioned methods for multiple exercise products have shown to be feasible in practice, the question whether a 'pure martingale' dual representation for the multiple stopping problem exists as a natural extension of the dual representation for the single exercise case, in terms of an infimum over martingales (only), was still open. The main result in this paper is such a dual representation and so fills this gap. Moreover we propose Monte Carlo simulation methods for this representation, which require at most one degree of nesting, just as in the one-exercise case. As such the proposed procedures are natural extensions of the corresponding ones for the single exercise case. In particular, one of them may be seen as a natural generalization of the primal-dual approach in Andersen and Broadie (2004). It is more or less clear that the numerical potential of the proposed simulation procedures for the multiple dual is inherited from the numerical qualities of the methods for the standard (additive) dual extensively documented in the literature. Therefore, we prefer to communicate the new multiple dual representation together with a brief description of its implementation in this paper, and consider an in depth numerical study to be more suitable for subsequent work. The main result, Theorem 2.5, is derived in Section 2, and the description of the simulation procedures is given in Section 3.

2 The multiple stopping problem and its dual representation

Let $(Z_i : i = 0, 1, \ldots, T)$ be a non-negative stochastic process in discrete time on a filtered probability space $(\Omega, \mathcal{F}, P)$, adapted to some filtration $\mathbb{F} := (\mathcal{F}_i : 0 \leq i \leq T)$ which satisfies

$$\sum_{i=1}^{T} E|Z_i| < \infty.$$

The process Z may be seen as a (discounted) cash-flow, which an investor may exercise L times, subjected to the additional constraint that it is not allowed to exercise more than one right at the same date. The goal of the investor is to maximize his expected gain by making optimal use of his L exercise rights. This goal may be formalized as a multiple stopping problem.

Definition 2.1 For notational convenience in our further analysis we extend the cash-flow process in a trivial way by $Z_i :\equiv 0$ and $\mathcal{F}_i :\equiv \mathcal{F}_T$ for $i > T$.

Let us define for each fixed $0 \leq i \leq T$ and L, $\mathcal{S}_i(L)$ as the set of $\mathbb{F}$-stopping vectors $\tau := (\tau^{(1)}, \ldots, \tau^{(L)})$ such that $i \leq \tau^{(1)}$ and, for all l, $1 < l \leq L$, $\tau^{(l-1)} + 1 \leq \tau^{(l)}$. The multiple stopping problem then comes down to find a family of stopping vectors $\tau_i^* \in \mathcal{S}_i(L)$ such that for $0 \leq i \leq T$,

$$E_i \sum_{l=1}^{L} Z_{\tau_i^{*l}} = \sup_{\tau \in \mathcal{S}_i(L)} E_i \sum_{l=1}^{L} Z_{\tau^{(l)}}, \tag{2.1}$$

where henceforth $E_i := E_{\mathcal{F}_i}$ denotes conditional expectation with respect to the σ-algebra $\mathcal{F}_i$, and where sup is to be understood as *essential supremum* (if it ranges over an uncountable family of random variables). The process on the right-hand-side of (2.1) is called the *Snell envelope* of Z under L exercise rights and we denote it by Y_i^{*L}. In the case of one exercise right we usually write $Y_i^* := Y_i^{*1}$. Note in view of Definition 2.1 that, (a) if $i > T$ then $Y_i^{*L} = 0$ for any $L \geq 0$, and, (b) if $L \geq T - i + 1$ then we may trivially take $\tau_i^{*l} = i + l - 1$ for $1 \leq l \leq L$ in (2.1).

We recall from Bender and Schoenmakers (2006) that the multiple stopping problem can be reduced to L nested stopping problems with one exercise right in the following way. $Y^{*0} := 0$, Y^{*1} is the Snell envelope of Z. For general L, $L \geq 1$, Y^{*L} is the Snell envelope of the process $Z_i + E_i Y_{i+1}^{*L-1}$ (seen as generalized cash-flow) under one exercise right. It is thus natural to define (as in Bender and Schoenmakers (2006)) for each $L = 1, 2, \ldots$, the stopping family

$$\sigma_i^{*L} = \inf\{j \geq i : \; Z_j + E_j Y_{j+1}^{*L-1} \geq Y_j^{*L}\}, \qquad i \geq 0, \tag{2.2}$$

i.e. the first optimal stopping family for exercising the first of L exercise rights. The family of optimal stopping vectors $\tau_i^{*L} \in \mathcal{S}_i(L)$ for the multiple stopping problem with L exercise rights and cash-flow Z is connected with (2.2) via

$$\begin{aligned} \tau_i^{*1,L} &= \sigma_i^{*L}, \\ \tau_i^{*l+1,L} &= \tau_{\sigma_i^{*L}+1}^{*l,L-1}, \quad 1 \le l < L. \end{aligned} \tag{2.3}$$

The reduction (2.2), (2.3) is intuitively clear: It basically says, that the investor has to choose the first stopping time of the stopping vector in the following way: Decide, at time i whether it is better to take the cash-flow Z_i and enter a new contract with $L-1$ exercise rights starting at $i+1$, or to keep the L exercise rights. Then, after entering the stopping problem with $L-1$ exercise rights, he proceeds in the same (optimal) way.

2.1 Case $L = 1$; the standard stopping problem

In the case of one exercise right $L = 1$ we have the standard stopping problem. Let us recall some well-known facts (e.g. see Neveu (1975)).

1. The Snell envelope Y^* of Z is the smallest super-martingale that dominates Z.
2. A family of optimal stopping times is given by

   $$\tau_i^* = \inf\{j : j \ge i, \quad Z_j \ge Y_j^*\}, \quad 0 \le i \le T.$$

   In particular, the above family is the family of first optimal stopping times if several optimal stopping families exist.

2.2 Dual representation for the standard stopping problem

For the standard stopping problem, that is one exercise right $L = 1$, we have the (additive) dual representation theorem which we state in a form suitable for our purposes:

Theorem 2.2 *Rogers (2002), Haugh and Kogan (2004)*
If $\mathcal{M}$ is the set of all $\mathbb{F}$-martingales, it holds

$$Y_i^{*,1} = Y_i^* = \inf_{M \in \mathcal{M}} E_i \max_{i \le j \le T} (Z_j + M_i - M_j) \tag{2.4}$$

$$= \max_{i \le j \le T} \left(Z_j + M_i^* - M_j^*\right) \ a.s. \tag{2.5}$$

with M^* being the unique Doob martingale of Y^*, that is $Y^* = Y_0^* + M^* - A^*$ where M^* is a martingale, A^* is predictable and nondecreasing, and $M_0^* = A_0^* = 0$.

For the results in this paper the almost sure statement (2.5) is very important. Therefore, and because of its appealing simplicity, let us shortly recall the proof:

Proof For any martingale M we have

$$\begin{aligned} Y_i^* &= \sup_{i \le \tau \le T} E_i Z_\tau = \sup_{i \le \tau \le T} E_i \left[Z_\tau + M_i - M_\tau \right] \\ &\le E_i \max_{i \le j \le T} \left(Z_j + M_i - M_j \right). \end{aligned}$$

For the martingale M^* it then holds

$$\begin{aligned} Y_i^* &\le E_i \max_{i \le j \le T} \left(Z_j + M_i^* - M_j^* \right) \\ &\le E_i \max_{i \le j \le T} \left(Z_j + Y_i^* + A_i^* - Y_j^* - A_j^* \right) \\ &\le Y_i^* + E_i \max_{i \le j \le T} \left(A_i^* - A_j^* \right) = Y_i^*, \end{aligned}$$

since for all j, $0 \le j \le T$, $Y_i^* - E_i Y_{i+1}^* = A_{i+1}^* - A_i^* \ge 0$, and thus

$$Y_i^* = E_i \max_{i \le j \le T} \left(Z_j + M_i^* - M_j^* \right). \tag{2.6}$$

Moreover, by

$$\begin{aligned} \max_{i \le j \le T} \left(Z_j + M_i^* - M_j^* \right) &= \max_{i \le j \le T} \left(Z_j + Y_i^* + A_i^* - Y_j^* - A_j^* \right) \\ &\le Y_i^* + \max_{i \le j \le T} \left(A_i^* - A_j^* \right) = Y_i^* \end{aligned}$$

and (2.6) we have (2.5).

The corner stone for generalizing Theorem 2.2 to the multiple stopping problem is the following simple proposition which is a slight extension of (2.5) in a sense.

Proposition 2.3 *Let $(Z_i : 0 \le i \le T)$ be a nonnegative integrable cash-flow process with Snell envelope Y^* and let $Y^* = Y_0^* + M^* - A^*$ be its Doob decomposition as in Theorem 2.2. It then holds for each j, $0 \le j < T$,*

$$E_j Y_{j+1}^* = E_j \max_{j < l \le T} \left(Z_l - M_l^* + M_j^* \right) = \max_{j < l \le T} \left(Z_l - M_l^* + M_j^* \right) \quad a.s.$$

Proof For fixed $0 \le j < T$, we have by (2.5)

$$Y_{j+1}^* = \max_{j < l \le T} \left(Z_l - M_l^* + M_{j+1}^* \right) \quad \text{a.s.}, \tag{2.7}$$

so on the one hand it holds

$$E_j Y_{j+1}^* = E_j \max_{j < l \le T} \left(Z_l - M_l^* + M_j^* \right).$$

By the Doob decomposition of Y^* and (2.7), we have

$$\begin{aligned} E_j Y_{j+1}^* &= Y_{j+1}^* + M_j^* - M_{j+1}^* = \max_{j < l \le T} \left(Z_l - M_l^* + M_{j+1}^* \right) + M_j^* - M_{j+1}^* \\ &= \max_{j < l \le T} \left(Z_l - M_l^* + M_j^* \right) \quad \text{a.s.} \end{aligned}$$

on the other hand.

Remark 2.4 (i) It is not difficult to see that a further generalization of Proposition 2.3 is not possible in the sense that in general

$$E_j \max_{p<l\leq T} \left(Z_l - M_l^* + M_j^*\right) \overset{\text{a.s.}}{\neq} \max_{p<l\leq T} \left(Z_l - M_l^* + M_j^*\right)$$

if $p > j$. (ii) Proposition 2.3 is related to the analysis of supersolutions in Chen and Glasserman (2007).

2.3 Dual representation for the multiple stopping problem

We are now ready for proving the following theorem which is a natural generalization of Theorem 2.2 and Proposition 2.3 to the multiple exercise case.

Theorem 2.5 *It holds for all* $0 \leq i \leq T$, $L = 1, 2, ...$

$$Y_i^{*L} = \inf_{M^{(1)},...,M^{(L)}\in\mathcal{M}} E_i \max_{i\leq j_1<\cdots<j_L} \sum_{k=1}^{L} \left(Z_{j_k} + M_{j_{k-1}}^{(k)} - M_{j_k}^{(k)}\right) \tag{2.8}$$

$$= \max_{i\leq j_1<\cdots<j_L} \sum_{k=1}^{L} \left(Z_{j_k} + M_{j_{k-1}}^{*L-k+1} - M_{j_k}^{*L-k+1}\right) \quad a.s., \tag{2.9}$$

and in addition

$$E_i Y_{i+1}^{*L} = \max_{i<j_1<\cdots<j_L} \sum_{k=1}^{L} \left(Z_{j_k} + M_{j_{k-1}}^{*L-k+1} - M_{j_k}^{*L-k+1}\right) \quad a.s., \tag{2.10}$$

with $j_0 := i$, *and where for* $k = 1, ..., L$, M^{*k} *is the Doob martingale of the Snell envelope for* k *exercise rights. That is*

$$Y^{*k} = Y_0^{*k} + M^{*k} - A^{*k}$$

with A^* *predictable, and* $M_0^{*k} = A_0^{*k} = 0$. *In particular, for each* k, A_i^{*k} *is nondecreasing in* i.

Proof For any set of martingales $M^{(1)}, ..., M^{(L)} \in \mathcal{M}$ we have
(with $\tau^{(0)} := j_0 := i$),

$$\begin{aligned}
Y_i^{*L} &= \sup_{i\leq\tau^{(1)}<...<\tau^{(L)}} E_i \sum_{k=1}^{L} Z_{\tau^{(k)}} \\
&= \sup_{i\leq\tau^{(1)}<...<\tau^{(L)}} E_i \sum_{k=1}^{L} \left(Z_{\tau^{(k)}} + M_{\tau^{(k-1)}}^{(k)} - M_{\tau^{(k)}}^{(k)}\right) \\
&\leq \sup_{i\leq\tau^{(1)}<...<\tau^{(L)}} E_i \max_{i\leq j_1<\cdots<j_L} \sum_{k=1}^{L} \left(Z_{j_k} + M_{j_{k-1}}^{(k)} - M_{j_k}^{(k)}\right) \\
&= E_i \max_{i\leq j_1<\cdots<j_L} \sum_{k=1}^{L} \left(Z_{j_k} + M_{j_{k-1}}^{(k)} - M_{j_k}^{(k)}\right),
\end{aligned}$$

from which it follows that Y_i^{*L} is less than or equal to the right-hand-side of (2.8). We will next show that this inequality is sharp and that moreover (2.9) and (2.10) hold, by induction on the number of exercise rights L.

Due to Definition 2.1, for all $j > T$ we have $Z_j \equiv 0$, and $M_j \equiv M_T$ for any $M \in \mathcal{M}$. Thus, for $L = 1$ the statements collapse to the statements of Theorem 2.2 and Proposition 2.3.

Now suppose the Theorem holds for L exercise rights. By the Bellman principle it holds,

$$Y_i^{*L+1} = \max\left[Z_i + E_i Y_{i+1}^{*L},\ E_i Y_{i+1}^{*L+1}\right],$$

hence Y_i^{*L+1} may be seen as the Snell envelope of the cash-flow $Z_i + E_i Y_{i+1}^{*L}$ under one exercise right. So by the standard dual representation Theorem 2.2 (and taking into account the conventions in Definition 2.1),

$$Y_i^{*L+1} = \max_{i \le j_1}\left(Z_{j_1} + E_{j_1} Y_{j_1+1}^{*L} + M_i^{*L+1} - M_{j_1}^{*L+1}\right) \quad \text{a.s.,} \tag{2.11}$$

where M^{*L+1} is the Doob martingale of Y_i^{*L+1} satisfying $M_0^{*L+1} = 0$. By the induction hypothesis on (2.10) it now follows that

$$\begin{aligned} Y_i^{*L+1} &= \max_{i \le j_1}\Bigg(Z_{j_1} + M_i^{*L+1} - M_{j_1}^{*L+1} \\ &\quad + \max_{j_1 < p_1 < \cdots < p_L} \sum_{k=1}^{L}\left(Z_{p_k} + M_{p_{k-1}}^{*L-k+1} - M_{p_k}^{*L-k+1}\right)\Bigg) \\ &\quad \text{(with } p_0 := j_1\text{)} \\ &= \max_{i \le j_1 < j_2 < \cdots < j_{L+1}} \sum_{k=1}^{L+1}\left(Z_{j_k} + M_{j_{k-1}}^{*L+1-k+1} - M_{j_k}^{*L+1-k+1}\right) \end{aligned}$$

(with $j_0 := i$). Hence we obtain (2.9) for $L + 1$ rights. Next, by applying Proposition 2.3 to the cash-flow $Z_i + E_i Y_{i+1}^{*L}$, and the induction hypothesis (2.10) for L exercise rights again we obtain,

$$\begin{aligned} E_i Y_{i+1}^{*L+1} &= \max_{i < j_1}\left(Z_{j_1} + E_{j_1} Y_{j_1+1}^{*L} + M_i^{*L+1} - M_{j_1}^{*L+1}\right) \\ &= \max_{i < j_1}\Bigg(Z_{j_1} + M_i^{*L+1} - M_{j_1}^{*L+1} \\ &\quad + \max_{j_1 < p_1 < \cdots < p_L} \sum_{k=1}^{L}\left(Z_{p_k} + M_{p_{k-1}}^{*L-k+1} - M_{p_k}^{*L-k+1}\right)\Bigg) \\ &\quad \text{(with } p_0 := j_1\text{)} \\ &= \max_{i \le j_1 < \cdots < j_{L+1}} \sum_{k=1}^{L+1}\left(Z_{j_k} + M_{j_{k-1}}^{*L+1-k+1} - M_{j_k}^{*L+1-k+1}\right) \end{aligned}$$

hence (2.10) for $L + 1$ rights. Finally, A^{*L+1} is nondecreasing as it is the predictable part of the Snell envelope of the generalized cashflow $Z_i + E_i Y_{i+1}^{*L}$.

Remark 2.6 In a typical application of Theorem 2.5 one has $i = 0$ and $L < T + 1$ (the case $L \geq T - i + 1$ is trivial as already noted in Section 2.2). Then, obviously, the domain of the maximum operator in (2.9) and (2.10) can be restricted to $0 \leq j_1 < \cdots < j_L \leq T$ and $0 < j_1 < \cdots < j_L \leq T$, respectively.

3 Primal-dual Monte Carlo methods for multiple stopping

In this section we show in particular how well known dual approaches for the one exercise case such as the primal-dual algorithm of Andersen and Broadie (2004) may be generalized to the multiple exercise case. In this context we assume that the cash-flow process Z is of the form (with slight abuse of notation)

$$Z_i = Z_i(X_i) \quad 0 \leq i \leq T, \tag{3.1}$$

for some underlying (possibly high-dimensional) Markovian process X. Moreover it is assumed that we are given approximations $Y_i^{(k)}$ of Y_i^{*k}, $k = 1, ..., L$, which are of the form

$$Y_i^{(k)} = Y_i^{(k)}(X_i), \quad 0 \leq i \leq T, \quad 1 \leq k \leq L. \tag{3.2}$$

Remark 3.1 As an immediate consequence of (3.1) the Snell envelopes are also of the form

$$Y_i^{*k} = Y_i^{*k}(X_i), \quad k = 1, ..., L, \tag{3.3}$$

so (3.2) is a quite natural assumption.

It is meanwhile industrial standard to obtain approximations of the form (3.2) by regression methods. For the single exercise case (Bermudan derivatives for example) the methods of Longstaff and Schwartz (2001) and Tsitsiklis and Van Roy (2001) are quite popular, and for the multiple exercise case (e.g. swing options) one may apply these methods recursively as explained in Section 3.1. One generally obtains in this way sub-optimal exercise strategies, hence lower bounds for the optimal value. In Section 3.2 it is described how to incorporate (e.g. regression based) approximations for the Snell envelope in a Monte Carlo procedure for dual upper bounds.

3.1 Recap of regression based approaches

Let us recap briefly how well-known regression methods such as the method of Longstaff and Schwartz (2001) and Tsitsiklis and Van Roy (2001) may be recursively applied to the multiple exercise problem. As these methods are broadly known, we do not explain them here in detail but merely recall that for the single exercise case, both methods end up with an expansion of the continuation function in terms of a properly chosen and 'rich enough' system of basis functions on the state space. That is, for an approximation of the (single exercise) Snell envelope one obtains formally

$$C_i^*(X_i) := E_i\, Y_{i+1}^*(X_{i+1}) \approx \sum_{r=0}^{R} \beta_{ir}\psi_r(X_i) =: C_i(X_i), \quad 0 \le i < T,$$

where $(\psi_r : \mathbb{R}^d \to \mathbb{R}^d,\ r = 0, 1, 2, ...)$ is a (countable) set of basis functions and R some number which determines the number of basis functions involved in the regression. (Note that due to Definition 2.1, $C_T :\equiv 0$.) The coefficients (β_{ir}) are obtained by a regression procedure applied to a Monte Carlo sample of trajectories of X. In Clement et al. (2002) it is analyzed that for a suitable set of basis functions under suitable conditions $C_i \to C_i^*$, when the number of trajectories and the number of basis functions involved go to infinity in a suitable relationship.

Application of the above regression method to the multiple exercise problem is described by the following inductive scheme:

- Step 1 : Construct with our favorite regression method for $0 \le i \le T$ (approximations to) the continuation functions $C_i^{(1)}(\cdot)$ of the single exercise problem.
- Step k : Let the continuation functions $C_i^{(p)}(\cdot)$, $0 \le i \le T$, of the (approximative) multiple exercise problem for p exercise rights be constructed for all $1 \le p \le k \le L$, that is,

$$C_i^{*p}(X_i) := E_i\, Y_{i+1}^{*p}(X_{i+1}) \approx \sum_{r=0}^{R} \beta_{ir}^{(p)}\psi_r(X_i) =: C_i^{(p)}(X_i)$$

  (with $C_T^{(p)} :\equiv 0$). Then,
  - If $k < L$, define the cash-flow process

$$\widetilde{Z}_i(X_i) := Z_i(X_i) + C_i^{(k)}(X_i)$$

    with $C_T^{(k)} :\equiv 0$, and apply our favorite regression method to obtain (approximations to) the continuation function $\widetilde{C}_i(X_i)$ corresponding to the Snell envelope of $\widetilde{Z}_i$ under one exercise right. Then set

$$C_i^{(k+1)}(X_i) := \widetilde{C}_i(X_i), \quad 0 \le i \le T.$$

  - if $k = L$, then stop.

Inductive application of the above scheme thus yields a system of (approximate) continuation functions $C_i^{(k)}(\cdot)$, $1 \le k \le L$, $1 \le i \le T$. At this stage one may take as approximations for the Snell envelopes $(Y_i^{*k} : 1 \le i \le T, 1 \le k \le L)$,

$$Y_i^{(k)}(X_i) := \max[Z_i(X_i) + C_i^{(k-1)}(X_i),\ C_i^{(k)}(X_i)] \tag{3.4}$$

with $C^{(0)} :\equiv 0$.

It is important to note that while approximations (3.4) may be accurate, they can be biased from above or below. For bounding the Snell envelopes from above *and* below later on, we also need lower bounds however. For this we construct for each k, $1 \leq k \leq L$, a system of (sub-optimal) exercise policies $(\tau_i^{p,k} : 1 \leq i \leq T,\, 1 \leq p \leq k)$ as follows. Define $\tau_i^{0,k} := i-1$, and for $0 < p \leq k$,

$$\tau_i^{p,k} = \inf\{j : \tau_i^{p-1,k} < j \leq T,\ Z_j(X_j) + C_j^{(k-p)}(X_i) \geq C_j^{(k-p+1)}(X_j)\}. \quad (3.5)$$

Then the process defined by

$$\underline{Y}_i^{(k)} := E_i \sum_{p=1}^{k} Z_{\tau_i^{p,k}} \quad (3.6)$$

due to the stopping family (3.5) is X_i measurable and is a lower bound process, i.e. $\underline{Y}_i^{(k)}(X_i) \leq Y_i^{*k}(X_i)$. Obviously, the stopping family $(\tau_i^{p,k})$ satisfies for each k the consistency relation

$$\tau_i^{1,k} > i \Longrightarrow \tau_i^{p,k} = \tau_{i+1}^{p,k}, \quad 1 \leq p \leq k. \quad (3.7)$$

Due to (3.7) we have in addition,

$$\begin{aligned}
\underline{Y}_i^{(k)} 1_{\{\tau_i^{1,k} > i\}} &= 1_{\{\tau_i^{1,k} > i\}} E_i \sum_{p=1}^{k} Z_{\tau_i^{p,k}} = 1_{\{\tau_i^{1,k} > i\}} E_i 1_{\{\tau_i^{1,k} > i\}} \sum_{p=1}^{k} Z_{\tau_i^{p,k}} \\
&= 1_{\{\tau_i^{1,k} > i\}} E_i \sum_{p=1}^{k} Z_{\tau_{i+1}^{p,k}} = 1_{\{\tau_i^{1,k} > i\}} E_i E_{i+1} \sum_{p=1}^{k} Z_{\tau_{i+1}^{p,k}} \\
&= 1_{\{\tau_i^{1,k} > i\}} E_i \underline{Y}_{i+1}^{(k)}, \qquad (3.8)
\end{aligned}$$

which is in the case $L = 1$ a corner stone of the primal-dual algorithm (Andersen and Broadie, 2004). The lower bounds $\underline{Y}$ may be constructed by a standard (non-nested) Monte Carlo simulation using (3.5).

3.2 Dual simulation procedures for the multiple stopping problem

For any approximation $Y^{(k)}$, $1 \leq k \leq L$, for example obtained from (3.4) or (3.6), we may construct the Doob martingale $M^{(k)}$ of $Y^{(k)}$, via

$$M_i^{(k)} - M_{i-1}^{(k)} = Y_i^{(k)} - E_{i-1} Y_i^{(k)}, \quad 1 < i \leq T,$$

and consider for each i, $0 \leq i \leq T$, the upper bound

$$
\begin{aligned}
&Y_i^{\mathrm{up},L} := E_i \max_{i \leq j_1 < \cdots < j_L} \sum_{k=1}^{L} \left(Z_{j_k} + M_{j_{k-1}}^{(L-k+1)} - M_{j_k}^{(L-k+1)} \right) \\
&= E_i \max_{i \leq j_1 < \cdots < j_L} \sum_{k=1}^{L} \left(Z_{j_k} - \sum_{r=j_{k-1}+1}^{j_k} Y_r^{(L-k+1)} + \sum_{r=j_{k-1}+1}^{j_k} E_{r-1} Y_r^{(L-k+1)} \right) \\
&\quad = E_i \max_{i \leq j_1 < \cdots < j_L} \sum_{k=1}^{L} \Bigg(Z_{j_k} + Y_{j_{k-1}}^{(L-k+1)} - Y_{j_k}^{(L-k+1)} \\
&\qquad + \sum_{r=j_{k-1}}^{j_k - 1} \left(E_r Y_{r+1}^{(L-k+1)} - Y_r^{(L-k+1)} \right) \Bigg) \\
&\quad = Y_i^{(L)} + E_i \max_{i \leq j_1 < \cdots < j_L} \sum_{k=1}^{L} \Bigg(Z_{j_k} + Y_{j_k}^{(L-k)} - Y_{j_k}^{(L-k+1)} \\
&\quad + \sum_{r=j_{k-1}}^{j_k - 1} \left(E_r Y_{r+1}^{(L-k+1)} - Y_r^{(L-k+1)} \right) \Bigg) =: Y_i^{(L)} + \Delta_i^{(L)} \qquad (3.9)
\end{aligned}
$$

(note that $Y^{(0)} \equiv 0$). The following proposition gives an estimate for the difference of the upper bound (3.9) due to an approximation to the Snell envelope and the approximation itself.

Proposition 3.2 *W.l.o.g. we may assume that $Y_T^{(k)} = Y_T^{*k} = Z_T$, $1 \leq k \leq L$. It then holds for $0 \leq i \leq T$,*

$$
\begin{aligned}
\Delta_i^{(L)} &= Y_i^{up,L} - Y_i^{(L)} \qquad (3.10) \\
&\leq E_i \sum_{k=1}^{L} \sum_{r=i}^{T-1} \left[\left(Z_r + E_r Y_{r+1}^{(k-1)} - Y_r^{(k)} \right)^+ + \left(E_r Y_{r+1}^{(k)} - Y_r^{(k)} \right)^+ \right].
\end{aligned}
$$

Proof From (3.9) we obtain by rearranging terms,

$$
\begin{aligned}
Y_i^{\mathrm{up},L} - Y_i^{(L)} = E_i \max_{i \le j_1 < \cdots < j_L} \Bigg(\sum_{k=1}^{L} \left(Z_{j_k} + E_{j_k} Y_{j_k+1}^{(L-k)} - Y_{j_k}^{(L-k+1)} \right) \\
+ \sum_{r=i}^{j_1-1} \left(E_r Y_{r+1}^{(L)} - Y_r^{(L)} \right) + \sum_{k=1}^{L} \left(Y_{j_k}^{(L-k)} - E_{j_k} Y_{j_k+1}^{(L-k)} \right) + \\
\sum_{k=2}^{L} \left(E_{j_{k-1}} Y_{j_{k-1}+1}^{(L-k+1)} - Y_{j_{k-1}}^{(L-k+1)} \right) + \sum_{k=2}^{L} \sum_{r=j_{k-1}+1}^{j_k-1} \left(E_r Y_{r+1}^{(L-k+1)} - Y_r^{(L-k+1)} \right) \Bigg) \\
= E_i \max_{i \le j_1 < \cdots < j_L} \Bigg(\sum_{k=1}^{L} \left(Z_{j_k} + E_{j_k} Y_{j_k+1}^{(L-k)} - Y_{j_k}^{(L-k+1)} \right) \\
+ \sum_{r=i}^{j_1-1} \left(E_r Y_{r+1}^{(L)} - Y_r^{(L)} \right) + \sum_{k=1}^{L-1} \sum_{r=j_k+1}^{j_{k+1}-1} \left(E_r Y_{r+1}^{(L-k)} - Y_r^{(L-k)} \right) \Bigg),
\end{aligned}
$$

and then the statement easily follows (mind the conventions in Definition 2.1).

Remark 3.3 If in Proposition 3.2 the approximation $Y_i^{(L)}$ is a lower bound, i.e. $Y_i^{(L)} \le Y_i^{*L}$, we have in (3.10) on the other hand $0 \le Y_i^{\mathrm{up},L} - Y_i^{(L)}$. If in particular $Y_\cdot^{(k)} = Y_\cdot^{*k}$ for $1 \le k \le L$, the right-hand-side of inequality (3.10) is zero, so then (3.10) yields $Y_i^{\mathrm{up},L} = Y_i^{(L)} = Y_i^{*k}$. In this sense the estimate (3.10) is sharp. Proposition 3.2 may be considered as a generalization of a similar result for the case $L = 1$ in Andersen and Broadie (2004) and Kolodko and Schoenmakers (2004).

For constructing multiple dual upper bounds we now have two options: the first one is a dual procedure based on approximations to the Snell envelope (for example (3.4)) and the second one is a dual procedure based on approximations to optimal stopping times (for example (3.6)). The procedures are briefly described below (for a more algorithmic description see the preprint version of this paper, Schoenmakers (2009)).

Procedure based on approximations to the Snell envelope

For arbitrary but given approximations to the Snell envelopes $Y^{(k)}$, $1 \le k \le L$ ((3.4) for example), the implementation of (3.9) within the Markovian framework around (3.1)–(3.3) naturally leads to a (one-degree) nested Monte Carlo simulation. Let us take w.l.o.g. $i = 0$ and assume $L \le T + 1$ to exclude trivialities. In fact, by splitting of the (given) approximation $Y_0^{(L)}$ in (3.9) , one needs to simulate the difference term $\Delta_0^{(L)}$. As a rule, this will reduce the variance of the estimation remarkably. In principle, the corresponding Monte Carlo simulation may be organized as follows:

Develop in a straightforward way a (formally non-nested) Monte Carlo procedure for $\Delta_0^{(L)}$ in which, besides the given functions $Y_r^{(l)}(X)$ (being approximations to the Snell envelope), the conditional expectation $E_r Y_{r+1}^{(l)}$ is generically declared as (an at this level known) function of the underlying state X, exercise date r, and number of exercise possibilities l. Next, the implementation of this conditional expectation function itself requires a one-step inner simulation procedure starting at X. After each one-step inner simulation, from $X_r^{\text{inner}} = X$ to X_{r+1}^{inner} say, one reads the value of the given (approximate) Snell envelope $Y_{r+1}^{(l)}(X_{r+1}^{\text{inner}})$ and then takes the average over the inner simulations. For an illustration in the case $L = 1$, $T = 3$, see Figure 1, upper picture.

Remark 3.4 It should be noted that, since for any (finite) set of random variables $(\varsigma_i : i \in I)$ it holds $E \max_{i\in I} \varsigma_i \geq \max_{i\in I} E\varsigma_i$, it follows analogously to Andersen and Broadie (2004) (and also Kolodko and Schoenmakers (2004)) that the above procedure yields a Monte Carlo estimate of $Y_0^{\text{up},L}$ which is biased up, i.e. the expectation of this estimate is an upper bound.

Remark 3.5 We underline that the above procedure is essentially different from the one in Meinshausen and Hambly (2004) as it may be applied directly to any approximation to the Snell-envelope (given in closed form for instance) and thus may be regarded as 'stopping time free'. In contrast, the Meinshausen and Hambly (2004) method always involves a set of 'good' stopping times for the multiple stopping problem besides a set of 'good' martingales.

Remark 3.6 In view of Proposition 3.2 we may modify the Monte Carlo procedure in an obvious way to obtain a procedure which estimates the upper bound

$$Y_0^{(L)} + E_i \sum_{k=1}^{L} \sum_{r=0}^{T-1} \left[\left(Z_r + E_r Y_{r+1}^{(k-1)} - Y_r^{(k)} \right)^+ + \left(E_r Y_{r+1}^{(k)} - Y_r^{(k)} \right)^+ \right] \geq Y_0^{\text{up},L}.$$

Clearly, in this method no path-wise maximization procedure is involved. The price one may have to pay however is a higher upper bound. But, due to Remark 3.3 this upper bound may be still tight if the input approximations of the Snell envelopes are good enough.

Procedure based on approximations to optimal stopping times

Let us now consider an arbitrary but given family of (sub-optimal) exercise policies $(\tau_i^{p,k} : 1 \leq i \leq T,\ 1 \leq p \leq k)$, $1 \leq k \leq L$, which satisfies the consistency condition (3.7) (for example the one obtained from (3.5)). Such a family yields by (3.6) a set of lower approximations to the Snell envelope for which, by (3.7), (3.9) (for $i=0$) takes the form,

$$\underline{Y}_0^{\mathrm{up},L} := \underline{Y}_0^{(L)} + E_0 \max_{0\le j_1<\cdots<j_L\le T} \sum_{k=1}^{L} \Bigg(Z_{j_k} + \underline{Y}_{j_k}^{(L-k)} - \underline{Y}_{j_k}^{(L-k+1)} \tag{3.11}$$

$$+ \sum_{r=j_{k-1}}^{j_k-1} 1_{\left\{\tau_r^{1,L-k+1}=r\right\}} \left(E_r \underline{Y}_{r+1}^{(L-k+1)} - \underline{Y}_r^{(L-k+1)} \right) \Bigg) =: \underline{Y}_0^{(L)} + \underline{\Delta}_0^{(L)},$$

where the $\underline{Y}_r^{(l)}$ are given by (3.6), and

$$E_r \underline{Y}_{r+1}^{(l)} = E_r \sum_{p=1}^{l} Z_{\tau_{r+1}^{p,l}}. \tag{3.12}$$

Naturally, the implementation of (3.11) leads to a nested Monte Carlo procedure again, where the lower bound $\underline{Y}_0^{(L)}$ can be accurately computed by a standard (non-nested) Monte Carlo simulation, using stopping rule (3.5) and a relatively large sample size. In the case $L=1$ this usually takes out about 90% of the variance, depending on the quality of the stopping rule of course. What is left is the estimation of the nonnegative gap term (see Remark 3.3) $\underline{\Delta}_0^{(L)} = \underline{Y}_0^{\mathrm{up},L} - \underline{Y}_0^{(L)}$.

Just as for the previous simulation procedure (based on approximations to the Snell envelope) we may develop a (formally non-nested) Monte Carlo procedure for $\underline{\Delta}_0^{(L)}$ in which the lower approximation to the Snell envelope $Y_r^{(l)}$ and the conditional expectation $E_r Y_{r+1}^{(l)}$ is generically declared as (an at this level known) function of the underlying state X, exercise date r, and number of exercise possibilities l. Here the implementation of $Y_r^{(l)}$ requires an inner simulation procedure based on (3.6) where, in contrast to the previous method, each inner simulation continuous until all the stopping rules $\tau_r^{p,l}$, $1\le p\le l$, have stopped. Likewise, the implementation of the function $E_r Y_{r+1}^{(l)}$ requires inner simulations based on (3.12) where each inner simulation continuous until all the stopping rules $\tau_{r+1}^{p,l}$, $1\le p\le l$, have stopped. According to (3.11) the latter function will only be called in the case where $\tau_r^{1,l}=r$. For an illustration in the case $L=1$, $T=3$, see Figure 1, lower picture.

Remark 3.7 (i) Remark 3.4 applies again; the above procedure yields a Monte Carlo estimate of $\underline{Y}_0^{\mathrm{up},L}$ which is biased up. (ii) For $L=1$, representation (3.11) and the corresponding Monte Carlo procedure collapses to the well-known Andersen-Broadie representation and Andersen-Broadie primal-dual algorithm, respectively. Indeed, for $L=1$ we get

$$\underline{Y}_0^{\mathrm{up},L}(X_0) = \underline{Y}_0(X_0) + E_0 \max_{0\le j\le T} \left(Z_j - \underline{Y}_j + \sum_{r=0}^{j-1} 1_{\{\tau=r\}} \left(E_r \underline{Y}_{r+1} - \underline{Y}_r \right) \right) \tag{3.13}$$

with the well demonstrated advantage that the term $\underline{Y}_0(X_0)$ may be computed using an accurate non-nested Monte Carlo simulation, and that the remaining gap term has typically low variance.

Remark 3.8 Given the success of the dual representation in the single exercise case as reported in the literature, it will be obvious that analogue simulation procedures for the multiple case as suggested in this paper are potentially promising as well. A detailed description and numerical study (including a comparison with the method of Meinshausen and Hambly (2004)) is the subject of a subsequent article which is currently in preparation.

Remark 3.9 In the case where the process X is adapted to a Brownian filtration it looks feasible to construct a linear Monte Carlo algorithm for the multiple dual in a similar way as presented in Belomestny et al. (2009a). This might be done in future work.

Remark 3.10 One may wonder whether it is possible to generalize in a similar way the multiplicative dual approach of Jamshidian (2007) to the multiple exercise case. Anyway, the additive multiple dual as constructed in this article inherits the nice almost sure property of the standard additive dual representation when the optimal martingale is plugged in. The multiplicative dual fails to have this property, see also the discussion in Chen and Glasserman (2007) (below Prop. 6.8) on this.

Acknowledgements The author is grateful to the organizers Mark Broadie, Friedrich Hubalek, Damien Lamberton, Peter Laurence, and the participants of the 'Conference on Numerical Methods for American and Bermudan Options' at Vienna, October 2008. In the discussion after Christian Bender's talk on multiple exercise options, Chris Rogers did a guess how a 'pure martingale' multiple dual might look like. His guess was

$$Y_0^{*L} = \inf_{\substack{M\in\mathcal{M},\\ M_0=0}} E_0 \max_{0\leq j_1<\cdots<j_L\leq T} \sum_{k=1}^{L} \left(Z_{j_k} - M_{j_k}\right).$$

Inspired to examine this guess the present paper resulted. The author is also grateful to Christian Bender who shortened the author's proof of Proposition 2.3, and Anastasia Kolodko for constructive remarks.

References

Aleksandrov, N., Hambly, B.M.: A dual approach to multiple exercise option problems under constraints. Preprint (2008). http://people.maths.ox.ac.uk/hambly/PDF/Papers/oms2rev.pdf

Andersen, L.: A Simple Approach to the Pricing of Bermudan Swaptions in a Multifactor LIBOR Market Model. J. Comp. Finance **3**, 5–32 (1999)

Andersen, L., Broadie, M.: A primal-dual simulation algorithm for pricing multidimensional American options. Management Sciences **50**, No. 9, 1222–1234 (2004)

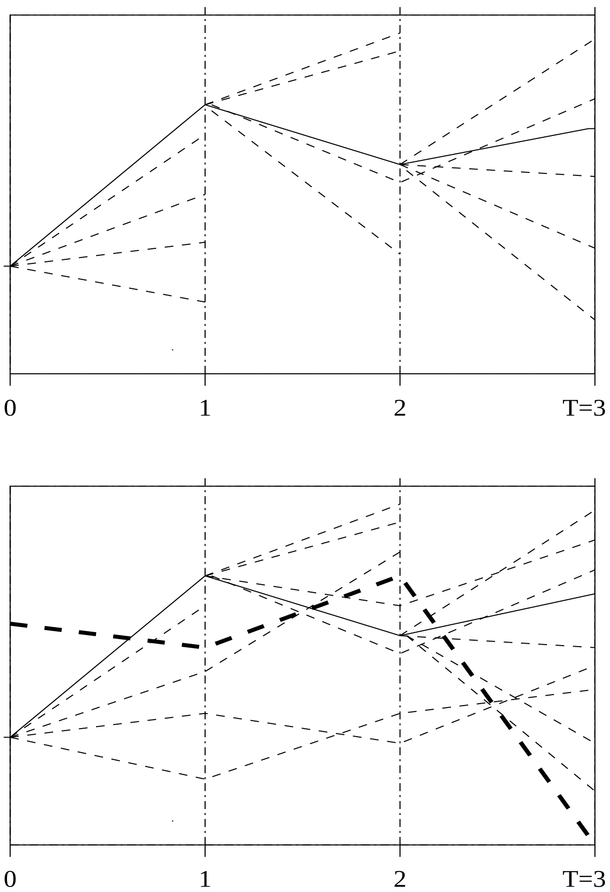

Fig. 3.1 Illustration for $L = 1$, $T = 3$, of inner simulations (dashed lines) from a fixed trajectory (solid line) for constructing an upper bound due to an approximation to the Snell envelope (upper picture), and due to an approximation to the optimal strategy, hence optimal exercise region (above the bold dashed line, lower picture).

Belomestny, D., Bender, C., Schoenmakers, J.: True upper bounds for Bermudan products via non-nested Monte Carlo. Math. Finance **19**, No. 1 , 53–71 (2009a)

Belomestny, D., Milstein, G.N.: Monte Carlo evaluation of American options using consumption processes. Int. J. Theor. Appl. Finance **9**, No. 4, 1–27 (2006)

Belomestny, D., Milstein, G.N., Spokoiny, V.: Regression methods in pricing American and Bermudan options using consumption processes. Quant. Fin. **9**, No. 3, 315–327 (2009b)

Bender, C.: Dual Pricing of Multi-Exercise Options under Volume Constraints. Preprint (2008). Finance Stoch. to appear.

Bender, C., Kolodko, A., Schoenmakers, J.: Enhanced policy iteration for American options via scenario selection. Quant. Finance **8**, No. 2, 135–146 (2008)

Bender, C., Kolodko, A., Schoenmakers, J.: Iterating cancellable snowballs and related exotics. Risk **8**, September issue, 135–146 (2006)

Bender, C., Schoenmakers, J.: An iterative algorithm for multiple stopping: Convergence and stability. Adv. Appl. Prob. **38**, No. 3, 729–749 (2006)

Broadie, M., Glasserman, P.: A stochastic mesh method for pricing high-dimensional American options. J. Comp. Finance **7**, 4, 35–72 (2004)

Carmona, R., Touzi, N.: Optimal Multiple-Stopping and Valuation of Swing Options. Math. Finance **18**, No. 2, 239–268 (2008)

Chen, N., Glasserman P.: Additive and Multiplicative Duals for American Option Pricing. *Finance Stoch.* **11**, 153–179 (2007)

Clément, E., Lamberton, D., Protter P.: An analysis of a least squares regression algorithm for American option pricing. Finance Stoch. **6**, 449–471 (2002)

Davis, M.H.A., Karatzas, I.: A Deterministic Approach to Optimal Stopping. In: Probability, Statistics and Optimisation (ed. F.P. Kelly). NewYork Chichester: John Wiley & Sons Ltd. 455–466 (1994)

Haugh, M. B., Kogan, L.: Pricing American Options: A Duality Approach. Operations Research **52**, 258–270 (2004)

Jaillet, P., Ronn, E. I., Tompaidis, S.: Valuation of Commodity Based Swing Options. Management Science **50**, 909–921 (2004)

Jamshidian, F.: The duality of optimal exercise and domineering claims: A Doob-Meyer decomposition approach to the Snell envelope. Stochastics **79**, No. 1–2, 27–60 (2007)

Kolodko, A., Schoenmakers, J.: Upper bounds for Bermudan style derivatives. Monte Carlo Methods and Appl. **10**, No. 3–4, 331–343 (2004)

Kolodko, A., Schoenmakers, J.: Iterative Construction of the Optimal Bermudan Stopping Time. Finance Stoch. **10**, 27–49 (2006)

Longstaff, F. A., Schwartz, R. S.: Valuing American Options by Simulation: A Simple Least-Square Approach. Review of Financial Studies **14**, 113–147 (2001)

Meinshausen, N., Hambly, B. M.: Monte Carlo Methods for the Valuation of Multiple-Exercise Options. Math. Finance, **14**, 557–583 (2004)

Neveu, J.: *Discrete Parameter Martingales.* North-Holland, Amsterdam (1975)

Rogers, L. C. G.: Monte Carlo Valuation of American Options. Math. Finance **12**, 271–286 (2002)

Schoenmakers J.: The real multiple dual. WIAS preprint 1438 (2009)

Tsitsiklis, J., Van Roy, B.: Regression methods for pricing complex American style options. IEEE Trans. Neural. Net., **12**, 694–703 (2001)