A new Monte Carlo method for American Options

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November 25, 2003

2000 Mathematical Subject Classification: 60H30, 65C30, 91B28.

Keywords: Pricing and hedging of American options, Monte Carlo simulation, Determination of the exercise boundary

Abstract

We introduce a new Monte Carlo method for constructing the exercise boundary of an American option in a generalized Black-Scholes framework. Based on a known exercise boundary, it is shown how to price and hedge the American option by Monte Carlo simulation of suitable probabilistic representations in connection with the respective parabolic boundary value problem. The method presented is supported by numerical experiments.

1 Introduction

We consider the general one-dimensional American style option in a generalized Black-Scholes framework

$$dX_t = X_t(a(t,X_t)dt + \sigma(t,X_t))dW_t), \quad X_0 = x, \tag{1}$$
$$dB_t = r(t,X_t)B_t dt, \quad B_0 = 1,\ 0 \le t \le T. \tag{2}$$

In (1), (2), the process X is the price of a risky asset, B is the price of a locally riskless asset, and r, a, σ are smooth and bounded functions from $[0,T] \times \mathbf{R}^+ \to \mathbf{R}$, such that the system (1) and (2) has a unique solution. Due to the American style option contract the holder has the right to exercise the option at any time t with $0 \le t \le T$, yielding a payoff $f(X_t)$, where f is a continuous function from $\mathbf{R}^+$ to $\mathbf{R}^+$. For example, an American put with strike price $K > 0$ is specified by $f(x) = (K - x)^+$.

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If we set $a = r$ in (1) we obtain the price process X in the risk neutral measure. We recall that with respect to the risk neutral measure the discounted process $\widetilde{X}(t) := e^{-\int_0^t r(s,X_s)ds}X(t)$ is a martingale and the price $u(t, X_t)$ of the option is given by

$$u(t,x) = \sup_{\tau \in \mathcal{T}_{t,T}} E[e^{-\int_t^\tau r(s,X_s^{t,x})ds} f(X_\tau^{t,x})] \tag{3}$$

where $\mathcal{T}_{t,T}$ represents the set of stopping times τ taking values in $[t, T]$, and $X_s^{t,x}$ is the solution of (1) with $X_t^{t,x} = x$, see e.g. [5]. It is well known that if the function f is bounded, non-increasing and convex, then $u(t, x)$ in (3) can be seen as the solution of a free boundary value problem where the free boundary γ is given by an equation $x = g(t)$, such that

$$\begin{aligned} u(t,x) &= f(x), \qquad \text{for } \ t = T \ \text{ or } \ x \leq g(t), \\ Lu := \frac{\partial u}{\partial t} &+ \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 u}{\partial x^2} + rx\frac{\partial u}{\partial x} - ru = 0\,, \qquad x > g(t), \end{aligned} \tag{4}$$

and that the so-called ,,smooth fitting (pasting) principle" is fulfilled at $\gamma(t)$, $t < T$:

$$\frac{\partial u}{\partial x}(t,x)\Big|_{x \downarrow g(t)} = \frac{\partial f}{\partial x}(x)\Big|_{x \uparrow g(t)}. \tag{5}$$

See e.g. [5, 12, 14] and the references therein for detailed studies of American options. The curve γ is called the exercise boundary or critical price curve in the sense that it is optimal to hold the option if $X_t > g(t)$ and to exercise when $X_t \leq g(t)$. The option price $u(t, x)$ in the domain $G := \{(t, x) : \ 0 \leq t < T, \ x > g(t)\}$ is the solution of the Dirichlet boundary value problem

$$Lu \ : \ = \frac{\partial u}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 u}{\partial x^2} + rx\frac{\partial u}{\partial x} - ru = 0\,, \tag{6}$$

$$u_{|\bar{\gamma}} \ = \ f(x)\ , \tag{7}$$

where the boundary $\bar{\gamma}$ consists of γ for $0 \leq t < T$ and the ray $\{(T, x) \mid x > g(T)\}$.

A hedging strategy for the American option is a self-financing portfolio $(\varphi_t, \ \psi_t)$, where φ_t and ψ_t are the amounts an option writer should hold in riskless B and risky asset S, respectively, in order to hedge the payoff of the option when the option holder exercises. It is known [5], that a self-financing hedging strategy is given by

$$\begin{aligned} \varphi_t &= \frac{1}{B_t}(u(t,X_t) - X_t\frac{\partial u}{\partial x}(t,X_t))\,, \\ \psi_t &= \frac{\partial u}{\partial x}(t,X_t), \end{aligned} \tag{8}$$

where

$$V_t = u(t, X_t) = \varphi_t B_t + \psi_t X_t$$

is the value of the replicating portfolio, i.e. at any time τ the holder exercises, it is guaranteed that $V(\tau) \geq f(S_\tau)$ and the portfolio satisfies the self-financing condition

$$dV_t = \varphi_t dB_t + \psi_t dX_t.$$

Moreover, due to (4) and (5) the function $v(t,x) := \frac{\partial u}{\partial x}(t,x)$ satisfies the boundary value problem

$$\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 v}{\partial x^2} + (\sigma^2 x + rx)\frac{\partial v}{\partial x} = 0\,, \quad (t,x) \in G, \tag{9}$$

$$v_{|\bar{\gamma}} = f'(x)\,. \tag{10}$$

In general, determination of the exercise boundary γ is a challenging task and, in particular, if γ is known, both the option value and the hedging strategy can be computed by Monte Carlo simulation of (6), (7), (9), (10).

For the standard American put where $f(x) = (K-x)^+$ with respect to the standard Black Scholes model, analytical approximations and asymptotic expressions for the exercise boundary near maturity have been studied extensively in the literature. For instance, see [1, 2, 4, 6, 13, 15] and the references therein. For the general case, however, the problem has to be solved by numerical methods. As a new alternative, we construct in Section 4 a Monte Carlo method for the determination of the critical exercise boundary γ. We will explain this method in detail for the one-dimensional case, and then we present a generalization for the multi-dimensional case.

Since for construction of a hedging strategy one needs at any time t the individual values $u(t,X_t)$ and $v(t,X_t) = \frac{\partial u}{\partial x}(t,X_t)$ at the known state X_t of the market, Monte Carlo methods are quite appealing, particularly in more dimensions. Of course, the computation time for attaining an accuracy ϵ by a standard Monte Carlo method which is typically of order $O(\varepsilon^{-2})$ independent of the dimension might be higher than the required time of some finite difference method for dimension one. However, due to ease of implementation, various possibilities of variance reduction (see Section 2), application of higher order integration schemes, and parallelizing opportunities, even in one dimension Monte Carlo simulation turns out to be a valuable tool. In Section 2, we give various probabilistic representations for solutions of boundary value problems (6)-(7) and (9)-(10) connected to respective stochastic differential equations (SDEs), provided that the critical price curve γ is known. There we also investigate some issues of variance reduction. In Section 3, we propose some algorithms for Monte Carlo evaluation of $u(t,x)$ and $v(t,x)$ under known exercise boundary. These algorithms are based on the results of [8], [9].

Usually, the exercise curve γ is not explicitly known and so for implementation of the methods presented in Section 2 and Section 3, one needs to construct γ first. For example, γ may be obtained by a finite difference method [5] which solves $u(t,x)$ by a parabolic system of differential inequalities. In a standard Black-Scholes environment, γ can also be constructed from the solution of a canonical optimal stopping problem (3). So the critical price curve plays a key role in pricing and hedging American options. In Section 4 we present a Monte Carlo construction of the curve γ without preliminary knowledge of the price $u(t,x)$ in the whole domain G. The critical price curve γ is built step-by-step where at each step we principally use the Snell envelope. The proposed procedure can be seen as an alternative to direct solutions of the corresponding Stefan problem [12, 14] (for example, by finite difference methods). Besides the fact that a pure Monte Carlo construction of the exercise boundary γ is interesting from a theoretical point of view, this procedure is easy to implement and requires only few storage capacity.

2 Probabilistic representations for price and hedge of the American option under known exercise boundary

The solution to the problem (6)-(7) has the following probabilistic representation

$$u(t,x) = E\left[e^{-\int_t^\tau r(s,X_s^{t,x})ds} f(X_\tau^{t,x})\right],\ (t,x) \in G, \tag{11}$$

where $X_s^{t,x}$ is the solution of the SDE

$$dX_s = X_s(r(s,X_s)ds + \sigma(s,X_s)dW_s), \quad X_t = x,\ s \geq t, \tag{12}$$

and

$$\tau = \tau^{t,x} = T \wedge \inf\{s : (s, X_s^{t,x}) \in \gamma\} \tag{13}$$

is a stopping time which is defined as the first time the process of (s, X_s) reaches the boundary $\bar{\gamma}$ (see Fig. 1). We should note that a more rigorous notation for (12) would be

$$dX_s = X_s(rds + \sigma dW_s)\, 1_{\{\tau > s\}},$$

but we use (12) as long as it does not lead to confusion.

As a probabilistic representation for the solution to problem (9)-(10) we have,

$$v(t,x) = E[f'(X_\tau^{t,x})], \quad (t,x) \in G, \tag{14}$$

where $X_s^{t,x}$ satisfies the equation

$$dX_s = X_s((r(s,X_s) + \sigma^2(s,X_s))ds + \sigma(s,X_s)dW_s),\ X_t = x,\ s \geq t, \tag{15}$$

and τ as in (13).

In general, the solution to problem (6)-(7) has various probabilistic representations:

$$u(t,x) = E[f(X_\tau^{t,x})Y_\tau^{t,x,1} + Z_\tau^{t,x,1,0}], \tag{16}$$

where $X,\ Y,\ Z$ satisfy the system of SDEs

$$\begin{aligned} dX_s &= X_s(r(s,X_s) - \mu(s,X_s)\sigma(s,X_s))ds + \sigma(s,X_s)X_s dW_s, \quad X_t = x, & (17)\\ dY_s &= -r(s,X_s)Y_s ds + \mu(s,X_s)Y_s dW_s, \quad Y_t = 1, & (18)\\ dZ_s &= F(s,X_s)Y_s dW_s, \quad Z_t = 0, & (19) \end{aligned}$$

where $\mu(\cdot,\cdot)$ and $F(\cdot,\cdot)$ are rather arbitrary functions, however, with good analytical properties and $\tau = \tau^{t,x}$ is the first time the process X in (17) hits the boundary $\bar{\gamma}$. The usual probabilistic representation (11) is a particular case of (16) – (19) with $\mu = 0$, $F = 0$, see, e.g. [3]. The representation for $\mu \neq 0$, $F = 0$ follows from Girsanov's theorem and then for $F \neq 0$ from $EZ = 0$.

Consider the random variable $\eta := f(X_\tau^{t,x})Y_\tau^{t,x,1} + Z_\tau^{t,x,1,0}$. While the mathematical expectation $E\eta$ does not depend on μ and F, the variance $\mathbf{Var}\ \eta = E\eta^2 - (E\eta)^2$ does. So, for a Monte Carlo estimation of (16) the variance may be reduced by suitably choosing the functions μ and F. In this respect two variance reduction methods are well known: The method of importance sampling where one takes $F = 0$ and seeks for a proper μ, and, the method of control variates where one takes $\mu = 0$ and seeks for a proper F. For both methods it is shown that, in principle, the variance can be reduced to zero. A generalization of these methods is obtained in [11]. We should note that, in fact, these variance reduction methods concern the Cauchy problem for equations of parabolic type, although the method of importance sampling is considered for boundary value problems as well in [7]. Here we carry over the results of [11] for the boundary value problem (6)-(7). We introduce the process

$$\eta_s := u(s \wedge \tau, X^{t,x}_{s\wedge\tau}) Y^{t,x,1}_{s\wedge\tau} + Z^{t,x,1,0}_{s\wedge\tau} .$$

Clearly

$$\eta_t = u(t,x),\ \eta_\tau = f(X^{t,x}_\tau)Y^{t,x,1}_\tau + Z^{t,x,1,0}_\tau = \eta_T = \eta.$$

Theorem 2.1 *Let μ and F be such that for any $x \in G$ there is a solution of the system* (17)-(19) *on the interval* $[t,\tau]$. *The variance* $\mathbf{Var}\ \eta$ *is equal to*

$$\mathbf{Var}\ \eta = E\int_t^{T\wedge\tau} Y_s^2(\sigma X_s \frac{\partial u}{\partial x}(s,X_s) + u(s,X_s)\mu(s,X_s) + F(s,X_s))^2 ds \qquad (20)$$

provided that the mathematical expectation in (20) *exists. In particular, if μ and F are such that*

$$\sigma x \frac{\partial u}{\partial x} + u\mu + F = 0\,, \qquad (21)$$

then $\mathbf{Var}\ \eta = 0$ *and η_s is deterministic and independent of $s \in [t,\tau]$.*

Proof. By Itô's formula, we obtain

$$\begin{aligned} d\eta_s &= 1_{\{\tau>s\}}[Lu(s,X_s)Y_s ds + \frac{\partial u}{\partial x}(s,X_s)X_sY_s\sigma(s,X_s)dW_s \\ &\quad + u(s,X_s)\mu(s,X_s)Y_s dW_s + F(s,X_s)Y_s dW_s] \\ &= 1_{\{\tau>s\}}[\frac{\partial u}{\partial x}(s,X_s)X_s\sigma(s,X_s) + u(s,X_s)\mu(s,X_s) + F(s,X_s)]Y_s dW_s, \end{aligned}$$

where $Lu = 0$ is taken into account. We thus get

$$\begin{aligned} \eta(s) &= u(t,x) + \\ &\int_t^s 1_{\{\tau>\alpha\}}\left[\frac{\partial u}{\partial x}(\alpha,X_\alpha)X_\alpha\sigma(\alpha,X_\alpha) + u(\alpha,X_\alpha)\mu(\alpha,X_\alpha) + F(\alpha,X_\alpha)\right] Y_\alpha dW_\alpha. \end{aligned}$$

Hence, (20) follows and the last assertion is obvious. □

Remark 2.2 Clearly μ and F from Theorem 2.1 cannot be constructed without knowing $u(s,x)$. Nevertheless, the theorem claims a general possibility of variance reduction by properly choosing the functions μ and F.

In the same way, we obtain via (14)-(15) the following representations for the solution of problem (9)-(10):

$$v(t,x) = E[f'(X_{t,x}(\tau))Y_{t,x,1}(\tau) + Z_{t,x,1,0}(\tau)], \qquad (22)$$

where $X,\ Y,\ Z$ satisfy the system of SDEs

$$\begin{aligned} dX_s &= X_s(r(s,X_s)+\sigma^2(s,X_s)-\tilde{\mu}(s,X_s)\sigma(s,X_s))ds+\sigma(s,X_s)X_s dW_s, && (23)\\ dY_s &= \tilde{\mu}(s,X_s)Y_s dW_s, && (24)\\ dZ_s &= \tilde{F}(s,X_s)Y_s dW_s, && (25)\end{aligned}$$

with the initial conditions

$$X_t = x, \qquad Y_t = 1, \qquad Z_t = 0. \tag{26}$$

Remark 2.3 It is interesting to see that for

$$\tilde{\mu} = \mu + \sigma,$$

(23) coincides with (17) and, as a consequence, their solutions X as well as the stopping times τ for hitting the boundary $\bar{\gamma}$ coincide as well. In particular, for $\mu = 0,\ \tilde{\mu} = \sigma,\ \tilde{F} = 0$ we obtain

$$\begin{aligned} u(t,x) &= E\ \exp\left(\int_0^\tau -r(s,X_s)ds\right) f(X_\tau^{t,x})\\ v(t,x) &= E\ \exp\left(\int_0^\tau -\frac{\sigma^2(s,X_s)}{2}ds+\sigma(s,X_s)dW_s\right) f'(X_\tau^{t,x}), \end{aligned} \tag{27}$$

where X satisfies SDE (12) and τ is defined by (13). Formulas (27) allow us to evaluate $u(t,x)$ and $v(t,x)$ by Monte Carlo simulation using the same trajectories for X.

Analogue to Theorem 2.1 we can proof

$$\mathbf{Var}(f'(X_\tau^{t,x})Y_\tau^{t,x,1} + Z_\tau^{t,x,1,0}) = 0 \tag{28}$$

if $\tilde{\mu}$ and $\tilde{F}$ are such that

$$\sigma x \frac{\partial v}{\partial x} + v\tilde{\mu} + \tilde{F} = 0. \tag{29}$$

Let μ and F be such that (21) is fulfilled. Differentiating (21) with respect to x then yields

$$\sigma x \frac{\partial v}{\partial x} + (\sigma+\mu)v + u\frac{\partial \mu}{\partial x} + \frac{\partial F}{\partial x} = 0\,. \tag{30}$$

Hence, by comparing (29) and (30) we see that for

$$\tilde{\mu} = \mu + \sigma,\ \tilde{F} = u\frac{\partial \mu}{\partial x} + \frac{\partial F}{\partial x}\,, \tag{31}$$

the variances of the Monte Carlo estimators of the probabilistic representations (16) and (22) for evaluation of u and v respectively are *both* equal to zero. Moreover, according to Remark 2.3, in both simulations the we can use the same trajectories for X.

Remark 2.4 In particular, if one reduces the variance (20) by the method of control variates, i.e. by taking $\mu = 0$ and choosing F suitably, then for $\tilde{\mu} = \sigma$ and $\tilde{F} = \partial F/\partial x$ we may expect for (28) reduced variance too.

Remark 2.5 It is known that an American option is equivalent to a European option with a consumption process involved, see [5, 12]. As a consequence, there exists a consumption function $c(t,x) \geq 0$ such that $u(t,x)$ in (4) satisfies

$$\frac{\partial u}{\partial t} + \frac{1}{2}\sigma^2(t,x)x^2\frac{\partial^2 u}{\partial x^2} + r(t,x)x\frac{\partial u}{\partial x} - r(t,x)u + c(t,x) = 0, \qquad u(T,x) = f(x)\,.$$

Due to (4) it follows that

$$c(t,x) \;=\; \begin{cases} 0 & \text{if } x \geq g(t), \\ -\frac{1}{2}\sigma^2(t,x)x^2 f''(x) - r(t,x)xf'(x) + r(t,x)f(x) & \text{if } x < g(t). \end{cases}$$

In particular, if $f(x) = (K-x)^+$, we get

$$c(t,x) \;=\; \begin{cases} 0 & \text{if } x \geq g(t), \\ Kr(t,x) & \text{if } x < g(t). \end{cases}$$

3 Numerical random walk algorithms under known critical price curve

For a European option we have to solve the Cauchy problem for a partial differential equation of parabolic type. In particular, in the European case we have $\tau = T$ in representations (11) and (16), and so we can use a Monte Carlo approach based on usual numerical schemes for SDEs both in mean-square and weak sense (see, e.g. [11]). For American options, however, we are faced with boundary value problems and then a number of complications arise. For example, $\tau - t$ in (11) may take arbitrarily small values and therefore numerical integration of (12) with a fixed step h is not appropriate. In particular, it is not possible to apply mean-square Euler approximations. Nonetheless, application of simple weak approximations is possible, when we take into account restrictions connected with the requirement that X cannot leave the domain G.

3.1 Methods of orders 1 and 1/2

Let consider the explicit weak Euler scheme applied to (17)-(19):

$$\begin{aligned} X_{t+h}^{t,x} &\approx X := x + h(r(t,x)x - \sigma(t,x)\mu(t,x)x) + h^{1/2}\sigma(t,x)x\xi\,, \\ Y_{t+h}^{t,x,y} &\approx Y := y - hr(t,x)y + h^{1/2}\mu(t,x)y\xi\,, \\ Z_{t+h}^{t,x,y,z} &\approx Z := z + h^{1/2}F(t,x)y\xi\,, \end{aligned} \qquad (32)$$

ξ is a random variable taking values ± 1 with probability 1/2: $P[\xi = -1] = P[\xi = 1] = 1/2$ and $h > 0$ is a time integration step being sufficiently small. We see that if x is close to $g(t)$ the variable X can be outside of $\bar{G}$ and therefore a random walk due to a scheme with fixed step h for all points of the t-layer $G_t := \{x : (t,x) \in \bar{G}\}$ is not quite suitable. As a better approach, which is essentially developed in [8], it is possible to control the step of numerical integration h when (t,x) is close to the boundary γ. In principle, we decrease the integration step such that the next state of the chain (32) remains in the domain $\bar{G}$. The idea is basically as follows.

First we follow a random walk based on (32) until we reach a narrow layer near the boundary ∂G of G where in particular the solution u may be approximated with sufficient accuracy by known boundary conditions. Then we proceed by suitably replacing the state x reached at the last step by either a state at the boundary or a state in the inside of G where the scheme (32) may be used again. Some methods based on this idea have been obtained in [8]. In [9] one constructs a random walk with respect to scheme (32) where a fixed time step h can be chosen for each t-layer. However, if a point (t_k, x_k) of the random walk is close to the boundary γ, we replace (t_k, x_k) in an appropriate way by a random point (t_k, X_k^*) where X_k^* can take two well specified values with certain probabilities: either $x_k^- = g(t_k)$, i.e. the random walk stops at the boundary, or a value x_k^+ inside G where (32) applies again. Below we explain this method more precisely.

Let us denote the two different states of X in (32) by x^{++} and x^{--}, $x^{--} < x^{++}$. Since the coefficients in (32) are bounded by assumption there exists (for each particular t-layer) a magnitude $\lambda > 0$ such that $x \geq g(t) + \lambda h^{1/2}$ implies $x^{--} \geq g(t+h)$. If (t,x) is such that $x^{--} \geq g(t+h)$, we perform a usual step according to (32). If $x^{--} < g(t+h)$ (and consequently $x < g(t)+\lambda h^{1/2}$) we introduce a random variable X^* which takes two values $x^- = g(t)$ and $x^+ = x + \lambda h^{1/2}$ with probabilities p and $q = 1-p$, respectively, where

$$p = \frac{\lambda h^{1/2}}{x + \lambda h^{1/2} - g(t)}\,. \tag{33}$$

We note that always $p > 1/2$, and if $x = g(t)$, then $p = 1$.

The idea behind is that for any function $V(x)$ with continuous second derivative we have,

$$E[V(X^*)] = pV(g(t)) + qV(x + \lambda h^{1/2}) = V(x) + O(h) \tag{34}$$

for p given by (33), $q = 1-p$. Hence, $E[V(X^*)]$ is given by linear interpolation at x of the function V between $g(t)$ and $x + \lambda h^{1/2}$. Now we are ready to present the complete algorithm.

Let $(t_0, x_0) \in G$ be a point at which the value $u(t_0, x_0)$ is required. We introduce a time discretization

$$t_0 < t_1 < ... < t_m = T,\ t_{k+1} - t_k = h_k,\ k = 0, ..., m-1.$$

By the following algorithm we construct a Markov chain (t_k, X_k, Y_k, Z_k) with (t_k, X_k) in the bounded domain $\bar{G}$, $k = 0, 1, ..., \kappa$, up to a random time t_κ , $\kappa \leq m$, where the chain is stopped, for solving the boundary value problem (6)-(7).

Algorithm 3.1

Initialisation: Set $(t_0, X_0, Y_0, Z_0) := (t_0, x_0, 1, 0)$;

If $X_0 = g(t_0)$ *then* $\kappa := 0$, i.e. $t_\kappa := t_0$, and stop.

While $(X_k > g(t_k)$ *and* $k < m)$ *do:*

Consider the values x^{++} and x^{--}, $x^{--} < x^{++}$
given by (32) for $\xi = \pm 1$, with $t = t_k$, $x = X_k$, and $h = h_k$.

If $x^{--} < g(t_{k+1})$ *then:*

Carry out the following step: With probability p, given by (33) with $t = t_k$, $x = X_k$, $h = h_k$, and an appropriate choice of λ_k, we assign,

$$(t_k, X_k, Y_k, Z_k) := (t_k, g(t_k), Y_k, Z_k), \qquad \kappa := k.$$

With probability $q = 1 - p$ we set

$$(t_k, X_k, Y_k, Z_k) := (t_k, X_k + \lambda_k h_k^{1/2}, Y_k, Z_k).$$

else: (hence if $x^{--} \geq g(t_{k+1})$):

Carry out (32) to obtain $(t_{k+1}, X_{k+1}, Y_{k+1}, Z_{k+1})$.

Logically, Algorithm 3.1 will end up with either $X_k = g(t_k)$ *and* $\kappa = k$, or $k = m$, where in the latter case we set $\kappa = m$. With respect to the above constructed Markov chain we have the following theorem.

Theorem 3.2 *It holds*

$$|E(f(X_\kappa)Y_\kappa + Z_\kappa) - u(t_0, x_0)| \leq Ch\,, \tag{35}$$

where $h = \max_{1\leq k\leq m} h_k$, *and* C *does not depend on* $t_0, x_0,\ h$.

We omit the proof (which can be done similar to [9]), but give some heuristic arguments justifying (35). The one-step error for the points which are not too close to ∂G ("usual" points) is $O(h^2)$ and because the number of all the steps does not exceed $O(1/h)$, the contribution of these steps to the global error is $O(h)$. Further, due to (34), the one-step error of the other points is $O(h)$. Fortunately, it turns out that the mean number of these large ($O(h)$) one-step errors is bounded by a constant which is independent of h. As a consequence, their total error contribution is $O(h)$ also and as a result the global error is $O(h)$, i.e. (35) holds.

Clearly, the result of Theorem 3.2 is also true for the function v solving the boundary value problem (9)-(10). For instance, if we take in (23)-(25) $\tilde{\mu} = \sigma$, $\tilde{F} = 0$, we get

$$|E[f'(X_\kappa)Y_\kappa] - v(t_0, x_0)| \leq Ch\,, \tag{36}$$

where the process X and in particular κ and X_κ, coincide with the solution of the first SDE in (32) under $\mu = 0$. So in this example we can use the paths of X obtained by Algorithm 3.1 for computing both u and v. However, the process Y in (36) has to be computed by the scheme (see (24))

$$Y_{k+1} = Y_k + h_k^{1/2}\sigma(t_k, X_k)Y_k\xi_{\,k}\,,\ Y_0 = 1. \tag{37}$$

Remark 3.3 If we simplify Algorithm 3.1 by stopping the chain, $\kappa := k$, hence $X_\kappa = X_k$, as soon as $x^{--} < g(t_{k+1})$, we obtain a more simple random walk. It can be shown that the method based on simulation of the expectation in (35) by this algorithm converges also, but, the order of convergence is then only $O(h^{1/2})$ (see [9]). However, if one takes advantage of the known fact that $\dfrac{\partial u(t, x^-)}{\partial x} = f'(x^-)$ at the curve γ we can obtain even with this simple random walk again a method of order 1 by Monte Carlo simulation of

$$E((f(x^-) + f'(x^-)(X_\kappa - x^-))Y_\kappa + Z_\kappa),$$

due to the fact that

$$|E((f(x^-) + f'(x^-)(X_\kappa - x^-))Y_\kappa + Z_\kappa) - u(t_0, x_0)| \leq Ch.$$

3.2 Methods of order $3/2$

For constructing a method of an order higher than one we use instead of the Euler scheme a weak second order scheme and use the fact that the derivative $\partial u(t,x)/\partial x = f'(x)$ is known on the critical price curve γ. It should be noted, however, that knowledge of this derivative is a special feature of American options which does not apply for general boundary value problems.

Let us write the first equation of the system (17)-(19) in the form

$$dX_t = X_t(\tilde{a}(t,X_t)dt + \sigma(t,X_t))dW_t). \tag{38}$$

Application of weak second order scheme (see, for example, [7]) to (17)-(19) gives the following one-step approximation for $X^{t,x}_{t+h}$, which we denote by X again,

$$\begin{aligned} X^{t,x}_{t+h} \approx\ & X := x + x\sigma\xi h^{1/2} \\ & + x\tilde{a}h + \frac{1}{2}(x\sigma^2 + x^2\sigma\frac{\partial\sigma}{\partial x})(\xi^2-1)h \\ & + \frac{1}{2}[x\frac{\partial\sigma}{\partial t} + x\tilde{a}(\sigma + x\frac{\partial\sigma}{\partial x}) + \frac{1}{2}x^2\sigma^2(2\frac{\partial\sigma}{\partial x} + x\frac{\partial^2\sigma}{\partial x^2}) + x\sigma(\tilde{a} + x\frac{\partial\tilde{a}}{\partial x})]\xi h^{3/2} \\ & + [x\frac{\partial\tilde{a}}{\partial t} + x\tilde{a}(\tilde{a} + x\frac{\partial\tilde{a}}{\partial x} + \frac{1}{2}x^2\sigma^2(2\frac{\partial\tilde{a}}{\partial x} + x\frac{\partial^2\tilde{a}}{\partial x^2}))]\frac{h^2}{2}. \end{aligned} \tag{39}$$

In (39) the functions $\tilde{a}$ and σ and their derivatives are computed at (t,x) and ξ is a three point random variable taking values $-\sqrt{3}$, 0, $\sqrt{3}$, with probabilities $P(\xi=0)=2/3$, $P(\xi=\pm\sqrt{3})=1/6$. For the corresponding approximations Y and Z of $Y^{t,x,y}_{t+h}$ and $Z^{t,x,y,z}_{t+h}$, respectively, we have similar expressions. For instance, if $\mu=0$, we obtain for Y:

$$\begin{aligned} Y^{t,x,y}_{t+h} \approx Y\ :=\ & y - ryh - \frac{1}{2}\sigma xy\frac{\partial r}{\partial x}\xi h^{3/2} \\ & + \frac{1}{2}(-\frac{\partial r}{\partial t} - \tilde{a}x\frac{\partial r}{\partial x} + r^2 - \frac{1}{2}\sigma^2x^2\frac{\partial^2 r}{\partial x^2})yh^2. \end{aligned} \tag{40}$$

For constant a and σ and $\mu = F = 0$ we obtain,

$$\begin{aligned} X^{t,x}_{t+h} &\approx X = x + x\sigma\xi h^{1/2} + xah + \frac{1}{2}x\sigma^2(\xi^2-1)h + xa\sigma\xi h^{3/2} + \frac{1}{2}xa^2h^2, \\ Y^{t,x,y}_{t+h} &\approx Y = y - yrh + \frac{1}{2}yr^2h^2, \\ Z^{t,x,y,z}_{t+h} &= Z = z. \end{aligned}$$

Thus, we now have three values for X corresponding to three values of ξ, which we denote by $x^{++} > x^{00} > x^{--}$. Clearly, again there exists a $\lambda > 0$ (λ may depend on t) such that if $x \geq g(t) + \lambda h^{1/2}$, then $x^{--} \geq g(t+h)$. If x is such that $x^{--} \geq g(t+h)$, we carry out a usual step according to (39). If x is such that $x^{--} < g(t+h)$ which implies $x < g(t) + \lambda h^{1/2}$, i.e. x is close to $g(t)$, we now consider a random variable X^* taking two values $x^- = g(t)$ and $x^+ = x + \lambda h^{1/2}$ with probabilities p and $q = 1-p$ given by

$$p = 1 - \frac{(x-x^-)^2}{(x^+-x^-)^2}, \qquad q = 1 - p = \frac{(x-x^-)^2}{(x^+-x^-)^2}, \tag{41}$$

respectively. The idea behind (41) is based on expansion of $u(t,\cdot)$ at x^- and utilizes the fact that $\partial u(t,x)/\partial x = f'(x)$ on the exercise curve γ as follows. For any p and q with $p+q=1$ we may write

$$\begin{aligned}
u(t,x) &= pu(t,x)+qu(t,x)\\
&= p[u(t,x^-)+\frac{\partial u}{\partial x}(t,x^-)(x-x^-)+\frac{1}{2}\frac{\partial^2 u}{\partial x^2}(t,x^-)(x-x^-)^2+\ldots]\\
&\quad +q[u(t,x^+)+\frac{\partial u}{\partial x}(t,x^+)(x-x^+)+\frac{1}{2}\frac{\partial^2 u}{\partial x^2}(t,x^+)(x-x^+)^2+\ldots]\\
&= p[u(t,x^-)+\frac{\partial u}{\partial x}(t,x^-)(x-x^-)+\frac{1}{2}\frac{\partial^2 u}{\partial x^2}(t,x^-)(x-x^-)^2+\ldots]\\
&\quad +q[u(t,x^+)+\frac{\partial u}{\partial x}(t,x^-)(x-x^+)+\frac{\partial^2 u}{\partial x^2}(t,x^-)(x^+-x^-)(x-x^+)\\
&\quad +\frac{1}{2}\frac{\partial^2 u}{\partial x^2}(t,x^-)(x-x^+)^2+\ldots]\\
&= pf(x^-)+qu(t,x^+)+pf'(x^-)(x-x^-)+qf'(x^-)(x-x^+)\\
&\quad +\frac{\partial^2 u}{\partial x^2}(t,x^-)[\frac{1}{2}p(x-x^-)^2+q(x^+-x^-)(x-x^+)+q\frac{1}{2}(x-x^+)^2]+\ldots
\end{aligned} \tag{42}$$

where the dots denote terms of order higher than one with respect to h. By next choosing p and q according to (41) the second order terms in (42) vanish and we then obtain

$$\begin{aligned}
u(t,x) &= pf(x^-)+qu(t,x^+)+pf'(x^-)(x-x^-)+qf'(x^-)(x-x^+)+\ldots\\
&= p[f(x^-)+f'(x^-)(x-x^-)+f'(x^-)(x-x^+)\frac{q}{p}]+qu(t,x^+)+\ldots\\
&= p[f(x^-)+f'(x^-)(x-x^-)-f'(x^-)\lambda h^{1/2}\frac{q}{p}]+qu(t,x^+)+O(h^{3/2}).
\end{aligned} \tag{43}$$

We are now ready to present a method of order 3/2 by the following algorithm. By Algorithm 3.4 we construct a Markov chain $(t_k, X_k', X_k, Y_k, Z_k)$ with (t_k, X_k) in the bounded domain $\bar{G}$ and X_k' being an auxiliary dummy process, for $k=0,1,...,\kappa$, up to a random time t_κ , $\kappa \le m$, where the chain is stopped:

Algorithm 3.4

Initialisation: Set $(t_0, X_0', X_0, Y_0, Z_0) := (t_0, x_0, x_0, 1, 0)$;

If $X_0 = g(t_0)$ *then* $\kappa := 0$, i.e. $t_\kappa := t_0$, and stop.

While $(X_k > g(t_k)$ *and* $k < m)$ *do:*

Consider the values x^{++}, x^{00}, x^{--} with $x^{++} > x^{00} > x^{--}$ given by (39), for $\xi = 0, \pm\sqrt{3}$, with $t = t_k$, $x = X_k$, and $h = h_k$.

If $x^{--} < g(t_{k+1})$ *then:*

Carry out the following step: With probability p, given by (41) with $t = t_k$, $x = X_k$, $h = h_k$ and an appropriate choice of λ_k, we assign,

$$(t_k, X_k', X_k, Y_k, Z_k) := (t_k, X_k', g(t_k), Y_k, Z_k), \qquad \kappa := k.$$

With probability $q = 1 - p$ we set

$$(t_k, X'_k, X_k, Y_k, Z_k) := (t_k, X_k + \lambda_k h_k^{1/2}, X_k + \lambda_k h_k^{1/2}, Y_k, Z_k).$$

else: (hence if $x^{--} \geq g(t_{k+1})$)

Carry out (39) and set $X'_{k+1} = X_{k+1}$ to obtain $(t_{k+1}, X'_{k+1}, X_{k+1}, Y_{k+1}, Z_{k+1})$.

Like Algorithm 3.1, the procedure 3.4 will end up with either $X_k = g(t_k)$ *and* $\kappa = k$, or $k = m$, where in the latter case we set $\kappa = m$.

For the Markov chain constructed in Algorithm 3.4 we then have the following theorem due to interpolation formula (43).

Theorem 3.5 *It holds*

$$|E(\widetilde{f}(X'_k, X_\kappa)Y_\kappa + Z_\kappa) - u(t_0, x_0)| \leq Ch^{3/2}\,, \tag{44}$$

where $h = \max_{1 \leq k \leq m} h_k$, *C does not depend on* $t_0, x_0,$ *h and the function* $\widetilde{f}$ *is defined by*

$$\widetilde{f}(X'_\kappa, X_\kappa) = \begin{cases} f(X_\kappa) + f'(X_\kappa)(X'_\kappa - X_\kappa) - f'(X_\kappa)\lambda_\kappa h_k^{1/2}\dfrac{q_\kappa}{p_\kappa}\ , & \text{if } \kappa < m, \\ f(X_\kappa)\ , & \text{if } \kappa = m. \end{cases} .$$

The proof is similar to the proof of Theorem 3.2

Remark 3.6 As we will see in Section 4, we also know the continuous extension of the second derivative $\partial^2 u(t,x)/\partial x^2$ inside of G to the boundary γ :

$$\frac{\partial^2 u}{\partial x^2}(t, g(t)) := \lim_{\substack{(s,x)\to(t,g(t)) \\ (s,x)\in G}} \frac{\partial^2 u}{\partial x^2}(s,x) = \frac{r(t,g(t))f(g(t)) - r(t,g(t))g(t)f'(g(t))}{\frac{1}{2}\sigma^2(t,g(t))g^2(t)}$$

We thus have

$$u(t,x) = f(x^-) + f'(x^-)(x - x^-) + \frac{r(t,x^-)f(x^-) - r(t,x^-)x^- f'(x^-)}{\sigma^2(t,x^-)(x^-)^2}(x - x^-)^2 + \ldots \tag{45}$$

By using (45) we then get a method of order 3/2 via Monte Carlo simulation of

$$E(\widehat{f}(X_\kappa)Y_\kappa + Z_\kappa),$$

with

$$\widehat{f}(X_\kappa) := f(x^-) + f'(x^-)(X_\kappa - x^-) + \frac{r(t,x^-)f(x^-) - r(t,x^-)x^- f'(x^-)}{\sigma^2(t,x^-)(x^-)^2}(X_\kappa - x^-)^2,$$

using a simplified random walk obtained by stopping Algorithm 3.4 as in Remark 3.3 when the guard $x^{--} < g(t)$ is true (of course the dummy X' can then be omitted).

Remark 3.7 Let us consider the case $h_k = h,\ k = 0, ..., m-1$, and assume that the global error R of Algorithm 3.1 admits a certain expansion in the time step h,

$$R = C_0 h + O(h^\eta) \tag{46}$$

for some $\eta > 1$. The conjecture is that at least $\eta \geq 3/2$, but, practical experiments even suggest $\eta = 2$. Assuming that the conjecture $\eta \geq 3/2$ is true we can use a kind of generalized Richardson extrapolation to obtain a method of order $O(h^{3/2})$ by applying two times the algorithm with different time steps. Namely, let $\overline{u}^{h_1}$ and $\overline{u}^{h_2}$ are approximations of $u(t_0, x_0)$ computed with Algorithm 3.1. Then, we obtain a more accurate approximation $\widetilde{u}$ via

$$\widetilde{u} := \overline{u}^{h_1} \frac{h_2}{h_2 - h_1} - \overline{u}^{h_2} \frac{h_1}{h_2 - h_1},\ \widetilde{u} = u(t_0, x_0) + O(h^{3/2})\,. \tag{47}$$

For further details see [16].

4 Monte Carlo construction of the critical price curve

In this section we propose a Monte Carlo method for determination of the exercise curve γ. For this we assume that γ is known on the interval $[\bar{t}, T]:\ \ x = g(t)$, $\bar{t} \leq t \leq T$ (see Fig. 1) and then proceed with evaluating $g(\bar{t} - h)$ for a small step h to the left.

The idea is based on evaluating $g'(\bar{t})$. If an approximation $\tilde{g}'(\bar{t})$ of $g'(\bar{t})$ is known and that $\tilde{g}'(\bar{t}) = g'(\bar{t}) + O(h^q)$, $q > 0$, then

$$g(\bar{t} - h) \ \ = \ \ g(\bar{t}) - g'(\bar{t})h + O(h^2) = g(\bar{t}) - \tilde{g}'(\bar{t})h + O(h^2) + O(h^{1+q}) \tag{48}$$

and thus obtain the curve γ on the extended interval $[\bar{t} - h, T]$.

So for a one-step extension of the exercise curve γ based on (48) the order of accuracy is equal to $\min(2, 1+q)$. Consequently, the order of accuracy for an approximation of γ on the whole interval $[0, T]$ is $\min(1, q)$. We will evaluate $g'(\bar{t})$ via computing the value of $u(\bar{t}, x)$ for some neighbouring point x, with $x > \bar{x} = g(\bar{t})$ using the expressions obtained in Section 4.1. The value $u(\bar{t}, x)$ will be computed by random walk algorithms proposed in Section 3, which is possible since the exercise curve is assumed to be known on $[\bar{t}, T]$. The connection between $g'(\bar{t})$ and $u(\bar{t}, x)$ is derived in Section 4.1. In Section 4.2 we consider particular algorithms of Section 3 for the computation of $g'(\bar{t})$ sketched above. Finally, we will show in Section 4.3 that the here developed pure Monte Carlo method for the evaluation of the American exercise boundary, hence the free boundary of a Stefan problem, can be naturally extended to the multi-dimensional case.

4.1 Some expressions for $g'(t)$

We first derive some useful relations on the curve γ by assuming that all derivatives of u within G extend continuously to the boundary at each point $(t, g(t))$ of γ with $t < T$. It should be noted that, while the first derivatives from the inside coincide with the derivative from the outside of G (see (5)), the second derivatives do not coincide in general. In what follows all derivatives of u on γ have to be considered as limits from the inside of G. By thus extending equations (5)-(7) and (9)-(10) to boundary points $(t, g(t))$ of γ with $t < T$, it follows that

$$\begin{aligned} \frac{\partial u}{\partial t}(t,g(t)) + \frac{1}{2}\sigma^2(t,g(t))g^2(t)\frac{\partial^2 u}{\partial x^2}(t,g(t)) + & \\ + r(t,g(t))g(t)\frac{\partial u}{\partial x}(t,g(t)) - r(t,g(t))u(t,g(t)) &= 0 \end{aligned} \tag{49}$$

$$u(t,g(t)) = f(g(t)), \tag{50}$$

$$\frac{\partial u}{\partial x}(t,g(t)) = f'(g(t)),\ 0 \le t < T. \tag{51}$$

Differentiating (50) with respect to t yields

$$\frac{\partial u}{\partial t}(t,g(t)) + \frac{\partial u}{\partial x}(t,g(t))g'(t) = f'(g(t))g'(t)\,, \tag{52}$$

so by taking (51) into account we obtain

$$\frac{\partial u}{\partial t}(t,g(t)) = 0\,. \tag{53}$$

Then, combining (49)-(53) gives

$$\frac{\partial^2 u}{\partial x^2}(t,g(t)) = 2\frac{r(t,g(t))f(g(t)) - r(t,g(t))g(t)f'(g(t))}{\sigma^2(t,g(t))g^2(t)} \tag{54}$$

and differentiating (51) with respect to t gives

$$\frac{\partial^2 u}{\partial t \partial x}(t,g(t)) + \frac{\partial^2 u}{\partial x^2}(t,g(t))g'(t) = f''(g(t))g'(t)\,, \tag{55}$$

whence – with notations shortened in an obvious way:

$$g'(t) = \frac{u''_{tx}(t,g(t))}{f''(g(t)) - u''_{xx}(t,g(t))}. \tag{56}$$

It is important to note that due to (50) and (51) the price and its derivative with respect to x ("delta") are continuous on γ. However, the second derivative u''_{xx} ("gamma" in financial terms) has on γ a jump of magnitude $f''(g(t)) - u''_{xx}(t,g(t))$. For example, for the standard American put where r and σ are constant and $f(x) = (K-x)^+$, this jump equals $2rK/(\sigma g(t))^2$.

Since $D_2(t) := u''_{xx}(t,g(t))$ is known from (54), we may determine $g'(t)$ from (56) by computing $u''_{tx}(t,g(t))$ only. For this purpose we differentiate the left-hand side of (6) with respect to x in the interior of G to get

$$u''_{tx} + \frac{1}{2}\sigma^2 x^2 u'''_{xxx} + (\sigma^2 x + rx + \frac{1}{2}x^2(\sigma^2)'_x)u''_{xx} + xr'_x u'_x - r'_x u = 0,$$

where the argument (t,x) is suppressed for convenience. Next, by taking the boundary limit to γ and using (50), (51) and (54) we obtain

$$u''_{tx}+\frac{1}{2}\sigma^2 g^2(t)u'''_{xxx}+(\sigma^2 g(t)+rg(t)+\frac{1}{2}g^2(t)(\sigma^2)'_x)D_2(t)+g(t)r'_x f'(g(t))-r'_x f(g(t))=0 \tag{57}$$

with partially suppressed argument $(t,g(t))$. Thus, to find $g'(\bar{t})$ by (56) we need $u''_{tx}(\bar{t},g(\bar{t}))$ which in turn may be computed from $u'''_{xxx}(\bar{t},g(\bar{t}))$ by (57).

Now let ρ and q be positive numbers to be specified later. For $\bar{x}:=g(\bar{t})$ we then have

$$\begin{aligned} u(\bar{t},\bar{x}+\rho h^q) &= u(\bar{t},\bar{x})+u'_x(\bar{t},\bar{x})\rho h^q+\frac{1}{2}u''_{xx}(\bar{t},\bar{x})\rho^2h^{2q}+\frac{1}{6}u'''_{xxx}(\bar{t},\bar{x})\rho^3h^{3q}+O(h^{4q}) \\ &= f(\bar{x})+f'(\bar{x})\rho h^q+\frac{1}{2}D_2(\bar{t})\rho^2h^{2q}+\frac{1}{6}u'''_{xxx}(\bar{t},\bar{x})\rho^3h^{3q}+O(h^{4q}). \end{aligned} \tag{58}$$

We are now going to compute $u(\bar{t},\bar{x}+\rho h^q)$ with accuracy of order $O(h^{4q})$ by one of the Monte Carlo methods discussed in Section 3, using the known part of the exercise boundary γ, see Figure 1.

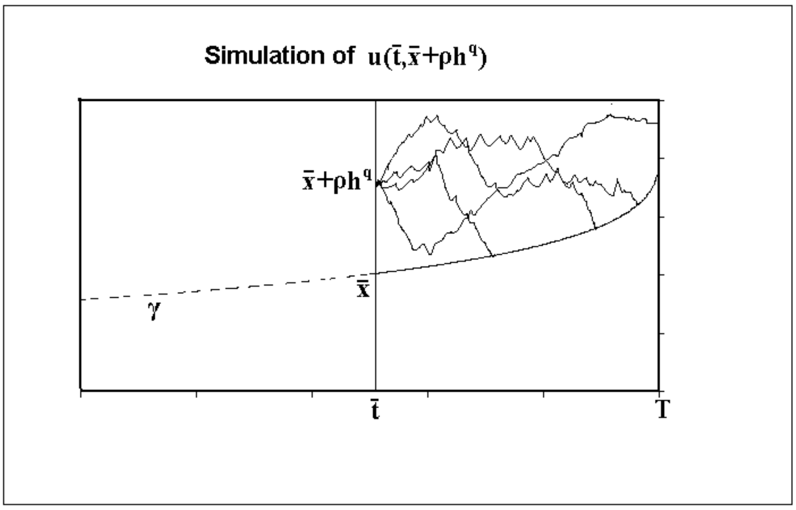

Figure 1: Backward construction of the exercise boundary

Then $u'''_{xxx}(\bar{t},\bar{x})$ can be obtained from (58) with accuracy $O(h^q)$. As a consequence, see (57) and (56), $u''_{tx}(\bar{t},\bar{x})$ and $g'(\bar{t})$ can then be found with accuracy $O(h^q)$ also. Then, since $g'(\bar{t})$ can thus be approximated as $\tilde{g}'(\bar{t})=g'(\bar{t})+O(h^q)$, we may extend the exercise curve one step h to the left with accuracy by

$$g(\bar{t}-h)=g(\bar{t})-\tilde{g}'(\bar{t})h+O(h^2)+O(h^{1+q})\,. \tag{59}$$

From (59) we see that it doesn't make sense to choose $q>1$. For $q\le 1$, the evaluation of $g(\bar{t}-h)$ by $g(\bar{t})-\tilde{g}'(\bar{t})h$ has accuracy $O(h^{1+q})$.

4.2 Random walk algorithms for evaluating $g'(t)$

Let us consider the case $q=1/4$. We may use Algorithm (3.1) with time steps h for simulating $u(\bar{t},\bar{x}+\rho h^{1/4})$ with accuracy $O(h)=O(h^{4q})$. Note that this simulation takes place in the time segment $[\bar{t}, T]$ where γ is known. The one-step error of the evaluation of $g(\bar{t} - h)$ is thus equal to $O(h^{5/4})$. Usually, if the one-step error of a method is $O(h^{1+q})$, $q > 0$, then the error on a whole interval is $O(h^q)$. Heuristically, this rule is evident, because the one-step errors have to be added and the numbers of steps is T/h. Applying this rule, we conclude that our method of backwards evaluating the whole critical price curve converges and its order of convergence is equal to $O(h^{1/4})$. We restrict ourselves to a numerical verification of the convergence of the method (Section 5), without giving a rigorous proof which would go beyond the scope of this paper. However, we note that rigorous proofs for related problems, both linear and non-linear, can be found in [9, 10].

By similar arguments it follows that by computing $u(\bar{t}, \bar{x} + \rho h^{3/8})$ via an 3/2-order algorithm with time steps h, for instance by Algorithm 3.4 (see Section 3.2), or more simply by a Richardson like method (47) assuming that the conjecture in Remark 3.7 holds true, we can obtain an algorithm for evaluating the exercise boundary with accuracy $O(h^{3/8})$.

As another alternative, we may follow an approach which is based on $u'''_{xxx}(\bar{t}, \bar{x}) = v''_{xx}(\bar{t}, \bar{x})$, the computation of $v''_{xx}(\bar{t}, \bar{x})$ from

$$v(\bar{t}, \bar{x} + \rho h^q) = v(\bar{t}, \bar{x}) + v'_x(\bar{t}, \bar{x})\rho h^q + \frac{1}{2} v''_{xx}(\bar{t}, \bar{x})\rho^2 h^{2q} + O(h^{3q}) \tag{60}$$

with accuracy $O(h^q)$, after the computation of $v(\bar{t}, \bar{x} + \rho h^q)$ from the boundary value problem (9) (10) with accuracy $O(h^{3q})$. For instance, by taking $q = 1/3$ and using the order 1 algorithm (3.1) with time steps h we can compute $v(\bar{t}, \bar{x} + \rho h^{1/3})$ with accuracy $O(h)$ to obtain $u'''_{xxx}(\bar{t}, \bar{x}) = v''_{xx}(\bar{t}, \bar{x})$ by (60) with accuracy $O(h^{1/3})$ and, as a result, a method of order $O(h^{1/3})$ for evaluating the exercise boundary. Furthermore, by using a Richardson like method (47) in Remark 3.7, or a method analogue to Section 3.2 based on the fact that $v'_x = u''_{xx}$ is known at the exercise boundary by (54), we may get order 3/2 Monte Carlo methods for the problem (9)-(10) as well. Using such a method we may simulate $v(\bar{t}, \bar{x} + \rho h^{1/2})$ with accuracy $O(h^{3/2})$ by taking time steps h and so obtain via (60) and $u'''_{xxx}(\bar{t}, \bar{x}) = v''_{xx}(\bar{t}, \bar{x})$, a method with accuracy of at least $O(h^{1/2})$.

We should note that, also due to the fact that $|\gamma'(T-)| = \infty$, the presented Monte Carlo method needs to be started up by some other method on a short interval, say $[T - \delta, T]$. In this respect one could apply on $[T - \delta, T]$ a PDE method, or one could use an in some sense "canonical" Monte Carlo method which is based on a Bermudan approximation: The interval $[T - \delta, T]$ is provided with a small time grid and it is assumed that the option may be exercised only at these grid points. Then, in the interval $[T - \delta, T]$ the exercise boundary is constructed backwardly at the grid points by a bi-section Monte Carlo search.

4.3 The multi-dimensional case

We now consider the general multi-dimensional American style option in a generalized Black-Scholes framework. This framework is given by (1)-(2) where now X is a price vector process of risky assets in $\mathbb{R}^d$, a is an $\mathbb{R}^d$ valued vector function, σ is an $\mathbb{R}^{d \times d}$ valued matrix function in $[0, T] \times \mathbf{R}^d_+$ with existing inverse σ^{-1}, and W_t is a standard Wiener process in $\mathbb{R}^d$. An American style option contract gives the holder the right to exercise the option at any $0 \leq t \leq T$ yielding a payoff $f(X_t)$, where f is a certain continuous function from $\mathbf{R}_+^d$ to $\mathbf{R}_+$ which is specified in the contract. As in Section 1 the price of the option is given by (3), where now the process X is d-dimensional and solves (1),(2) with $a_k \equiv r$ for $k = 1, \ldots, d$. In the multi-dimensional case it is known that for a function f, μ in (3) can be seen as the solution of a free boundary value problem where an open continuation region $\mathcal{G}$ and the free boundary $\Gamma = \partial\mathcal{G} \cap \{t < T\}$ are to be determined such that,

$$Lu := \frac{\partial u}{\partial t} + \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(t,x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(t,x)\frac{\partial u}{\partial x_i} - r(t,x)u = 0\,, \qquad (t,x) \in \mathcal{G},$$
$$u(t,x) = f(x), \qquad \frac{\partial u}{\partial x_i}(t,x) = \frac{\partial f}{\partial x_i}(x), \quad i = 1, \ldots, d, \text{ and } t = T \text{ or } (t,x) \in \Gamma,$$
$$u(t,x) = f(x), \qquad (t,x) \notin \mathcal{G}. \tag{61}$$

In (61) we have $a_{ij}(t,x) := x_i x_j \sum_{k=1}^d \sigma_{ik}(t,x)\sigma_{jk}(t,x)$ and $b_i(t,x) := r(t,x)x_i$, $i,j = 1,...,d$.

Let us assume that Γ has a parametrisation $x = g(t, v_1, ..., v_{d-1}) = g(t,v)$, where $d \geq 2$. By (61) we so have the following identities in (t,v),

$$u(t, g(t,v)) = f(g(t,v)), \tag{62}$$
$$\frac{\partial u}{\partial x_i}(t, g(t,v)) = \frac{\partial f}{\partial x_i}(g(t,v)) \qquad i = 1, \ldots, d, \tag{63}$$

where (63) is the multi-dimensional smooth fit property which essentially determines together with (61) and (62) the exercise boundary. Analogue to the one-dimensional case we assume that all derivatives of u within $\mathcal{G}$ extend continuously to the boundary at each point $(t, g(t,v))$ of Γ with $t < T$ and all derivatives below are to be understood as limits from the inside of $\mathcal{G}$. We obtain by differentiating (62),

$$\frac{\partial u}{\partial t}(t, g(t,v)) + \sum_{i=1}^{d} \frac{\partial u}{\partial x_i}(t, g(t,v))\frac{\partial g_i}{\partial t}(t,v) = \sum_{i=1}^{d} \frac{\partial f}{\partial x_i}(g(t,v))\frac{\partial g_i}{\partial t}(t,v)$$

and so by (63)

$$\frac{\partial u}{\partial t}(t, g(t,v)) = 0. \tag{64}$$

We now differentiate (63) with respect to t to get for $i = 1, \ldots, d$,

$$\frac{\partial^2 u}{\partial t \partial x_i} + \sum_{j=1}^{d} \frac{\partial^2 u}{\partial x_i \partial x_j}\frac{\partial g_j}{\partial t} = \sum_{j=1}^{d} \frac{\partial^2 f}{\partial x_i \partial x_j}\frac{\partial g_j}{\partial t}, \tag{65}$$

where for readability the arguments t and $g(t,v)$ are suppressed. Differentiating (63) with respect to v_α yields

$$\sum_{j=1}^{d} \frac{\partial^2 u}{\partial x_i \partial x_j}\frac{\partial g_j}{\partial v_\alpha} = \sum_{j=1}^{d} \frac{\partial^2 f}{\partial x_i \partial x_j}\frac{\partial g_j}{\partial v_\alpha}, \qquad i = 1, \ldots, d, \quad \alpha = 1, \ldots, d-1. \tag{66}$$

Taking the boundary limit of the PDE (61) to the boundary Γ yields by (62)-(64),

$$\frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(t,g(t,v))\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(t,g(t,v))\frac{\partial f}{\partial x_i} - r(t,g(t,v))f = 0. \tag{67}$$

We solve the second derivatives $\frac{\partial^2 u}{\partial x_i \partial x_j}(t,g(t,v))$ from the system (66),(67) as follows. Let for a fixed (t,v) the vector $\mathbf{n}(t,v)$ be orthogonal to the surface $v \to g(t,v)$ in $\mathbf{R}^d$-space at the point $g(t,v)$ (we assume this surface to be regular). Then, according to (66) there exists for each i a scalar $\xi_i(t,v)$ such that

$$\left[\frac{\partial^2 u}{\partial x_i \partial x_j}(t,g(t,v)) - \frac{\partial^2 f}{\partial x_i \partial x_j}(g(t,v))\right]_{j=1,\ldots,d} = \xi_i(t,v)\mathbf{n}(t,v). \tag{68}$$

By the symmetry of the matrix

$$\left[\frac{\partial^2 u}{\partial x_i \partial x_j} - \frac{\partial^2 f}{\partial x_i \partial x_j}\right]_{i,j=1,\ldots,d} \tag{69}$$

it follows that $\xi_i n_j = \xi_j n_i$ with n_i being the i-th component of $\mathbf{n}$. Hence, there is a scalar $\eta(t,v)$ such that $\xi_i = \eta n_i$. We thus get

$$\frac{\partial^2 u}{\partial x_i \partial x_j}(t,g(t,v)) = \frac{\partial^2 f}{\partial x_i \partial x_j}(g(t,v)) + \eta(t,v)n_i(t,v)n_j(t,v). \tag{70}$$

By now plugging (70) into (67) we get

$$\begin{aligned}&\tfrac{1}{2}\eta(t,v)\textstyle\sum_{i,j=1}^{d} a_{ij}(t,g(t,v))n_i(t,v)n_j(t,v)+\\ &+\tfrac{1}{2}\textstyle\sum_{i,j=1}^{d} a_{ij}(t,g(t,v))\frac{\partial^2 f}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(t,g(t,v))\frac{\partial f}{\partial x_i} - r(t,g(t,v))f = 0.\end{aligned} \tag{71}$$

We thus have to determine the normal $\mathbf{n}$ of the surface $v \to g(t,v)$ which is supposed to be known at time t and then after solving $\eta(t,v)$ from (71) we obtain all second order space derivatives of u via (70). Combining (65) with (70) yields

$$\frac{\partial^2 u}{\partial t \partial x_i} + \sum_{j=1}^{d} \eta(t,v)n_i(t,v)n_j(t,v)\frac{\partial g_j}{\partial t} = 0, \quad i = 1,\ldots,d. \tag{72}$$

So, once the mixed space-time derivatives $\frac{\partial^2 u}{\partial t \partial x_i}$ are computed we may solve for the time derivatives $\frac{\partial g_j}{\partial t}$ in (72) under the assumption that the matrix (69) is invertible and then extend the exercise boundary to the interval $[t-h,T]$ by

$$g_i(t-h,v) \approx g_i(t,v) - \frac{\partial g_i}{\partial t}(t,v)\,h, \quad i = 1,\ldots,d, \tag{73}$$

analogue to the one-dimensional case (48). Therefore we now consider the space-time derivatives in (65). For this we differentiate the PDE (61) with respect to x_k, yielding at the boundary Γ,

$$\begin{aligned}&\frac{\partial^2 u}{\partial t \partial x_k} + \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}\frac{\partial^3 u}{\partial x_i \partial x_j \partial x_k} + \sum_{i=1}^{d} b_i \frac{\partial^2 u}{\partial x_i \partial x_k} - r\frac{\partial f}{\partial x_k}\\ &+\frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial^2 u}{\partial x_i \partial x_j}\frac{\partial a_{ij}}{\partial x_k} + \sum_{i=1}^{d} \frac{\partial f}{\partial x_i}\frac{\partial b_i}{\partial x_k} - f\frac{\partial r}{\partial x_k} = 0\,.\end{aligned} \tag{74}$$

We next obtain the derivatives $\frac{\partial^2 u}{\partial t \partial x_k}(t, g(t,v))$ via (74) by evaluating the third order derivatives $\frac{\partial^3 u}{\partial x_i \partial x_j \partial x_k}(t, g(t,v))$, using the following procedure.

Let us fix (t,v), $x := g(t,v)$, $q > 0$, and $\rho > 0$. For a tripel (i,j,k) with different i,j,k and increments h_i, h_j, h_k, we define the vector

$$\mathbf{h}^q_{i,j,k} := (0,..,0, \overset{i\text{-th pos.}}{h^q_i}, 0,..,0, \overset{j\text{-th pos.}}{h^q_j}, 0,..,0, \overset{k\text{-th pos.}}{h^q_k}, 0,..0),$$

such that $(t, x + \rho \mathbf{h}^q_{i,j,k}) \in \mathcal{G}$. By a third order Taylor expansion of u at (t,x) and taking into account (62) and (63) we get

$$\begin{aligned} u(t, x + \rho \mathbf{h}^q_{i,j,k}) &= f(x) + \frac{\partial f}{\partial x_i}\rho h^q_i + \frac{\partial f}{\partial x_j}\rho h^q_j + \frac{\partial f}{\partial x_k}\rho h^q_k \\ &+ \frac{1}{2}\frac{\partial^2 u}{\partial x_i^2}\rho^2 h_i^{2q} + \frac{1}{2}\frac{\partial^2 u}{\partial x_j^2}\rho^2 h_j^{2q} + \frac{1}{2}\frac{\partial^2 u}{\partial x_k^2}\rho^2 h_k^{2q} \\ &+ \frac{\partial^2 u}{\partial x_i \partial x_j}\rho^2 h^q_i h^q_j + \frac{\partial^2 u}{\partial x_i \partial x_k}\rho^2 h^q_i h^q_k + \frac{\partial^2 u}{\partial x_j \partial x_k}\rho^2 h^q_j h^q_k \\ &+ \frac{1}{6}\frac{\partial^3 u}{\partial x_i^3}\rho^3 h_i^{3q} + \frac{1}{6}\frac{\partial^3 u}{\partial x_j^3}\rho^3 h_j^{3q} + \frac{1}{6}\frac{\partial^3 u}{\partial x_k^3}\rho^3 h_k^{3q} \\ &+ \frac{1}{2}\frac{\partial^3 u}{\partial x_i^2 \partial x_j}\rho^3 h_i^{2q} h^q_j + \frac{1}{2}\frac{\partial^3 u}{\partial x_j^2 \partial x_i}\rho^3 h^q_i h_j^{2q} \\ &+ \frac{1}{2}\frac{\partial^3 u}{\partial x_i^2 \partial x_k}\rho^3 h_i^{2q} h^q_k + \frac{1}{2}\frac{\partial^3 u}{\partial x_k^2 \partial x_i}\rho^3 h^q_i h_k^{2q} \\ &+ \frac{1}{2}\frac{\partial^3 u}{\partial x_j^2 \partial x_k}\rho^3 h_j^{2q} h^q_k + \frac{1}{2}\frac{\partial^3 u}{\partial x_k^2 \partial x_j}\rho^3 h^q_j h_k^{2q} \\ &+ \frac{\partial^3 u}{\partial x_i \partial x_j \partial x_k}\rho^3 h^q_i h^q_j h^q_k + O(h^{4q}), \end{aligned} \tag{75}$$

where $h := \max(|h_i|, |h_j|, |h_k|)$. Note that in (75) all second order derivatives are known from (70). By choosing $h_j = h_k = 0$ and $h_i = h$, we may compute $u(t, x + \rho \mathbf{h}^q_{i,j,k})$ with order $O(h^{4q})$ by Monte Carlo simulation to obtain $\frac{\partial^3 u}{\partial x_i^3}$ via (75) with order $O(h^q)$. In this way we compute all derivatives $\frac{\partial^3 u}{\partial x_i^3}$. We next set $h_k = 0$ and $h_i = h_j$ to compute similarly from (75)

$$\frac{\partial^3 u}{\partial x_i^2 \partial x_j} + \frac{\partial^3 u}{\partial x_j^2 \partial x_i}. \tag{76}$$

By next taking $h_i = 2h_j$, $h_k = 0$ we compute with (75)

$$2^q \frac{\partial^3 u}{\partial x_i^2 \partial x_j} + \frac{\partial^3 u}{\partial x_j^2 \partial x_i} \tag{77}$$

and then solve for $\frac{\partial^3 u}{\partial x_i^2 \partial x_j}$ and $\frac{\partial^3 u}{\partial x_j^2 \partial x_i}$ from (76) and (77). In the same way we compute all derivatives of this form. Finally, by taking $h_i = h_j = h_k$ we compute the derivatives of the form $\frac{\partial^3 u}{\partial x_i \partial x_j \partial x_i}$ and thus end up with all third order space derivatives of u with accuracy $O(h^q)$.

5 A numerical experiment for the American put

In this section we present an experimental study of the Monte Carlo procedure in Section 4 for the computation of the exercise boundary of the standard American put option. The results computed with our new Monte Carlo procedure will be compared with benchmark solutions obtained by a standard PDE method.

For the standard American put in a Black Scholes model, r and σ in (1)-(2), are constant, and $f(x) = \max(K - x, 0)$ in (3), with K being the strike of the option. For a particular choice of the parameters r, σ and K, the "exact" exercise boundary γ is computed by the projected Successive Over Relaxation (SOR) algorithm, a standard PDE method for solving American options, see e.g. [5], [17]. Note that the problem considered here is autonomous, only the time $T - t$ to maturity of the option is relevant, rather than specification of the maturity date T itself. The result is shown in Figure 2.

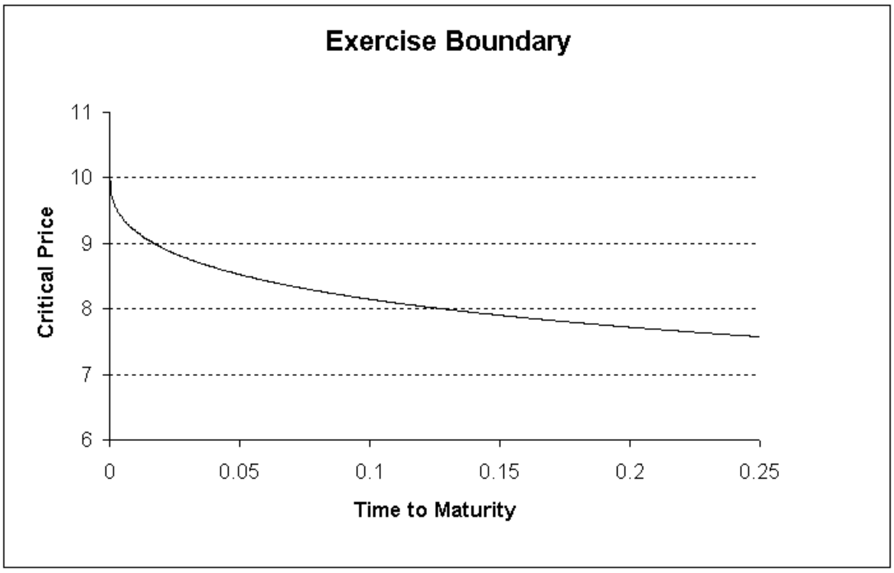

Figure 2: "Exact" exercise boundary computed by a PDE method; $K = 10$, $r = 0.1$, $\sigma = 0.4$.

As remarked in Section 4.2, the presented algorithm to construct the exercise boundary requires the boundary for small maturities as input. For our numerical study, we started the algorithm from the point $(t, g(t)) = (T - 0.05, 8.5239)$, which has been computed with high accuracy by a PDE method. We construct the exercise boundary backwards to $t = T - 0.25$ by the two Monte Carlo methods described in Section 4. The results are given in Table 1.

Remark 5.1 Another way to construct the boundary for short maturities is based on [2, 15], where approximations for the exercise boundary of the American put in the case of small maturities are analysed. For instance, we have the following approximation formula:

$$\gamma(t) \quad \approx \quad K \exp\left(-\sqrt{-\sigma^2(T-t)\ln\left(\frac{8\pi r^2}{\sigma^2}(T-t)\right)}\right).$$

$T-t$	$g^{PDE}(t)$	$g^{MC}_{1/4}(t)$	$err := \frac{g^{MC}_{1/4} - g^{PDE}}{g^{PDE}}$	$g^{MC}_{1/3}(t)$	$err := \frac{g^{MC}_{1/3} - g^{PDE}}{g^{PDE}}$
0	10.0000				
0.025	8.8439				
0.05	8.5239				
0.075	8.3102	8.2685	-0.0050	8.3122	0.00025
0.10	8.1470	8.1073	-0.0049	8.1292	-0.0022
0.125	8.0145	7.9784	-0.0045	7.9766	-0.0047
0.15	7.9027	7.8729	-0.0038	7.8724	-0.0038
0.175	7.8058	7.7780	-0.0036	7.8058	-0.0040
0.2	7.7202	7.6939	-0.0034	7.68340	-0.0047
0.225	7.6436	7.6198	-0.0031	7.6025	-0.0054
0.25	7.5745	7.5538	-0.0027	7.5265	-0.0063

Table 1. The exercise boundary computed by several algorithms and the corresponding errors.

Remarkably, for the example of the American put, the accuracy of both methods is much better than one would expect from Section 4. Even more, the $O(h^{1/4})$-method seems to be more accurate than the method of order $O(h^{1/3})$. It is possible to give an heuristic explanation for these phenomenon, which rely on the special structure of the pay-off function f and the fact that the parameters r and σ are taken to be constant. However, a detailed investigation concerning accuracy and convergence of the proposed methods requires considerable further study.

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