Coupling local currency Libor models to FX Libor models

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Abstract

We focus on the coupling of two existing and calibrated single currency Libor models into a joint Libor model that allows for pricing of multiple currency based structured interest rate products. Our main contribution is twofold: On the one hand we provide a method for synthesizing two local currency based correlation structures into a correctly defined joint correlation structure that describes the cross Libor correlations between the two currencies in a realistic way. On the other hand we introduce an (necessary) FX related factor X in order to describe the unified model with respect to one particular numéraire measure. In addition we propose to calibrate this factor to FX instruments in the case where X is modeled via Heston type dynamics.

1 Introduction

Libor interest rate modeling, initially developed by [16], [6], and [12] almost two decades ago, is still considered to be the universal tool for evaluation of structured interest rate products. One of the main reasons is the great flexibility in the choice of the Libor volatilities in the Libor framework. Starting from deterministic volatility structures leading to the Libor market model, many enhancements have been proposed in order to match implied volatility patterns of liquid products such as caps and swaptions. In this respect we mention (among other approaches) the Lévy Libor model by [8] (see for example [3] and [17] for a numerical treatments and practical implementation), displaced diffusion, CEV Libor models, log-normal mixture models, and even random parameter Libor models (e.g. [5] and the references therein for an overview). Another important line of research is the development of (one factor) stochastic volatility models based on CIR type scalar volatilities by [1], [19], and their multi factor extensions by [2] and more recently [14]. Further we mention SABR related Libor models (e.g. [10]) that are based on a different types of scalar volatilities. SABR based Libor models gained popularity because they allow for pricing of European liquids by relatively simple approximation formulas based on heat kernel expansion techniques.

For pricing of structured interest rate products that involve different currencies (quanto style products), Libor models that are jointly defined with respect to these currencies are called for. Although there already appeared some approaches in the literature more recently (e.g. [4]), by now we have not seen a generic approach to connect two existing generally specified and calibrated single currency Libor models.

We here present a generic approach to melt two given Libor rate models with respect to two different currencies (domestic and foreign) into a unified Libor model. As a key issue we propose a tractable approach to synthesize the Libor correlation structures given in their respective currencies into a joint correlation structure from which the initial domestic and foreign correlations may be retrieved, and moreover the cross correlations between domestic and foreign Libors are modeled as a (re-scaled) suitably defined average of the domestic and foreign correlations. This averaging procedure is based on coupling of a particular square root of the domestic structure with another particular square root of the foreign structure. The coupling is carried out in such a way that joint matrix is a real correlation matrix in the sense that it is positive and has diagonal entries that are all equal to one. In order to describe the unified model with respect to a unified measure, for instance the terminal domestic bond or the terminal foreign bond measure, an additional FX related factor X has to be incorporated. We finally outline an FFT based procedure by [7] for pricing liquidly traded FX options, in the case where X is driven by a Heston type stochastic volatility process. This procedure may then be used in order to calibrate the dynamics of X. The method is fairly general in the sense that it can be applied to virtually all Libor models driven by a Wiener environment.

2 Resume of Wiener based Libor modeling

For a fixed sequence of tenor dates $0 =: T_0 < T_1 < \ldots < T_n$, called a tenor structure, we consider zero bond processes $B_i,\ i = 1,\ldots,n$, where each B_i is defined on the interval $[0, T_i]$ and ends up with terminal face value $B_i(T_i) = 1$. We now define a set of forward Libors on the tenor structure by

$$L_i(t) := \frac{1}{\delta_i}\left(\frac{B_i(t)}{B_{i+1}(t)} - 1\right), \quad 0 \le t \le T_i,\ 1 \le i < n, \tag{1}$$

where the $\delta_i := T_{i+1} - T_i,\ i = 1,\ldots,n-1$, denoting the periods between two subsequent tenor dates, are called day-count fractions. L_i is in fact the annualized effective rate corresponding to a forward rate agreement (FRA) contracted at time t, for the period $[T_i, T_{i+1}]$. Here we assume that according to this agreement, the interest rate $\delta_i L_i(T_i)$ par notional 1 has to be payed at T_{i+1}.

In this paper we consider a framework where the zero-bonds $(B_i)_{i=1,\ldots,n}$ that define the Libors are adapted processes which are defined on a filtered probability space $(\Omega,\ (\mathcal{F}_t)_{0\le t\le T_\infty},\ P)$ with $T_\infty \ge T_n$ being some finite time horizon. Throughout it is assumed that the filtration $(\mathcal{F}_t)$ is generated by some d-dimensional standard Brownian motion $\mathcal{W}$ (thus excluding jump type models). Furthermore, we consider predictable (column) processes σ_i with state $\mathbb{R}^d$, that denote the volatility of the bonds B_i respectively, predictable (scalar) drift processes μ_i denoting the drifts of the B_i, and a (scalar) market price of risk process λ, all adapted to the driving Brownian motion $\mathcal{W}$. That is, in the objective measure the zero bond dynamics are of the form

$$\frac{dB_i}{B_i} = \mu_i dt + \sigma_i^\top d\mathcal{W} \quad \text{with} \quad \mu_i = \sigma_i^\top \lambda. \tag{2}$$

Under some further mild technical conditions (see [12] and [13] for details) there now exists for each i, $1 \leq i < n$, an $\mathbb{R}^d$-valued predictable volatility process Γ_i such that the Libor dynamics are given by

$$\frac{dL_i}{L_i} = - \sum_{j=i+1}^{n-1} \frac{\delta_j L_j}{1 + \delta_j L_j} \Gamma_i^\top \Gamma_j dt + \Gamma_i^\top d\mathcal{W}^{(n)}, \quad 0 \leq t \leq T_i, \ 1 \leq i < n, \tag{3}$$

where $\mathcal{W}^{(n)}$ is an equivalent standard Brownian motion under the terminal numéraire measure P_n induced by the terminal zero coupon bond B_n. That is, for all j, B_j/B_n are P_n-martingales. (We do not dwell on issues of local versus true martingales in this paper.) In particular it holds

$$\Gamma_i := \delta_i^{-1} L_i^{-1} (1 + \delta_i L_i) \left(\sigma_i - \sigma_{i+1}\right)_i, \ 1 \leq i < n. \tag{4}$$

For some general fixed i, $1 \leq i < n$ we may consider instead the numéraire measure P_{i+1} induced by the bond B_{j+1}, and then for $1 \leq j \leq i$ we obtain from (3) the dynamics

$$\begin{aligned}
\frac{dL_j}{L_j} &= \Gamma_j^\top \left(- \sum_{k=j+1}^{n-1} \frac{\delta_k L_k}{1 + \delta_k L_k} \Gamma_k dt + d\mathcal{W}^{(n)} \right) \\
&= - \sum_{k=j+1}^{i} \frac{\delta_k L_k}{1 + \delta_k L_k} \Gamma_j^\top \Gamma_k dt + \Gamma_j^\top \left(- \sum_{k=i+1}^{n-1} \frac{\delta_k L_k}{1 + \delta_k L_k} \Gamma_k dt + d\mathcal{W}^{(n)} \right) \\
&=: - \sum_{k=j+1}^{i} \frac{\delta_k L_k}{1 + \delta_k L_k} \Gamma_j^\top \Gamma_k dt + \Gamma_j^\top d\mathcal{W}^{(i+1)}, \quad 1 \leq j \leq i. \qquad (5)
\end{aligned}$$

Since due to (1) L_i is a martingale under P_{i+1}, it automatically follows that $\mathcal{W}^{(i+1)}$ in (5) is a standard Brownian motion under the equivalent measure P_{i+1}. Finally we note that in the case where the Γ_j are *deterministic* we have the well documented Libor Market Model (LMM) (see for example [5] and [18] and the references therein).

3 Multi currency extension of the Libor model

In this section we will melt two markets, the domestic and the foreign interest rate market, into just one. That is, we are going to consider zero bonds and more general traded assets in this extended market and determine their *unified* dynamics described by an SDE. Let $(B_{1,...,}B_n, B_1^*, ..., B_{n^*}^*)$ be an arbitrage free joint system of domestic zero bonds B_i and foreign zero bonds B_i^* expressed *in domestic currency,* corresponding to a domestic and foreign tenor structure $0 =: T_0 < T_1 < \ldots < T_n$, and $0 =: T_0^* < T_1^* < \ldots < T_{n^*}^*$, respectively. Since it only make sense to consider the domestic and foreign bond system on the same joint time interval, we make the following structural assumption,

$$T_n = T_{n^*}^* = T_\infty,$$

that is, we allow both tenor structures to be different, but they both span the same time period. In view of (2) we consider the coupled dynamics

$$\frac{dB_i}{B_i} = \mu_i dt + \sigma_i^\top d\mathbf{W}, \quad 1 \le i \le n, \qquad (6)$$
$$\frac{dB_i^*}{B_i^*} = \mu_i^* dt + \sigma_i^{*\top} d\mathbf{W}\,, \quad 1 \le i \le n^*,$$

where now $\mathbf{W}$ is a D-dimensional standard Brownian motion with D being sufficiently large. Connected with (6) we so introduce a general FX-Libor system $(L_1, ..., L_{n-1}, L_1^*, ..., L_{n^*-1}^*, X)$ defined by

$$L_i = \frac{1}{\delta_i}(\frac{B_i}{B_{i+1}} - 1), \quad L_i^* = \frac{1}{\delta_i^*}(\frac{B_i^*}{B_{i+1}^*} - 1), \quad X = \frac{B_{n^*}^*}{B_n} \qquad (7)$$

with $\delta_i^* := T_{i+1}^* - T_i^*$. Then with respect to B_n as numéraire we obtain under P_n the joint dynamics

$$\frac{dL_i}{L_i} = -\sum_{j=i+1}^{n-1} \frac{\delta_j L_j}{1+\delta_j L_j} \Gamma_i^\top \Gamma_j dt + \Gamma_i^\top d\mathbf{W}^{(n)}, \quad 1 \le i < n,$$
$$\frac{dL_i^*}{L_i^*} = -\Gamma_i^{*\top}\Gamma_X dt - \sum_{j=i+1}^{n^*-1} \frac{\delta_j^* L_j^*}{1+\delta_j^* L_j^*} \Gamma_i^{*\top} \Gamma_j^* dt + \Gamma_i^{*\top} d\mathbf{W}^{(n)}, \quad 1 \le i < n^*,$$
$$\frac{dX}{X} = \Gamma_X^\top d\mathbf{W}^{(n)}, \qquad (8)$$

where $\Gamma_X := \sigma_{n^*}^* - \sigma_n$, $\mathbf{W}^{(n)}$ is standard Brownian motion under P_n, and due to (4)

$$\Gamma_i = (\delta_i L_i)^{-1} (1 + \delta_i L_i) (\sigma_i - \sigma_{i+1})_i\,, \quad 1 \le i < n,$$
$$\Gamma_i^* = (\delta_i^* L_i^*)^{-1} (1 + \delta_i^* L_i^*) \left(\sigma_i^* - \sigma_{i+1}^*\right)\,, \quad 1 \le i < n^*.$$

Similarly, with respect to $B^*_{n^*}$ as numéraire we get

$$\frac{dL_i}{L_i} = \Gamma_i^\top \Gamma_{X_n} dt - \sum_{j=i+1}^{n-1} \frac{\delta_j L_j}{1+\delta_j L_j} \Gamma_i^\top \Gamma_j dt + \Gamma_i^\top d\mathbf{W}^{(n^*)}, \quad 1 \le i < n,$$

$$\frac{dL_i^*}{L_i^*} = -\sum_{j=i+1}^{n^*-1} \frac{\delta_j^* L_j^*}{1+\delta_j^* L_j^*} \Gamma_i^{*\top} \Gamma_j^* dt + \Gamma_i^{*\top} d\mathbf{W}^{(n^*)}, \quad 1 \le i < n^*,$$

$$\frac{dX}{X} = \|\Gamma_X\|^2 dt + \Gamma_X^\top d\mathbf{W}^{(n^*)}, \tag{9}$$

where $\mathbf{W}^{(n^*)}$ is a D-dimensional standard Brownian motion under the measure P_{n^*}, corresponding to $B^*_{n^*}$ that satisfies

$$d\mathbf{W}^{(n^*)} = d\mathbf{W}^{(n)} - \Gamma_X dt. \tag{10}$$

The connecting relation (10) is easily verified in the following way. Since X^{-1} is a martingale under P_{n^*}, we derive by Ito's formula and using (8),

$$\frac{dX^{-1}}{X^{-1}} = X\left(-\frac{1}{X^2} dX + \frac{1}{X^3} d\langle X, X\rangle\right) = -\Gamma_X^\top d\mathbf{W}^{(n)} + \|\Gamma_X\|^2 dt$$
$$= -\Gamma_X^\top \left(d\mathbf{W}^{(n)} - \Gamma_X dt\right) = -\Gamma_X^\top d\mathbf{W}^{(n^*)},$$

from which (10) follows.

Due to the above approach, the domestic zero bonds B_i are in general correlated with the foreign zero bonds B_i^* (in domestic currency). We think that this is a natural way of modeling and, indeed, leaving this possibility out of consideration would give rise to a very controversial discussion among practitioners. However, as a consequence, the volatility structures of both the domestic and foreign Libors in (8) (respectively (9)) need to be determined, and moreover also the volatility process Γ_X.

4 Connecting a local general model with a foreign extended market model

Let us assume that we are given a local Libor model (3) where the local volatility processes are of the form

$$\Gamma_i(t) = \|\Gamma_i\|(t) e_i(t), \quad e_i \in \mathbb{R}^d, \quad 1 \le i < n, \tag{11}$$

where the e_i are unit vectors that at most deterministically depend on t, and the $\|\Gamma_i\|(t)$ are given scalar volatility processes adapted to a d-dimensional Brownian motion $\mathcal{W}$. W.l.o.g. we assume that the correlation structure introduced by

$$R_{ij} := \left[(e_i)^\top e_j\right](t), \quad 1 \le i, j < n,$$

has constant rank d. In addition we assume that we are also given a foreign Libor model where the foreign volatility processes are of the more special form

$$\Gamma_i^*(t) = g_i(t, L^*)e_i^*(t), \quad e_i^* \in \mathbb{R}^{d^*}, \quad 1 \le i < n^*, \tag{12}$$

where e_i^* are unit vectors deterministically depending on t, and $g_i(t,\cdot)$ are non-negative deterministic (scalar) volatility functions. Hence the foreign Libors follow a so called extended market model in the foreign terminal bond measure P_{n^*}. For suitably chosen g_i, (12) might represent for example a CEV model, some displaced diffusion model, or a standard market model. It is well known that the distribution of such a model, that is the distribution of $\left(L_j^*\right)_{1\le j<n^*}$ in the P_{n^*} measure, is completely determined by these volatility functions g_i and the correlation structure R^* determined by

$$R_{ij}^* = \left[(e_i^*)^\top e_j^*\right](t) \quad 1 \le i,j < n^*. \tag{13}$$

W.l.o.g. we assume that R^* has constant rank d^*. Our goal is now to construct a new joint model (8) such that the distribution of $(L_j)_{1\le j<n}$ and $\left(L_j^*\right)_{1\le j<n^*}$ coincide with the respective initial ones in their respective measures.

Let us define $C \in \mathbb{R}^{(n-1)\times d}$ by $C_{ik} = e_{i,k}$ $1 \le i < n, 1 \le k \le d$, hence $CC^\top = R$. Next, let $F \in \mathbb{R}^{(n^*-1)\times d}$, and for some suitable $p \ge 0$ to be determined, $G \in \mathbb{R}^{(n^*-1)\times p}$ with $p = rank(G)$ (with $\mathbb{R}^{(n^*-1)\times 0} := \varnothing$, i.e. an empty matrix), such that F and G solve the following matrix equation,

$$FF^\top + GG^\top = R^*. \tag{14}$$

We then have

$$\begin{aligned}\Sigma &:= \begin{pmatrix} C & \emptyset \\ F & G \end{pmatrix}\begin{pmatrix} C & \emptyset \\ F & G \end{pmatrix}^\top = \begin{pmatrix} C & \emptyset \\ F & G \end{pmatrix}\begin{pmatrix} C^\top & F^\top \\ \emptyset & G^\top \end{pmatrix} \\ &= \begin{pmatrix} R & CF^\top \\ FC^\top & R^* \end{pmatrix}\end{aligned}$$

and it holds

$$\begin{aligned}\frac{dL_i}{L_i} &= -\sum_{j=i+1}^{n-1} \frac{\delta_j L_j}{1+\delta_j L_j}\Gamma_i^\top \Gamma_j dt + \Gamma_i^\top d\mathbf{W}^{(n)}, \quad 1 \le i < n, \\ \frac{dL_i^*}{L_i^*} &= -\sum_{j=i+1}^{n^*-1} \frac{\delta_j^* L_j^*}{1+\delta_j^* L_j^*}\Gamma_i^{*\top} \Gamma_j^* dt + \Gamma_i^{*\top} d\mathbf{W}^{(n^*)}, \quad 1 \le i < n^*,\end{aligned} \tag{15}$$

with respect to an extended Brownian motion $\mathbf{W}^{(n)} := (\mathcal{W}^{(n)}, \widetilde{\mathcal{W}}^{(n)}) \in \mathbb{R}^D$ and $\mathbf{W}^{(n^*)} := (\mathcal{W}^{(n^*)}, \widetilde{\mathcal{W}}^{(n^*)}) \in \mathbb{R}^D$, under the measures P_n and P_{n^*}, respectively, with $D = d + p$ and

$$\Gamma_i = \|\Gamma_i\| \, \mathbf{e}_i, \quad 1 \le i < n, \quad \text{and} \quad \Gamma_i^* = g_i \mathbf{e}_i^*, \quad 1 \le i < n^*, \tag{16}$$

where

$$\mathbf{e}_{i,k} := e_{i,k}, \quad 1 \le k \le d, \quad \mathbf{e}_{i,d+k} = 0, \quad 1 \le k \le p, \quad 1 \le i < n,$$
$$\mathbf{e}^*_{i,k} := F_{ik}, \quad 1 \le k \le d, \quad \mathbf{e}^*_{i,d+k} = G_{ik}, \quad 1 \le k \le p, \quad 1 \le i < n^*. \tag{17}$$

There are two edge solutions: i) $F = 0$ implying that $GG^\top = R^*$ (so $p = d^*$ hence $D = d + d^*$) and

$$\Sigma = \begin{pmatrix} R & \emptyset \\ \emptyset & R^* \end{pmatrix},$$

i.e. L and L^* are independent, ii) $G = \varnothing$ ($p = 0$) implying that $FF^\top = R^*$ (so $D = d \ge d^*$),

$$\Sigma = \begin{pmatrix} R & CF^\top \\ FC^\top & R^* \end{pmatrix}, \tag{18}$$

and both the local and the foreign model are driven by $\mathcal{W}^{(n)} \in \mathbb{R}^d$ in fact.

A simple pragmatic solution

By taking $p = d^*$ in the case $d \ge d^*$ and letting F_1 and G_1 be matrices as specified above with $F_1F_1^\top = R^*$ and $G_1G_1^\top = R^*$, we have that for any $|\varrho| \le 1$, $F_\varrho = \varrho F_1$ and $G_\varrho = \sqrt{1-\varrho^2}G_1$ solve the matrix equation (14). Let us specialize to the case $d = d^* = p$: A matrix G with $GG^\top = R^*$ is determined up to a orthogonal transformation. Indeed, let $G_1 \in \mathbb{R}^{(n^*-1)\times p}$ be the unique lower triangular (Cholesky) root of R^* with $G_{1,ii} > 0$, and $G_1G_1^\top = R^*$, then any G with $G = G_1Q$ for orthogonal $Q \in \mathbb{R}^{p\times p}$, satisfies $GG^\top = R^*$. Now let further C_1 be the unique lower triangular (Cholesky) root of R, hence $C_1C_1^\top = R$, then determine $Q_C \in \mathbb{R}^{p\times p}$ such that $C = C_1Q_C$, and take $F_1 = G_1Q_C$. For the joint FX Libor model we then take for F and G in (17),

$$F_\varrho = \varrho G_1 Q_C \quad \text{and} \quad G_\varrho = \sqrt{1-\varrho^2}G_1, \tag{19}$$

respectively, and the volatilities in (16) accordingly. For the cross currency Libor correlation matrix we then obtain in (18)

$$CF_\varrho^\top = \varrho C Q_C^\top G_1^\top = \varrho C_1 G_1^\top, \quad |\varrho| \le 1,$$

hence

$$\Sigma_\varrho := \begin{pmatrix} R & \varrho C_1 G_1^\top \\ \varrho G_1 C_1^\top & R^* \end{pmatrix} \tag{20}$$

being a valid correlation matrix for any $|\varrho| \le 1$. In the particular case where $R = R^*$ (hence $n = n^*$) we thus obtain by this construction $CF_\varrho^\top = \varrho R$. In the general case where $R \ne R^*$ and possibly $n \ne n^*$ (but $d = d^* = p$) we may consider the matrix $C_1G_1^\top$ as some kind of average between R and R^*. In the next section we will outline the calibration of ϱ to FX rate vanilla options.

5 Calibration to FX market

Let us consider $X := B^*_{n^*}/B_n$. Note that (cf. [9]),

$$B_n(t) = B_{\eta(t)}(t) \prod_{j=\eta(t)}^{n-1} \frac{1}{1+\delta_j L_j(t)}$$

for $0 \le t \le T_n = T_\infty$. and $\eta(t) := \min\{m : T_m \ge t\}$. Thus, by (15),

$$\begin{aligned}
\frac{dB_n}{B_n} &= (...)dt + d\ln B_n \\
&= (...)dt + \frac{dB_{\eta(t)}}{B_{\eta(t)}} - \sum_{j=\eta(t)}^{n-1} \frac{\delta_j L_j}{1+\delta_j L_j} \Gamma_j^\top d\mathbf{W}^{(\cdot)} \\
&=: (...)dt + \sigma_{\eta(t)}^\top d\mathbf{W}^{(\cdot)} - \sum_{j=\eta(t)}^{n-1} \frac{\delta_j L_j}{1+\delta_j L_j} \Gamma_j^\top d\mathbf{W}^{(\cdot)}.
\end{aligned}$$

In the same way, for $t \le T^*_{n^*} = T_\infty$,

$$B^*_{n^*}(t) = B^*_{\eta^*(t)}(t) \prod_{j=\eta^*(t)}^{n^*-1} \frac{1}{1+\delta^*_j L^*_j(t)}$$

with $\eta^*(t) := \min\{m : T^*_m \ge t\}$ and so

$$\begin{aligned}
\frac{dB^*_{n^*}(t)}{B^*_{n^*}(t)} &= (...)dt + d\ln B^*_{n^*} \\
&= (...)dt + \sigma^{*\top}_{\eta^*(t)} d\mathbf{W}^{(\cdot)} - \sum_{j=\eta^*(t)}^{n^*-1} \frac{\delta^*_j L^*_j}{1+\delta^*_j L^*_j} \Gamma_j^{*\top} d\mathbf{W}^{(\cdot)}.
\end{aligned}$$

Note that $B^*_{n^*}(t)$ is the foreign terminal bond expressed in domestic currency. We so may set for $t \le T_\infty$, $B^*_{n^*}(t) =: \zeta(t)\widetilde{B}_{n^*}(t)$, where $\widetilde{B}_{n^*}(t)$ is a foreign bond expressed in the foreign currency and $\zeta(t)$ is the FX spot rate. In particular we have $B^*_{n^*}(T_{n^*}) = \zeta(T_{n^*}) = \zeta(T_n) = \zeta(T_\infty)$, and

$$\zeta(t) = \frac{B^*_{n^*}(t)}{\widetilde{B}_{n^*}(t)} = \frac{B^*_{\eta^*(t)}(t)}{\widetilde{B}_{\eta^*(t)}(t)}.$$

We thus have

$$\begin{aligned}
\frac{dX}{X} &= (...)dt + \left(\sigma^{*\top}_{\eta^*(t)} - \sigma^\top_{\eta(t)}\right) d\mathbf{W}^{(\cdot)} \\
&+ \left(\sum_{j=\eta(t)}^{n-1} \frac{\delta_j L_j}{1+\delta_j L_j} \Gamma_j^\top - \sum_{j=\eta^*(t)}^{n^*-1} \frac{\delta^*_j L^*_j}{1+\delta^*_j L^*_j} \Gamma_j^{*\top} \right) d\mathbf{W}^{(\cdot)} \\
&=: (...)dt + \Gamma_X^\top d\mathbf{W}^{(\cdot)} = \Gamma_X^\top d\mathbf{W}^{(n)}
\end{aligned}$$

with

$$\Gamma_X = \sigma^*_{\eta^*(t)} - \sigma_{\eta(t)} + \sum_{j=\eta(t)}^{n-1} \frac{\delta_j L_j}{1+\delta_j L_j}\Gamma_j - \sum_{j=\eta^*(t)}^{n^*-1} \frac{\delta^*_j L^*_j}{1+\delta^*_j L^*_j}\Gamma^*_j. \tag{21}$$

Let us further assume that

$$\frac{d\zeta}{\zeta} = (\cdots)dt + \left(\sigma^{fx}\right)^\top d\mathbf{W}^{(\cdot)}.$$

Since $B^*_{\eta^*(t)}(t) = \zeta(t)\widetilde{B}_{\eta^*(t)}(t)$ we then have $\sigma^*_{\eta^*(t)} = \sigma^{fx} + \widetilde{\sigma}_{\eta^*(t)}$ with $\widetilde{\sigma}_j$ being the volatility of the foreign zero bond maturing at T^*_j. From this we observe that Γ_X is completely determined by specification of σ^{fx} and the difference $\widetilde{\sigma}_{\eta^*(t)} - \sigma_{\eta(t)}$. Conversely, specifying Γ_X implicitly determines $\sigma^{fx} + \widetilde{\sigma}_{\eta^*(t)} - \sigma_{\eta(t)}$ via (21).

Remark 1 *Moreover, in practice one may neglect the volatility of $B_{\eta(t)}(t)$ and $\widetilde{B}_{\eta^*(t)}(t)$, respectively, being the volatilities of zero bonds less than one period before maturity. We then have in approximation*

$$\sigma^*_{\eta^*(t)} - \sigma_{\eta(t)} \approx \sigma^{fx}(t) \tag{22}$$

in (21).

More generally, for $i \le n^*$ and $j \le n$ we may consider the process $X_{i,j} := B^*_i(t)/B_j(t)$, $0 \le t \le T^*_i \wedge T_j$ (hence $X \equiv X_{n^*,n}$), and for its volatility $X_{i,j}$ we derive in a similar way,

$$\begin{aligned}\Gamma_{X_{i,j}} &= \sigma^*_{\eta^*(t)} - \sigma_{\eta(t)} + \sum_{k=\eta(t)}^{j-1} \frac{\delta_k L_k}{1+\delta_k L_k}\Gamma_k - \sum_{k=\eta^*(t)}^{i-1} \frac{\delta^*_k L^*_k}{1+\delta^*_k L^*_k}\Gamma^*_k \\ &= \Gamma_X + \sum_{k=i}^{n^*-1} \frac{\delta^*_k L^*_k}{1+\delta^*_k L^*_k}\Gamma^*_k - \sum_{k=j}^{n-1} \frac{\delta_k L_k}{1+\delta_k L_k}\Gamma_k, \quad t \le T^*_{i-1} \wedge T_{j-1}. \end{aligned} \tag{23}$$

So any $\Gamma_{X_{i,j}}$ is determined by Γ_X via (23).

Let us now consider an option to buy one unit of foreign currency for K units of domestic currency at time T^*_i, $i \le n^*$, and assume that $T^*_i = T_{i'}$ for a certain i'. Clearly, the net payoff of this option is

$$(\zeta(T^*_i) - K)^+ = (B^*_i(T^*_i) - K)^+ = \left(\frac{B^*_i(T_{i'})}{B_{i'}(T_{i'})} - K\right)^+ = \left(X_{i,i'}(T_{i'}) - K\right)^+,$$

and the option value in domestic currency at time $t = 0$ is given by

$$\begin{aligned} C_i(K) &:= B_n(0)E_n \frac{(\zeta(T^*_i) - K)^+}{B_n(T^*_i)} \\ &= B_{i'}(0)E_{i'}\left(X_{i,i'}(T_{i'}) - K\right)^+. \end{aligned}$$

For $i = n^*$, $i' = n$, we thus obtain by $T^*_{n^*} = T_n = T_\infty$,

$$\begin{aligned} C_{n^*}(K) &:= B_n(0) E_n \frac{(\zeta(T^*_{n^*}) - K)^+}{B_n(T_n)} \\ &= B_n(0) E_n \left(X(T_\infty) - K \right)^+ . \end{aligned} \tag{24}$$

We thus conclude that any standard FX option maturing on a joint tenor date $T^*_i = T_{i'}$ as described above may be priced, once the volatility process Γ_X is specified, via the formula

$$C_i(K) = B_{i'}(0) E_{i'} \exp \left[-\frac{1}{2} \int_0^{T_{i'}} \left\| \Gamma_{X_{i,j}} \right\|^2 dt + \int_0^{T_{i'}} \Gamma^\top_{X_{i,j}} d\mathbf{W}^{(i')} \right], \tag{25}$$

where $\Gamma_{X_{i,j}}$ follows from (23). Needless to say that a particular evaluation procedure for (25) largely depends on the specific structure of the respective volatility specifications for Γ, Γ^*, and Γ_X.

Example 2 *In the special case of a (multi-factor) domestic and foreign* Libor Market Model, *that is Γ and Γ^* are deterministic vector functions, the $\Gamma_{X_{i,j}}$ may be obtained from Γ_X by standardly freezing the Libors in (23). If moreover Γ_X is taken to be deterministic as well, we may then compute all prices (25) by the Black 76 formula.*

Remark 3 *We further observe, for instance, that when Γ_X has a Heston type structure, like $\Gamma_X =: \beta_X \sqrt{V} \mathbf{e}_X$ for some deterministic β_X, unit vector $\mathbf{e}_X$, and square-root volatility process V, then due to (23) $\Gamma_{X_{i,j}}$ is essentially* ***not*** *of Heston type for $i < n^*$. In this respect we should note that the approach in [4], where simultaneously all the $FX_i = \Gamma_{X_{i,i}}$ have a Heston type volatility structure seems to be inconsistent with this observation.*

Remark 4 *In the case where both the domestic and foreign model is a* one-factor Libor Market Model, *i.e. both Γ and Γ^* are deterministic scalar volatilities connected to a one dimensional Brownian motion, we are in a setting related to the one in [4] in a sense.*

We continue with a further mild structural assumption on the process Γ_X, namely that it is of the form

$$\Gamma_X = \|\Gamma_X\| \left(\sum_{j=1}^{n-1} \rho_X \mathbf{e}_j + \sum_{j=1}^{n^*-1} \rho^*_X \mathbf{e}^*_j \right) =: \|\Gamma_X\| \, \mathbf{e}_X, \tag{26}$$

under the normalization condition

$$\|\mathbf{e}_X\|^2 = \left\| \sum_{j=1}^{n-1} \rho_X \mathbf{e}_j + \sum_{j=1}^{n^*-1} \rho^*_X \mathbf{e}^*_j \right\|^2 =$$

$$\rho_X^2 \sum_{j,j'=1}^{n-1} R_{jj'} + 2\varrho \rho_X \rho^*_X \sum_{j=1}^{n-1} \sum_{j'=1}^{n^*-1} \left[C_1 G_1^\top \right]_{jj'} + (\rho^*_X)^2 \sum_{j=1}^{n^*-1} \sum_{j'=1}^{n^*-1} R^*_{jj'} = 1, \tag{27}$$

where ρ_X and ρ_X^* are considered to be some kind of uniform partial correlation of the FX market with the domestic and foreign Libor rates, respectively. Further, in (26) $\|\Gamma_X\|$ is in general a scalar stochastic process that is still to be specified. In (8) we now have by (26) and (19),

$$\Gamma_i^{*\top}\Gamma_X = \|\Gamma_X\| \left(\sum_{j=1}^{n-1} \rho_X \Gamma_i^{*\top}\mathbf{e}_j + \sum_{j=1}^{n^*-1} \rho_X^* \Gamma_i^{*\top}\mathbf{e}_j^* \right)$$

$$\|\Gamma_X\| \left(\varrho\rho_X \|\Gamma_i^*\| \sum_{j=1}^{n-1} \left[C_1 G_1^\top\right]_{ji} + \rho_X^* \|\Gamma_i^*\| \sum_{j=1}^{n^*-1} R_{ij}^* \right). \tag{28}$$

In particular the correlations of X with the domestic and foreign Libors are given by

$$\begin{aligned} \mathrm{Corr}_{X,L_i} &= \sum_{j=1}^{n-1} \rho_X \mathbf{e}_j \cdot \mathbf{e}_i + \sum_{j=1}^{n^*-1} \rho_X^* \mathbf{e}_j^* \cdot \mathbf{e}_i \qquad (29) \\ &= \rho_X \sum_{j=1}^{n-1} R_{ij} + \varrho\rho_X^* \sum_{j=1}^{n^*-1} \left[C_1 G_1^\top\right]_{ij} \\ &=: \rho_X P_i + \varrho\rho_X^* Q_i, \quad 1 \le i < n, \quad \text{and} \\ \mathrm{Corr}_{X,L_i^*} &= \sum_{j=1}^{n-1} \rho_X \mathbf{e}_j \cdot \mathbf{e}_i^* + \sum_{j=1}^{n^*-1} \rho_X^* \mathbf{e}_j^* \cdot \mathbf{e}_i^* \qquad (30) \\ &= \varrho\rho_X \sum_{j=1}^{n-1} \left[C_1 G_1^\top\right]_{ji} + \rho_X^* \sum_{j=1}^{n^*-1} R_{ij}^* \\ &=: \varrho\rho_X Q_i^* + \rho_X^* P_i^*, \quad 1 \le i < n^*, \end{aligned}$$

respectively. Since all processes X, L_i, and L_i^* are observable at the market and the constants P_i, Q_i, and P_i^*, Q_i^*, in (29) and (30) are in principle known from the respective calibrations of the domestic and foreign Libor system, it seems natural to estimate the correlations in (29) and (30) from historical data. This may be done by minimizing the total square distance

$$\sum_{i=1}^{n} \left(\varrho\rho_X Q_i^* + \rho_X^* P_i^* - \widehat{\mathrm{Corr}}_{X,L_i} \right)^2 + \sum_{i=1}^{n^*} \left(\varrho\rho_X Q_i^* + \rho_X^* P_i^* - \widehat{\mathrm{Corr}}_{X,L_i^*} \right)^2 \to \min_{\varrho,\rho_X,\rho_X^*}$$

with $\widehat{\mathrm{Corr}}_{X,L_i}$ and $\widehat{\mathrm{Corr}}_{X,L_i^*}$ being the respectively estimated correlations, under the normalization restriction

$$\rho_X^2 \sum_{i=1}^{n-1} P_i + 2\varrho\rho_X\rho_X^* \sum_{i=1}^{n-1} Q_i + (\rho_X^*)^2 \sum_{i=1}^{n-1} P_i^* = 1$$

(note that $\sum_{i=1}^{n-1} Q_i = \sum_{i=1}^{n^*-1} Q_i^*$ and cf. (27)). Note that after determination of Γ_X, the dynamics of the FX rate ζ are implicitly determined by (21) and (22), and are in particular driven by the total Brownian motion $\mathbf{W}$.

After identifying $\mathbf{e}_X$ in (26) in the above way, the norm process $\|\Gamma_X\|$ has to be modeled appropriately, such that calibration to a suitably large set of plain vanilla FX options is feasible. The most simple way is to assume that $\|\Gamma_X\|$ is deterministic (cf. Example 2). However the typically observed skew patterns in implied volatilities of vanilla FX options may not be captured in this way. Therefore more sophisticated choices are called for. Below we will sketch the procedure in the context of a Heston type model for X and a fairly generally structured domestic and foreign Libor model as described in Section 4.

Let us assume that

$$\frac{dX}{X} = \beta_X \sqrt{V} \mathbf{e}_X^\top d\mathbf{W}^{(n)}, \tag{31}$$

where V follows the square-root dynamics

$$dV = \kappa_X(\theta_X - V)dt + \sigma_X \sqrt{V} \left(\rho_X \mathbf{e}_X^\top d\mathbf{W}^{(n)} + \sqrt{1-\rho_X^2} dW_X \right). \tag{32}$$

In (31) and (32) the parameters β_X, κ_X, θ_X, σ_X, and ρ_X are assumed to be constants, and W_X is an additional independent standard Brownian motion to inforce decorrelation between X and V. (Formally one might extend the vector $\mathbf{W}$ with an extra Brownian component and extend correspondingly all the unit vectors $\mathbf{e}_i$, $\mathbf{e}_i^*$, $\mathbf{e}_X$, with an extra zero component.) Subsequently we may calibrate the system (31)-(32) to a family of vanilla FX options (24) with different strikes and common maturity $T_n = T_{n*}^* = T_\infty$. This may be done in a standard way by using a relatively fast Fourier based pricing procedure. Although this pricing procedure is more or less standard, we still present it here for the convenience of the reader (cf. also [14]).

Let us write (24) as

$$C(K) := C_{n^*}(K) = B_n(0) E_n \left(X(0) e^{\ln \frac{X(T_\infty)}{X(0)}} - K \right)^+. \tag{33}$$

We may then apply the Fourier pricing method of Carr-Madan (spelled out later on) to the triple

$$\varphi(z\,; v), \quad X(0), \quad K,$$

where the characteristic function

$$\varphi(z\,; v) := E_n \left[e^{\mathrm{i}z \ln \frac{X(T_\infty)}{X(0)}} \middle| V(0) = v \right] \tag{34}$$

may be obtained as follows. Consider the logarithm of (31),

$$d \ln X = -\frac{1}{2} \beta_X^2 V dt + \beta_X \sqrt{V} \mathbf{e}_X^\top d\mathbf{W}^{(n)}, \tag{35}$$

along with the square-root dynamics (32). Let us then abbreviate $Y^{0,y,v}(t) := \ln X(t)$ with $Y^{0,y,v}(0) = \ln X(0) =: y$, and $V^{0,y,v}(t) := V(t)$ with $V^{0,y,v}(0) = V(0) =: v$. Then by (35), the generator of the vector process (Y, V) is given by

$$\begin{aligned} A := A_{y,v} := &-\frac{1}{2}\beta_X^2 v dt \frac{\partial}{\partial y} + \kappa_X \left(\theta_X - v\right) \frac{\partial}{\partial v} \\ &+ \frac{1}{2} v \beta_X^2 \frac{\partial^2}{\partial y^2} + v \beta_X \sigma_X \rho_X \frac{\partial^2}{\partial y \partial v} + \frac{1}{2}\sigma_X^2 v \frac{\partial^2}{\partial v^2}. \end{aligned}$$

Let $\widehat{p}(z, z'; t, y, v)$ satisfy the Cauchy problem

$$\frac{\partial \widehat{p}}{\partial t} = A\widehat{p}, \qquad \widehat{p}(z, z'; 0, y, v) = e^{\mathrm{i}\left(zy + z'v\right)}. \tag{36}$$

Then

$$\widehat{p}(z, z'; t, y, v) = Ee^{\mathrm{i}\left(zY^{0,y,v}(t) + z'V^{0,x,v}(t)\right)}.$$

We are only interested in the solution for $z' = 0$. Let us therefore consider the ansatz

$$\widehat{p}(z; t, y, v) = \exp\left(A(z;t) + B_0(z;t)y + B(z;t)v\right)$$

with

$$A(z;0) = 0, \quad B_0(z;0) = \mathrm{i}z, \quad B(z;0) = 0. \tag{37}$$

Substitution this ansatz into (36) yields,

$$\begin{aligned}\left(\frac{\partial A}{\partial t} + \frac{\partial B_0}{\partial t}x + \frac{\partial B}{\partial t}v\right) &= -\frac{1}{2}v\beta_X^2 B_0 + \kappa_X\left(\theta_X - v\right)B \\ &\quad + \frac{1}{2}v\beta_X^2 B_0^2 + v\beta_X\sigma_X\rho_X B_0 B + \frac{1}{2}\sigma_X^2 v B^2,\end{aligned}$$

and we so obtain the Riccati system

$$\begin{aligned}\frac{\partial A}{\partial t} &= \kappa_X\theta_X B \\ \frac{\partial B_0}{\partial t} &= 0 \\ \frac{\partial B}{\partial t} &= -\frac{1}{2}\beta_X^2 B_0 - \kappa_X B + \frac{1}{2}\beta_X^2 B_0^2 + \beta_X\sigma_X\rho_X B_0 B + \frac{1}{2}\sigma_X^2 B^2.\end{aligned}$$

In view of (37) we then get

$$\begin{aligned}\frac{\partial A}{\partial t} &= \kappa_X\theta_X B \\ \frac{\partial B}{\partial t} &= -\frac{1}{2}\beta_X^2\left(\mathrm{i}z + z^2\right) - \left(\kappa_X - \mathrm{i}z\beta_X\sigma_X\rho_X\right)B + \frac{1}{2}\sigma_X^2 B^2.\end{aligned}$$

As a well known fact (see [11]) this system can be explicitly solved, but there are different representations for its solution depending on the chosen branch of the complex logarithm. We here use Lord and Kahl's representation due to the principal branch, see [15][1], to get

$$B(z;t) = \frac{a+d}{\sigma_X^2}\frac{1 - e^{dt}}{1 - ge^{dt}}$$

and

$$A(z;t) = \frac{\kappa_X\theta_X}{\sigma_X^2}\left\{(a-d)\,t - 2\ln\left[\frac{e^{-dt} - g}{1-g}\right]\right\},$$

[1] Roger Lord confirmed a typo in the published version in a personal communication and therefore referred to the preprint version.

where

$$
\begin{aligned}
a &= \kappa_X - \mathrm{i}z\beta_X\sigma_X\rho_X \\
d &= \sqrt{a^2 + (\mathrm{i}z + z^2)\,\beta_X^2\sigma_X^2} \\
g &= \frac{a+d}{a-d}.
\end{aligned}
$$

Taking all together we have with $t = T_\infty$ for (34),

$$
\varphi(z\,;v) = e^{-\mathrm{i}z\ln X(0)}\widehat{p}\,(z\,;T_\infty, \ln X(0), v) = \exp\left(\widetilde{A}(z;T_\infty) + B(z;T_\infty)v\right) \quad (38)
$$

with

$$
\begin{aligned}
B(z;T_\infty) &= \frac{a+d}{\sigma_X^2}\frac{1-e^{dT_\infty}}{1-ge^{dT_\infty}}, \quad \text{and} \\
\widetilde{A}(z;t) &:= \frac{\kappa_X\theta_X}{\sigma_X^2}\left\{(a-d)\,T_\infty - 2\ln\left[\frac{e^{-dT_\infty}-g}{1-g}\right]\right\}.
\end{aligned}
$$

Carr & Madan inversion formula

Due to Carr and Madan [7], the FX vanilla option price may be obtained by the following inversion formula,

$$
\begin{aligned}
C(K) = B_n(0)(X(0)-K)^+ + \\
\frac{B_n(0)X(0)}{2\pi}\int_{-\infty}^{\infty}\frac{1-\varphi(z-\mathrm{i};V(0))}{z(z-\mathrm{i})}e^{-\mathrm{i}z\ln\frac{K}{X(0)}}\,dz, \qquad (39)
\end{aligned}
$$

where φ is given by (38). The integrand in (39) decays with rate z^{-2} if $|z| \to \infty$, which is relatively slow from a numerical point of view. Therefore it is better to modify the inversion formula in the following way. Let $\varphi^{\mathcal{B}}$ be the characteristic function (34) due to some Black model,

$$
X(T_\infty) = X(0)e^{-\frac{1}{2}\left(\sigma^B\right)^2 T_\infty + \sigma^B\sqrt{T_\infty}\varsigma}, \quad \varsigma \in N(0,1)
$$

in the measure P_n, for a particular suitably chosen volatility σ^B. We then have (cf. Black's 76 formula)

$$
E_n\left(X(T_\infty)-K\right)^+ = \mathcal{B}(X(0), T_\infty, \sigma^B, K),
$$

where

$$
\begin{aligned}
\mathcal{B}(X,T,\sigma,K) &:= X\mathcal{N}\left(d_+\right) - K\mathcal{N}\left(d_-\right), \quad \text{with} \\
d_\pm &:= \frac{\ln\frac{X}{K} \pm \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}, \quad \text{and} \\
\varphi^{\mathcal{B}}(z\,;v) = \varphi^{\mathcal{B}}(z) &= E_n e^{\mathrm{i}z\left(-\frac{1}{2}\left(\sigma^B\right)^2 T_\infty + \sigma^B\sqrt{T_\infty}\varsigma\right)} \\
&= e^{-\frac{1}{2}\left(\sigma^B\right)^2 T_\infty\left(z^2+\mathrm{i}z\right)}.
\end{aligned}
$$

By application of Carr and Madan's formula to the Black model we get,

$$C^{\mathcal{B}}(K) := B_n(0)\mathcal{B}(X(0), T_\infty, \sigma^B, K) = B_n(0)(X(0) - K)^+ \qquad (40)$$
$$+ \quad \frac{B_n(0)X(0)}{2\pi} \int_{-\infty}^{\infty} \frac{1 - \varphi^{\mathcal{B}}(z - \mathfrak{i})}{z(z - \mathfrak{i})} e^{-\mathfrak{i}z \ln \frac{K}{X(0)}} dz,$$

and then by subtracting (40) from (39) we obtain,

$$C(K) = C^{\mathcal{B}}(K)+ \qquad (41)$$
$$\frac{B_n(0)X(0)}{2\pi} \int_{-\infty}^{\infty} \frac{\varphi^{\mathcal{B}}(z - \mathfrak{i}; \cdot) - \varphi(z - \mathfrak{i}; V(0))}{z(z - \mathfrak{i})} e^{-\mathfrak{i}z \ln \frac{K}{X(0)}} dz.$$

Inversion formula (41) is usually much more efficient due to the typically much faster decaying integrand in comparison with (39).

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