Optimal stopping via pathwise dual empirical maximisation

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Abstract

The optimal stopping problem arising in the pricing of American options can be tackled by the so called dual martingale approach. In this approach, a dual problem is formulated over the space of adapted martingales. A feasible solution of the dual problem yields an upper bound for the solution of the original primal problem. In practice, the optimization is performed over a finite-dimensional subspace of martingales. A sample of paths of the underlying stochastic process is produced by a Monte-Carlo simulation, and the expectation is replaced by the empirical mean. As a rule the resulting optimization problem, which can be written as a linear program, yields a martingale such that the variance of the obtained estimator can be large. In order to decrease this variance, a penalizing term can be added to the objective function of the pathwise optimization problem. In this paper, we provide a rigorous analysis of the optimization problems obtained by adding different penalty functions. In particular, a convergence analysis implies that it is better to minimize the empirical maximum instead of the empirical mean. Numerical simulations confirm the variance reduction effect of the new approach.

Keywords: optimal stopping problem, dual martingale, variance reduction
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1 Introduction

In this paper we consider the following optimal stopping problem. Let $\mathcal{F}_t$ be a filtration, and let Z_t, $t = 0, \dots, T$ be a discrete-time adapted process with bounded variance. We wish to maximize

$$Y^* = \max_{\tau \in \mathcal{T}} \mathbb{E}[Z_\tau], \tag{1}$$

where $\mathcal{T}$ is the set of stopping times on $\{0, \dots, T\}$. Desai et al. [3] introduced a *pathwise optimization* method for solving this kind of optimal stopping problems. It is based on the *dual martingale* approach, which was developed in [6] and [4], see also [2] which in fact contained the dual approach in germ. The dual problem to the original optimal stopping problem can be written as an optimization problem over the space of martingales M with zero initial value. More precisely,

$$Y^* = \inf_M \mathbb{E} \max_{t=0,\dots,T} (Z_t - M_t) = \max_{t=0,\dots,T} (Z_t - M_t^*) \quad \text{a.s.} \tag{2}$$

where M^* is the martingale part of the Doob-Meyer decomposition of the Snell envelope of Z_t, and the optimal values of the primal and dual problem coincide.

As the optimal stopping problem, the dual problem is infinite-dimensional. In order to reduce it to a finite-dimensional one, it was proposed in [3] to optimize over a finite-dimensional section of the space

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of martingales. In other words, a finite number of basis martingales is chosen and the optimization is performed over all linear combinations of these basis functions, the coefficients playing the role of the decision variables. Since the optimal martingale M^* is as a rule not contained in the linear span of the chosen basis functions, the procedure yields a suboptimal solution leading to an upper bound of the optimal value of the original optimal stopping problem. Another practical problem in solving dual problem (2) is the presence of the expectation operator. This can be circumvented by replacing the expectation by the empirical mean over a sample of N paths [3]. As a consequence, we obtain the optimization problem

$$\inf_{\alpha} \frac{1}{N} \sum_{n=1}^{N} \max_{t=0,\dots,T} (Z_t^{(n)} - \sum_{k=1}^{K} \alpha_k M_t^{k,(n)}). \tag{3}$$

Here $\alpha = (\alpha_1, \dots, \alpha_K)^T$ is the vector of decision variables, $Z_t^{(n)}$ are the sample paths of the process Z_t, and $M_t^{k,(n)}$ are the paths of the basis martingales. The index $k = 1, \dots, K$ denotes the index of the basis martingale, and $n = 1, \dots, N$ denotes the index of the path.

For an independent new simulation with $\tilde{N}$ samples the quantity

$$\frac{1}{\tilde{N}} \sum_{\tilde{n}=1}^{\tilde{N}} \max_{t=0,\dots,T} \left(Z_t^{(\tilde{n})} - \sum_{k=1}^{K} \alpha_k M_t^{k,(\tilde{n})} \right)$$

gives an upper biased estimate of (2).

As mentioned before, the martingale part M^* of the Doob-Meyer decomposition of the Snell envelope of Z_t yields an optimal solution to (2). While this solution may not be unique, it has the distinguished property that the random variable $Z(M^*) = \max_{t=0,\dots,T}(Z_t - M_t^*)$ has variance zero (see [1]). Martingales having this property have been named *surely optimal* and have been characterized in [10]. When seeking a solution to the dual problem (2), one is not only interested in a martingale M that gives a tight upper bound on the optimal value of the optimal stopping problem, but also in a martingale that is close to a surely optimal one, in the sense that the random variable $Z(M) = \max_{t=0,\dots,T}(Z_t - M_t)$ has a low variance. This second condition, however, is usually not met by the optimal solutions of the approximated problem (3), as evidenced in [10].

To counter this problem, Belomestny proposed in [1] to add the empirical standard deviation as a penalty term to the objective in (3), leading to the optimization problem

$$\inf_{\alpha} \left(\frac{1}{N} \sum_{n=1}^{N} Z^{(n)}(M(\alpha)) + \lambda \sqrt{\frac{1}{N-1} \sum_{n=1}^{N} \left(Z^{(n)}(M(\alpha)) - \frac{1}{N} \sum_{l=1}^{N} Z^{(l)}(M(\alpha)) \right)^2} \right), \tag{4}$$

where $M(\alpha) = \sum_{j=1}^{K} \alpha_j M^j$ and

$$Z^{(n)}(M(\alpha)) = \max_{t=0,\dots,T} \left(Z_t^{(n)} - \sum_{j=1}^{K} \alpha_j M_t^{j,(n)} \right). \tag{5}$$

Here $\lambda \geq 0$ is a scalar determining the weight of the penalty term. However, while problem (3) can be cast as a linear program, problem (4) may fail to be convex if λ is too large, and thus difficult to solve to a global optimum.

The subject of this paper is to analyze the effect of the penalty term in augmented problems of type (4). As a penalty we assume a general continuous convex homogeneous function F of degree 1 of the vector $\hat{Z}(M(\alpha)) = (\hat{Z}^{(n)}(M(\alpha)))_{n=1,\dots,N}$, where

$$\hat{Z}^{(n)}(M(\alpha)) = Z^{(n)}(M(\alpha)) - \frac{1}{N} \sum_{l=1}^{N} Z^{(l)}(M(\alpha)). \tag{6}$$

Note that the vector $\hat{Z}(M(\alpha))$ resides in a subspace $L \subset \mathbb{R}^N$ of codimension 1, and hence F is effectively a function from $\mathbb{R}^{N-1}$ to $\mathbb{R}$. We will, however, consider convex homogeneous functions $F : \mathbb{R}^N \to \mathbb{R}$ and have in mind that only the values of F on the $(N-1)$-dimensional subspace L are relevant. As F should penalize deviations of the vector $\hat{Z}(M(\alpha))$ from zero, we shall assume that

$$F(x) > 0 \qquad \forall\, x \in L \setminus \{0\}. \tag{7}$$

The problem considered in this paper is hence

$$\inf_{\alpha}\left(\frac{1}{N}\sum_{k=1}^{N} Z^{(k)}(M(\alpha)) + \lambda F(\hat{Z}(M(\alpha)))\right). \tag{8}$$

The choice $F(x_1,\dots,x_N) = \sqrt{\frac{1}{N-1}\sum_{n=1}^{N} x_n^2}$ leads, in particular, to problem (4).

We will construct a convex conic relaxation to this augmented problem and, for given $\lambda > 0$, provide a sufficient condition on the function F such that this relaxation is exact, i.e., yields the same optimal objective value as the original problem (8).

The aim of the next section is to show that there is a largest function F satisfying this condition, namely the function given by $\mathbb{R}^N \ni x \mapsto \lambda^{-1}\max_{n=1,\dots,N} x_n$. The augmented optimization problem corresponding to this particular choice of F is given by

$$\inf_{\alpha}\max_{n=1,\dots,N} Z^{(n)}(M(\alpha)) = \inf_{\alpha}\max_{t=0,\dots,T;n=1,\dots,N}\left(Z_t^{(n)} - \sum_{j=1}^{K}\alpha_j M_t^{j,(n)}\right) \tag{9}$$

and can also be cast as a linear program. In the subsequent section we analyze the convergence of (9) to Y^* as $N, K \to \infty$.

2 Penalties in the optimal stopping problem

In this section we first investigate under which condition problem (8) can be equivalently rewritten as a conic program. We reformulate this condition as an easily checkable condition on the penalty function F and give several examples of functions F satisfying this condition. We then show that among all such functions F there exists a maximal one, and this particular choice of F leads to the linear program (9) formulated above.

We shall identify the vector space $\mathbb{R}^N$ with its dual by means of the standard Euclidean scalar product. Then the orthogonal projection operator Π onto the subspace $L = \{x \in \mathbb{R}^N \mid \langle \mathbf{1}_N, x\rangle = 0\}$ equals its adjoint and is given by the matrix $\Pi = I - \frac{1}{N}\mathbf{1}_N\mathbf{1}_N^T$. Note that the vector $\hat{Z}$ defined in (6) then becomes the projection $\hat{Z} = \Pi Z$ of Z. The function of Z which is to minimize in (8) can be written as $g(Z) = \frac{1}{N}\langle \mathbf{1}_N, Z\rangle + \lambda F(\Pi Z)$. We may then rewrite (8) equivalently as

$$\inf_{\alpha} g(Z) \;:\; Z^{(n)} = \max_{t=0,\dots,T}\left(Z_t^{(n)} - \sum_{j=1}^{K}\alpha_j M_t^{j,(n)}\right),\; n = 1,\dots,N, \tag{10}$$

where $Z^{(n)}$, $n = 1,\dots,N$ are the components of the vector Z.

In order to write this problem as a conic program, we shall need the epigraph $\mathcal{K} = \{(a,c) \in \mathbb{R}\times\mathbb{R}^N \mid a \ge g(c)\} \subset \mathbb{R}^{N+1}$ of the function g. Note that g is continuous, convex, and homogeneous of degree 1, because F was supposed to be so. These properties imply that $\mathcal{K}$ is a closed convex cone.

The conic program we consider is given by

$$\inf_{\alpha,a,c} a \quad:\quad (a,c) \in \mathcal{K},\; c_n \ge Z_t^{(n)} + \sum_{j=1}^{K}\alpha_j M_t^{j,(n)},\; n = 1,\dots,N,\; t = 0,\dots,T, \tag{11}$$

where a is an additional scalar auxiliary variable and $c = (c_1,\dots,c_N)^T$ is a vector-valued auxiliary variable.

Problem (11) has a linear objective function and involves the conic constraint $(a,c) \in \mathcal{K}$ as well as $N\cdot(T+1)$ linear inequalities in quantities which are linear in the decision variables α, a, c. Therefore problem (11) can indeed be written as a conic program over the closed convex cone $\mathcal{K}\times\mathbb{R}_+^{N(T+1)}$. If $\mathcal{K}$ is polyhedral, it can be written as a linear program.

Lemma 2.1 *The optimal value of problem* (11) *is not greater than the optimal value of* (10).

Proof. Let $\alpha \in \mathbb{R}^K$ be arbitrary, and set $c_n = Z^{(n)}(M(\alpha))$, $n = 1,\dots,N$, as defined in (5), and $a = g(c)$. Then the pair (α, a, c) is a feasible point for problem (11). Moreover, (α, a, c) gives the same value to the cost function in (11) as α gives to the cost function in (10). This proves our claim. ∎

Lemma 2.1 shows that conic program (11) is a *relaxation* of the original problem (10). In order for this relaxation to be *exact*, i.e., for problems (10) and (11) to be equivalent, we need further assumptions on the penalty function F, or equivalently, on the objective function g.

Lemma 2.2 *Suppose the function $g : \mathbb{R}^N \to \mathbb{R}$ is such that for every $c \in \mathbb{R}^N$ there exists a subgradient $y \in \partial g(c)$ which is nonnegative element-wise. Then relaxation* (11) *is exact, i.e., it yields the same optimal value as the original problem* (10).

Proof. Assume the conditions of the lemma, and let (α, a, c) be an arbitrary feasible point for (11). Set $Z = (Z^{(1)}, \dots, Z^{(N)})$ with $Z^{(n)} = Z^{(n)}(M(\alpha))$, $n = 1, \dots, N$.

The inequality constraints in (11) imply that $c \geq Z$ element-wise, and the conic constraint implies that $a \geq g(c)$. Let $y \geq 0$ be a subgradient of g at Z. Then $a \geq g(c) \geq g(Z) + \langle y, c - Z\rangle \geq g(Z)$. However, $g(Z)$ is the value of the objective in (10) corresponding to α. It follows that the optimal value in (11) is not smaller than the optimal value in (10).

Combination with Lemma 2.1 completes the proof. ∎

The condition imposed on g, and hitherto on F, in Lemma 2.2 is in no way artificial. In Section B in the Appendix we show that in the absence of this condition the penalized problem (8) is in general non-convex, which is a computationally highly undesirable property.

As it stands, the condition in Lemma 2.2 is not easy to check for a given function F. We can, however, equivalently rewrite it as the following condition on the polar F_1^o of the 1-level subset $F_1 = \{c \in \mathbb{R}^N \mid F(c) \leq 1\}$ of F:

$$\lambda \Pi v \geq -\frac{1}{N}\mathbf{1}_N \qquad \forall\ v \in F_1^o. \tag{12}$$

The proof of this equivalence is relegated to Section C in the Appendix.

Hence the condition in Lemma 2.2 becomes $\Pi v \geq -\frac{1}{\lambda N}\mathbf{1}_N$ for all $v \in F_1^o$. In other words, the largest allowed value for λ is such that a shift of the projection $\Pi[F_1^o]$ by the vector $\frac{1}{\lambda N}\mathbf{1}_N$ still moves it into the nonnegative orthant.

We shall now consider different examples of penalty functions F. Set $\tilde{F}_1 = F_1 \cap L$.

1. $F(x) = \max_n x_n$. The set F_1 is given by $\{x \mid x_n \leq 1\}$. Its polar is given by $F_1^o = \{y \geq 0 \mid \langle y, \mathbf{1}^N\rangle \leq 1\}$. The polar $\tilde{F}_1^o = \Pi[F_1^o]$ is then spanned by the projections Πe_n of the unit vectors, namely the vector $(-\frac{1}{N}, \dots, -\frac{1}{N}, \frac{N-1}{N})$ and the vectors obtained by permutation of the elements from it. The condition on λ becomes $(-\frac{1}{N}, \dots, -\frac{1}{N}, \frac{N-1}{N})^T \geq -\frac{1}{\lambda N}\mathbf{1}_N$, which yields $\lambda \leq 1$.

2. $F(x) = \max_n |x_n|$. The set F_1 is the unit cube, its polar F_1^o the unit hyper-octahedron. The polar $\tilde{F}_1^o$ is then spanned by the projections $\pm\Pi e_n$, namely the vectors $\pm(-\frac{1}{N}, \dots, -\frac{1}{N}, \frac{N-1}{N})$ and the vectors obtained by permutation of the elements from it. The condition on λ becomes $\lambda \leq \frac{1}{N-1}$.

3. $F(x) = \sum_n |x_n|$. The set F_1 is the unit hyper-octahedron, its polar the unit cube. The set $\tilde{F}_1^o$ is then spanned by the projections of the vertices of the cube. These projections are given by $(2 - \frac{2n}{N}, \dots, 2 - \frac{2n}{N}, -\frac{2n}{N}, \dots, -\frac{2n}{N})$ and their permutations, where the first number appears n times and the second one $N - n$ times, $n = 0, \dots, N$. The condition on λ becomes $-2N + 2n \leq \frac{1}{\lambda}$ for all $n = 1, \dots, N$, and $2n \leq \frac{1}{\lambda}$ for all $n = 0, \dots, N-1$, yielding $\lambda \leq \frac{1}{2(N-1)}$.

4. $F(x) = ||x||_2$. Then both F_1 and F_1^o are the unit ball, and $\tilde{F}_1^o$ is the intersection of the unit ball with L. The condition on λ is determined by the unit length vector in L with the smallest element, which is $(\frac{1}{\sqrt{N(N-1)}}, \dots, \frac{1}{\sqrt{N(N-1)}}, -\sqrt{\frac{N-1}{N}})$. We hence get $-\sqrt{\frac{N-1}{N}} \geq -\frac{1}{\lambda N}$, yielding $\lambda \leq \frac{1}{\sqrt{N(N-1)}}$.

While problem (11) with F given by cases $1-3$ is a linear program, it is a conic quadratic program with one conic quadratic constraint when F is given by 4. In general, (11) is a linear program if and only if $\mathcal{K}$ is a polyhedral cone.

Finally we shall show that among the penalty functions F which allow a weighting value of $\lambda = 1$, the function $F(x) = \max_n x_n$ is maximal.

Lemma 2.3 *Suppose the function F satisfies condition* (12) *with $\lambda = 1$. Then for every $x \in \mathbb{R}^n$ such that $\langle \mathbf{1}_N, x\rangle = 0$ we have $F(x) \leq \max_n x_n$.*

Proof. Define the set $C = \{x \in \mathbb{R}^N \mid \max_k x_k \leq 1\}$ and the set $\tilde{C} = \{x \in C \mid \langle \mathbf{1}_N, x\rangle = 0\}$. From case 1 above we have that the polar $\tilde{C}^o$ is given by $\{y \geq -\frac{1}{N} \mid \langle y, \mathbf{1}^N\rangle = 0\}$. By assumption we have that $\frac{1}{N}\mathbf{1}_N^T + \Pi[F_1^o]$ is contained in the intersection of the subspace $\{y \mid \langle y, \mathbf{1}^N\rangle = 1\}$ with the nonnegative orthant, i.e., in the convex hull of the unit vectors. Hence we have the inclusion $\Pi[F_1^o] = \tilde{F}_1^o \subset \tilde{C}^o$. It follows that $\tilde{C} \subset \tilde{F}_1$. From this the claim of the lemma easily follows. ∎

In this sequel we henceforth concentrate on the case where $\lambda = 1$ and $F(x) = \max_{n=1,\ldots,N} x_n$. In this case the equivalent problems (10) and (11) simplify to the easily solvable linear program (9), whose properties will be the subject of the next sections.

3 Convergence analysis of the maximally penalized problem

In this section we analyse the convergence of the solutions of the sequence of optimisation problems (9) if both the dimension K of the subspace of martingales and the number of paths N tend to infinity. We establish bounds on the growth rate of N in dependence on K ensuring the convergence to the solution of the original problem (2). Consider for a sequence of basis martingales M^k, $k = 1, 2, \ldots$ with $M_0^k = 0$, the linear span

$$\Lambda_K := \left\{ M_\cdot(\alpha) = \sum_{k=1}^{K} \alpha_k M_\cdot^j : \ \alpha_1, \ldots, \alpha_K \in \mathbb{R} \right\}$$

for any $K \in \mathbb{N}_+$, and then study the convex optimization problem

$$\alpha^{K,N} := \underset{\alpha:\, M(\alpha)\in\Lambda_K}{\arg\inf} \max_{n=1,\ldots,N} \mathcal{Z}^{(n)}(M(\alpha)). \tag{13}$$

with $\mathcal{Z}^{(n)}(M(\alpha)) := \max_{t=0,\ldots,T} \left(Z_t^{(n)} - \sum_{k=1}^{K} \alpha_k M_t^{k,(n)} \right)$. The following result is proved in the Appendix.

Theorem 3.1 *Suppose that an almost surely optimal martingale M_t^* from (2) is square integrable and has a representation*

$$M_t^* := \sum_{k=1}^{\infty} \alpha_k^* M_t^k, \quad t \in [0, T]$$

in L^2 satisfying

$$\mathbb{E}\left[\left| \sum_{k=K+1}^{\infty} \alpha_k^* M_T^k \right|^p \right] \le \eta K^{-\rho}, \quad \forall\ K > K_0 \tag{14}$$

for some $p > 1$, $K_0 > 0$, $\eta > 0$, and $\rho > 0$. Let t_* be a random variable satisfying

$$Y^* = (Z_{t_*} - M_{t_*}^*), \qquad a.s.$$

Then it holds for any $c > 0$, $\varepsilon > 0$ that

$$\mathbb{P}\Big(\big\{ \|\alpha^{*,K} - \alpha^{K,N}\| \ge \varepsilon \big\} \cap \mathcal{E}_{c,K,N} \Big) \le A_p\, N K^{-\rho} / (c\varepsilon)^p, \tag{15}$$

where $\alpha^{*,K} := (\alpha_1^*, \ldots, \alpha_K^*)$, A_p is a constant depending on p, and

$$\mathcal{E}_{c,K,N} := \left\{ \max_{n=1,\ldots,N} \sum_{k=1}^{K} \delta_k M_{t_*^{(n)}}^{k,(n)} \ge c\, \|\delta\| \ \textit{for all } \delta \in \mathbb{R}^K \right\}.$$

One of the main issues is the estimation of the probability of the event $\mathcal{E}_{c,K,N}$. Clearly we have

$$\mathbb{P}\{\mathcal{E}_{c,K,N}\} \ge \mathbb{P}\left\{ \max_{n=1,\ldots,N} \min_{t=1,\ldots,T} \sum_{k=1}^{K} z_k M_t^{k,(n)} \ge c\, \|z\| \ \text{ for all } z \in \mathbb{R}^K \right\}. \tag{16}$$

Note that the right-hand side depends only on the choice of the basis martingales M^k. In order to estimate this probability, we introduce the random set

$$S = \left\{ z \in \mathsf{S}^{K-1} \ \middle|\ \min_{t=1,\ldots,T} \sum_{k=1}^{K} z_k M_t^k \ge c \right\},$$

where S^{K-1} is the unit sphere in $\mathbb{R}^K$. The random set S is the intersection of T random spherical caps $S_1, \ldots, S_T$ with opening angles $\varphi_1, \ldots, \varphi_T$ given by $\cos \frac{\varphi_t}{2} = \frac{c}{||M_t||}$, or equivalently via

$$\varphi_t = 2 \arccos \frac{c}{||M_t||}, \qquad t = 1, \ldots, T,$$

centered on $\frac{M_t}{||M_t||}$, respectively. Here $M_t = (M_t^1, \ldots, M_t^K)$ denotes the vector formed of the basis martingales at time instant t. The cap S_t is nonempty if and only if $||M_t|| \geq c$. Let now

$$S^{(n)} = \left\{ z \in \mathsf{S}^{K-1} \,\middle|\, \min_{t=1,\ldots,T} \sum_{k=1}^{K} z_k M_t^{k,(n)} \geq c \right\}, \qquad n = 1, \ldots, N,$$

be N independent copies of S corresponding to the N independent realisations $M_t^{k,(n)}$ of M_t^k. Then the event on the right-hand side of (16) happens if and only if $\bigcup_{n=1}^{N} S^{(n)} = \mathsf{S}^{K-1}$, i.e., if the random subsets $S^{(n)}$ cover the whole unit sphere.

In order to estimate the probability of this event, we shall employ an idea from [7]. Fix $\delta > 0$, and introduce random spherical caps $S_t^\delta \subset S_t$, centered on $\frac{M_t}{||M_t||}$ and with opening angle $\varphi_t - 2\delta$ if $\varphi_t \geq 2\delta$, and empty otherwise. In other words, S_t^δ is the subset of points of S_t which lie not closer than δ to the boundary of S_t. Define the random subset

$$S^\delta = \bigcap_{t=1,\ldots,T} S_t^\delta$$

of S^{K-1}. Since $S^\delta \subset S^{\delta'}$ for $\delta \geq \delta'$, for every $z \in \mathsf{S}^{K-1}$ the probability $\mathbb{P}\left\{z \in S^\delta\right\}$ is monotonously decreasing in δ. We have the following result.

Theorem 3.2 *Suppose that* $\min_{z \in \mathsf{S}^{K-1}} \mathbb{P}\left\{ z \in \bigcap_{t=1,\ldots,T} S_t^\delta \right\} \geq \pi_0 - \pi_1 \delta$ *for some* $\pi_0 \in (0,1)$ *and* $\pi_1 > 0$. *Then*

$$\mathbb{P}\left\{ \bigcup_{n=1}^{N} S^{(n)} = \mathsf{S}^{K-1} \right\} \geq 1 - 2\sqrt{\frac{2K}{\pi}} (1 - \pi_0)^{N-K+1} \left(\frac{\pi \pi_1 N e}{2(K-1)} \right)^{K-1}.$$

A proof of this theorem is given in the Appendix.

Remark 3.3 *The asymptotics of the bound for $N \to \infty$ is of the same order as in the exact results for the isotropic sphere covering problem by random sets obtained in [11] and [12].*

Combining with (16), we then get the following estimate for the probability of $\mathcal{E}_{c,K,N}$.

Corollary 3.4 *Under the assumptions of Theorem 3.2, we have the bound*

$$\mathbb{P}\{\mathcal{E}_{c,K,N}\} \geq 1 - 2\sqrt{\frac{2K}{\pi}} (1 - \pi_0)^{N-K+1} \left(\frac{\pi \pi_1 N e}{2(K-1)} \right)^{K-1}. \tag{17}$$

Remark 3.5 *The bound* (17) *depends on the parameter c via the coefficients π_0, π_1. If the basis martingales are chosen properly (see the next section for a discussion on this), then π_0 is bounded away from zero and π_1 grows linearly with growing dimension K. Hence the r.h.s. of* (17) *tends to* 1, *if N is growing not slower than $K^{1+\alpha}$ for some $\alpha > 0$. In this case we get that the probability for $\mathcal{E}_{c,K,N}$ not to occur is exponentially small in N.*

We stress that in the case when the event $\mathcal{E}_{c,K,N}$ does not occur, the optimization problem (3) will be unbounded. In other words, the complement of $\mathcal{E}_{c,K,N}$ will entail the failure of the solver to converge to a solution. In this case the Monte-Carlo simulation has to be repeated. In what follows, we shall assume that the event $\mathcal{E}_{c,K,N}$ has occurred. The results below are hence to be understood as conditioned on $\mathcal{E}_{c,K,N}$, in particular, from (15) it follows that,

$$\mathbb{P}\left\{ \|\alpha^{*,K} - \alpha^{K,N}\| \geq \varepsilon \right\} \leq A_p \, N K^{-\rho} / (c\varepsilon)^p, \tag{18}$$

Assume we have simulated a new set of trajectories (independent of those used to construct $\alpha^{K,N}$) $(Z_\cdot^{(n)}, M_\cdot^{(n)})$, $n = 1, \ldots, N_1$. Consider the estimate

$$Y_{K,N,N_1} := \frac{1}{N_1} \sum_{n=1}^{N_1} \max_{t=0,\ldots,T} \left(Z_t^{(n)} - \sum_{k=1}^{K} \alpha_k^{K,N} M_t^{k,(n)} \right).$$

Using the Doob inequality, we get

$$\mathbb{E}\Big[|Y_{K,N,N_1} - Y^*|^2\Big] \quad \leq \quad \frac{8}{N_1} \left[\mathbb{E} \left(\left| \sum_{k=1}^{K} \left(\alpha_k^{K,N} - \alpha_k^* \right) M_T^k \right|^2 + \left| \sum_{k=K+1}^{\infty} \alpha_k^* M_T^k \right|^2 \right) \right].$$

Hence the following proposition holds.

Proposition 3.6 *Suppose that M_t^k is a sequence of continuous square integrable martingales with $\mathbb{E}[M_T^i M_T^k] = \delta_{ik}$ for all $i, k \in \{1, \ldots, K\}$, and that (14), (18) are fulfilled with $p > 2$ and $\rho > 1$, then*

$$\begin{aligned} \mathbb{E}\Big[|Y_{K,N,N_1} - Y^*|^2\Big] \quad &\leq \quad \frac{8}{N_1} \left[\mathbb{E} \left\| \alpha^{K,N} - \alpha^{*,K} \right\|^2 + \sum_{k=K+1}^{\infty} (\alpha_k^*)^2 \right] \\ &\leq \quad \frac{8}{N_1} \left[\frac{p(A_p N K^{-\rho})^{2/p}}{(p-2)c^2} + (\eta K^{-\rho})^{2/p} \right]. \end{aligned}$$

Thus if N grows slower than K^ρ, we have a strong variance reduction effect by using the maximal penalty in (8) for $N, K \to \infty$.

3.1 Discussion of conditions

Suppose that $Z_t = G_t(X_t)$, where $G_t : \mathbb{R}^d \to \mathbb{R}$ is a Hölder function on $[0, T] \times \mathbb{R}$ and X_t is a d-dimensional Markov process solving the following system of SDE's:

$$dX_t \quad = \quad \mu(t, X_t)\, dt + \sigma(t, X_t)\, dW_t, \quad X_0 = x. \tag{19}$$

The coefficient functions $\mu : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : [0, T] \times \mathbb{R}^d \to \mathbb{R}^{d \times m}$ are supposed to be Lipschitz in space and 1/2-Hölder continuous in time, with m denoting the dimension of the Brownian motion $W = (W^1, \ldots, W^m)^\top$. It is well-known that under the assumption that a martingale M_t is square integrable and is adapted to the filtration generated by W_t, there is a square integrable (row vector valued) process $H_t = (H_t^1, \ldots, H_t^m)$ satisfying

$$M_t = \int_0^t H_s dW_s. \tag{20}$$

It is not hard to see that in the Markovian setting and under some rather weak assumptions, the optimal (almost sure) Doob martingale M^* can be represented as

$$M_t^* = \int_0^t u(s, X_s) dW_s. \tag{21}$$

for some vector function $u(s, x) = (u_1(s, x), \ldots, u_m(s, x))$ satisfying

$$\int_0^T \mathbb{E}[|u(s, X_s)|^2]\, ds < \infty.$$

In such a situation, we can consider a class of adapted square-integrable martingales which can be "parameterized" by the set $L_{2,P}([0, T] \times \mathbb{R}^d)$ of square-integrable m-dimensional vector functions ψ on $[0, T] \times \mathbb{R}^d$ that satisfy $\|\psi\|_{2,P}^2 := \int_0^T \mathbb{E}[|\psi(s, X_s)|^2]\, ds < \infty$. Choose a family of finite-dimensional linear models of functions, called sieves, with good approximation properties with respect to u. We can consider, for example, linear sieves of the form:

$$\Psi_K := \{\beta_1 \phi_1 + \ldots + \beta_K \phi_K : \beta_1, \ldots, \beta_K \in \mathbb{R}\},$$

where $\phi_1, \ldots, \phi_K$ are some given vector basis functions from $L_{2,P}([0,T] \times \mathbb{R}^d)$. Then the basis martingales in (9) can be defined via $M_t^k = M_t(\phi_k) = \int_0^t \phi_k(s, X_s) dW_s,\ k = 1, \ldots, K$. In this case the condition (14) can transformed (by using Burkholder-Davis-Gundy's inequalities) to the following one

$$\mathbb{E}\left[\left|\int_0^T (u(s, X_s) - \alpha_1^* \phi_1(s, X_s) - \ldots - \alpha_K^* \phi_K(s, X_s))^2 \, ds\right|^{p/2}\right] \leq \eta K^{-\rho} \tag{22}$$

which measures the quality of "best projection" of u on the linear subspace Ψ_K.

We now provide an example of martingale basis allowing for an explicit estimates of the quantities π_0, π_1. Let us assume that the process X is Markovian and is adapted to the filtration generated by the Brownian motion $W_s,\ s \in [0,T]$. Let us take basis martingales in the form

$$M_t^l = \sum_{j=1}^{T} \int_0^t 1_{\{t_{j-1} \leq s \leq t_j\}} \phi_l(s, X_{t_{j-1}}) \, dW_s \tag{23}$$

for some increasing set of times $0 = t_0 < t_1 < \ldots < t_T$. The form (23) can be viewed as an approximation for the integral $\int_0^t \phi_l(s, X_s) dW_s$. Note that the integrals

$$\Delta M_j^l := \int_{t_{j-1}}^{t_j} \phi_l(s, X_{t_{j-1}}) \, dW_s$$

are zero mean Gaussian conditional on $X_{t_{j-1}}$, with conditional covariance matrix

$$\Gamma_{ll'}^j(X_{t_{j-1}}) := \int_{t_{j-1}}^{t_j} \phi_l(s, X_{t_{j-1}}) \phi_{l'}(s, X_{t_{j-1}}) \, ds.$$

We have

$$M_{t_i}^l = \sum_{r=1}^{i} \Delta M_r^l \quad \text{and} \quad \|M_{t_i}\| \leq \sum_{r=1}^{i} \|\Delta M_r\|.$$

Hence the inequality

$$\sum_{r=1}^{i} \left(\sum_{l=1}^{K} z_l \Delta M_r^l - \delta \|\Delta M_r\| \right) \geq c \quad \text{for} \quad i = 1, ..., T,$$

implies

$$\sum_{l=1}^{K} z_l M_{t_i}^l \geq c \ + \delta \|M_{t_i}\| \quad \text{for} \quad i = 1, ..., T,$$

which in turn implies

$$\sum_{l=1}^{K} z_l M_t^l \geq c \cos \delta + \delta \|M_{t_i}\| \geq c \cos \delta + \sqrt{\|M_{t_i}\|^2 - c^2} \sin \delta = ||M_t|| \cos(\frac{\varphi_t}{2} - \delta) \quad \text{for} \quad i = 1, ..., T$$

and therefore $z \in S^\delta$. Hence for any $c > 0$,

$$\begin{aligned}
&\mathbb{P}\left(z \in S^\delta\right) \geq \mathbb{P}\left(\sum_{r=1}^{i} \left(\sum_{l=1}^{K} z_l \Delta M_r^l - \delta \|\Delta M_r\| \right) \geq c \quad \text{for} \quad i = 1, ..., T \right) \geq \\
&\mathbb{P}\left(\left\{ \sum_{l=1}^{K} z_l \Delta M_1^l - \delta \|\Delta M_1\| \geq c \right\} \bigcap_{r=2}^{T} \left\{ \sum_{l=1}^{K} z_l \Delta M_r^l - \delta \|\Delta M_r\| \geq 0 \right\} \right) = \\
&\mathbb{E}\, \mathbb{P}^{X_{t_{T-1}}} \left(\left\{ \sum_{l=1}^{K} z_l \Delta M_1^l - \delta \|\Delta M_1\| \geq c \right\} \bigcap_{r=2}^{T-1} \left\{ \sum_{l=1}^{K} z_l \Delta M_r^l - \delta \|\Delta M_r\| \geq 0 \right\} \right) \\
&\times \mathbb{P}^{X_{t_{T-1}}} \left(\sum_{l=1}^{K} z_l \Delta M_T^l - \delta \|\Delta M_T\| \geq 0 \right).
\end{aligned}$$

Now note that the (regular) conditional probability $\mathbb{P}^{X_{t_{T-1}}}\left(\sum_{l=1}^K z_l \Delta M_T^l \geq 0\right) = \frac{1}{2}$ almost surely, since $\sum_{l=1}^K z_l \Delta M_T^l$ are Gaussian. Let $\Gamma^T(y)$ be a covariance matrix of the vector $\Delta M_T = (\Delta M_T^1, \dots, \Delta M_T^K)$ given $X_{t_{T-1}} = y$. Denote by $\lambda_{\min}(A), \lambda_{\max}(A)$ the minimal and maximal eigenvalue of a matrix A, respectively. If the basis functions are such that the condition number

$$\lambda_{\max}(\Gamma^T(y))/\lambda_{\min}(\Gamma^T(y))$$

is bounded in y, there exists $\delta_0 > 0$ such that for all $\delta < \delta_0$ we have

$$\mathbb{P}^y\left(\sum_{l=1}^K z_l \Delta M_T^l - \delta \|\Delta M_T\| \geq 0\right) \geq \frac{1}{4}$$

uniformly in y. Iterating and arguing analogously backwards in time, we get for any $c > 0$ and small enough $\delta > 0$

$$\mathbb{P}\left(\bigcap_{i=1}^T \left\{\sum_{r=1}^i \left(\sum_{l=1}^K z_l \Delta M_r^l - \delta \|\Delta M_r\|\right) \geq c\right\}\right) \geq \frac{1}{4^{T-1}} \mathbb{P}\left(\sum_{l=1}^K z_l \Delta M_1^l - \delta \|\Delta M_1\| \geq c\right). \tag{24}$$

The probability density of the Gaussian vector $\Delta M_1 = (\Delta M_1^1, \dots, \Delta M_1^K)$ can be bounded from below by

$$p(x) = \frac{1}{(2\pi)^{K/2}\sqrt{\det(\Gamma^1(x_0))}} e^{-||x||^2/(2\lambda_{\min}(\Gamma^1(x_0)))},$$

where $\Gamma^1(x_0)$ is the covariance matrix of ΔM_1 given $X_0 = x_0$. Note that for $y_1, y, \gamma > 0$ and $\delta \in [0,1)$ the inequality $y_1 - \delta\sqrt{y_1^2 + y} \geq \gamma$ is equivalent to the inequality $y_1 - \frac{\gamma}{1-\delta^2} \geq \frac{\delta}{\sqrt{1-\delta^2}}\sqrt{\frac{\gamma^2}{1-\delta^2} + y}$. The function on the right-hand side of the latter is concave in y and hence majorized by its linear Taylor polynomial at $y = 0$. It follows that this inequality is implied by the inequality $y_1 \geq \frac{\gamma}{1-\delta} + \frac{\delta}{2\gamma} y$. Setting $y_1 = \frac{\sum_{l=1}^K z_l \Delta M_1^l}{\sqrt{\lambda_{\min}(\Gamma^1(x_0))}}$, $\gamma = \frac{c}{\sqrt{\lambda_{\min}(\Gamma^1(x_0))}}$, and $y = \frac{||\Delta M_1||^2 - y_1^2}{\lambda_{\min}(\Gamma^1(x_0))}$, we then have

$$\begin{aligned}
&\mathbb{P}\left(\sum_{l=1}^K z_l \Delta M_1^l - \delta \|\Delta M_1\| \geq c\right) \geq \frac{\lambda_{\min}(\Gamma^1(x_0))^{K/2}}{\sqrt{\det(\Gamma^1(x_0))}} \times \\
&\quad \mathbb{P}\left(\xi \geq \frac{c}{\sqrt{\lambda_{\min}(\Gamma^1(x_0))}(1-\delta)} + \frac{\sqrt{\lambda_{\min}(\Gamma^1(x_0))}\delta}{2c}\eta\right) \\
&= \frac{\lambda_{\min}(\Gamma^1(x_0))^{K/2}}{\sqrt{\det(\Gamma^1(x_0))}} \times \\
&\quad \left(\mathcal{N}\left(-\frac{c}{\sqrt{\lambda_{\min}(\Gamma^1(x_0))}}\right) - \delta \frac{\exp(-\frac{c^2}{2\lambda_{\min}(\Gamma^1(x_0))})}{\sqrt{2\pi}} \left(\frac{c}{\sqrt{\lambda_{\min}(\Gamma^1(x_0))}} + \frac{(K-1)\sqrt{\lambda_{\min}(\Gamma^1(x_0))}}{2c}\right) + O(\delta^2)\right)
\end{aligned}$$

where ξ and η are independent variables with distributions $\mathcal{N}(0,1)$ and χ^2_{K-1}, respectively. For small enough δ we then may choose

$$\begin{aligned}
\pi_0 &= \frac{\lambda_{\min}(\Gamma^1(x_0))^{K/2}}{4^{T-1}\sqrt{\det(\Gamma^1(x_0))}} \mathcal{N}\left(-\frac{c}{\sqrt{\lambda_{\min}(\Gamma^1(x_0))}}\right), \\
\pi_1 &= \frac{2\lambda_{\min}(\Gamma^1(x_0))^{K/2}}{4^{T-1}\sqrt{\det(\Gamma^1(x_0))}} \frac{\exp(-\frac{c^2}{2\lambda_{\min}(\Gamma^1(x_0))})}{\sqrt{2\pi}} \left(\frac{c}{\sqrt{\lambda_{\min}(\Gamma^1(x_0))}} + \frac{(K-1)\sqrt{\lambda_{\min}(\Gamma^1(x_0))}}{2c}\right).
\end{aligned}$$

If the basis functions $\phi_l(s, X)$ are chosen such that the ratio $\frac{\lambda_{\min}(\Gamma^1(x_0))^{K/2}}{\sqrt{\det(\Gamma^1(x_0))}}$ as well as the minimal eigenvalue $\lambda_{\min}(\Gamma^1(x_0))$ stay bounded from zero with growing K, then π_0 stays bounded away from zero and π_1 grows linearly with K (cf. Remark 3.5).

4 Simulation example

Consider the example given in [10, Section 8]. We have $T = 2$, $Z_0 = 0$, $Z_2 = 1$, and $Z_1 = \xi$ is a random variable which is uniformly distributed on the interval $[0, 2]$. The optimal stopping time τ^* is given by

$$\tau^* = \begin{cases} 1, & \xi \geq 1, \\ 2, & \xi < 1. \end{cases}$$

and the optimal value of problems (1) and (2) by $Y^* = \mathbb{E}\max(\xi, 1) = \frac{5}{4}$.

The martingale M in problem (2) can be assumed of the general form $M_0 = 0$, $M_1 = M_2 = h(\xi)$, where $h : [0, 2] \to \mathbb{R}$ is a function satisfying $\mathbb{E}_\xi h(\xi) = 0$. It follows that

$$\max_{t=0,1,2}(Z_t - M_t) = \max(h(\xi), \xi, 1) - h(\xi)$$

and hence $\mathbb{E}\max_{t=0,1,2}(Z_t - M_t) = \mathbb{E}\max(h(\xi), \xi, 1) \geq \mathbb{E}\max(\xi, 1) = \frac{5}{4}$. Any martingale given by a function h satisfying $h(\xi) \leq \max(\xi, 1)$ almost surely is hence an optimal solution for problem (2). Such an optimal solution then yields $\max_{t=0,1,2}(Z_t - M_t) = \max(\xi, 1) - h(\xi)$ almost surely.

However, not every such martingale is surely optimal in the sense defined in [10]. A surely optimal martingale is defined by a function $h(\xi)$ satisfying $\max_{t=0,1,2}(Z_t - M_t) = \max(\xi, 1) - h(\xi) = \frac{5}{4}$ almost surely, which gives $h(\xi) = \max(\xi, 1) - \frac{5}{4}$ almost surely. Define the function $h^*(\xi) = \max(\xi, 1) - \frac{5}{4}$ and denote the martingale defined by this function by M^*.

We shall now try to find the function h^* by Monte-Carlo methods. We search over a finite-dimensional subspace L_K of functions $h(\xi)$, which will depend on an even integer parameter K. Namely, L_K consists of functions of the form

$$h(\xi) = \sum_{k=1}^{K/2} c_k \cos(k\xi\pi) + s_k \sin(k\xi\pi),$$

where c_k, s_k, $k = 1, \dots, K/2$ are real coefficients. The dimension of the subspace L_K equals K. Note that $h^*(\xi)$ is not contained in L_K for any K. We rather have

$$h^*(\xi) = \sum_{k=1}^{\infty} c_k^* \cos(k\xi\pi) + s_k^* \sin(k\xi\pi)$$

with $s_k^* = -\frac{1}{k\pi}$, $c_k^* = 0$ for even k, and $c_k^* = \frac{2}{k^2\pi^2}$ for odd k.

Note further that for fixed K we have $\mathbb{P}(\mathcal{E}_{c,K,N}) > 0$ for $c > 0$ small enough and N large enough, and that for fixed $c > 0$ small enough we have $\lim_{N\to\infty} \mathbb{P}(\mathcal{E}_{c,K,N}) = 1$.

We solve the two optimization problems (3) and (9) for $K = 2, 4, \dots, 20$ and with the number of samples N being 50,100, and 200, respectively. For the martingale $\hat{M}$ which gives the optimal solution of problems (3) and (9), respectively, we compute the expected value and the variance of the expression $\max_{t=0,1,2}(Z_t - \hat{M}_t)$. Note that both this expected value and the variance are random variables, because they depend on the random realization of the paths $(Z_t^{(n)}, M_t^{j,(n)})$. For each pair (K, N), we perform 100 independent runs. On Fig. 1 we plot the distribution function of the logarithms of these quantities.

It is clear from Fig. 1 that the variance of $\max_{t=0,1,2}(Z_t - \hat{M}_t)$ drops dramatically if problem (9) is solved in place of problem (3). However, it can also be seen that the expectation $\mathbb{E}\max_{t=0,1,2}(Z_t - \hat{M}_t)$ improves. The conclusion is that the presence of the penalty term in (9) not only decreases the variance, but also leads to a robustification against the uncertainty introduced by the sampling procedure.

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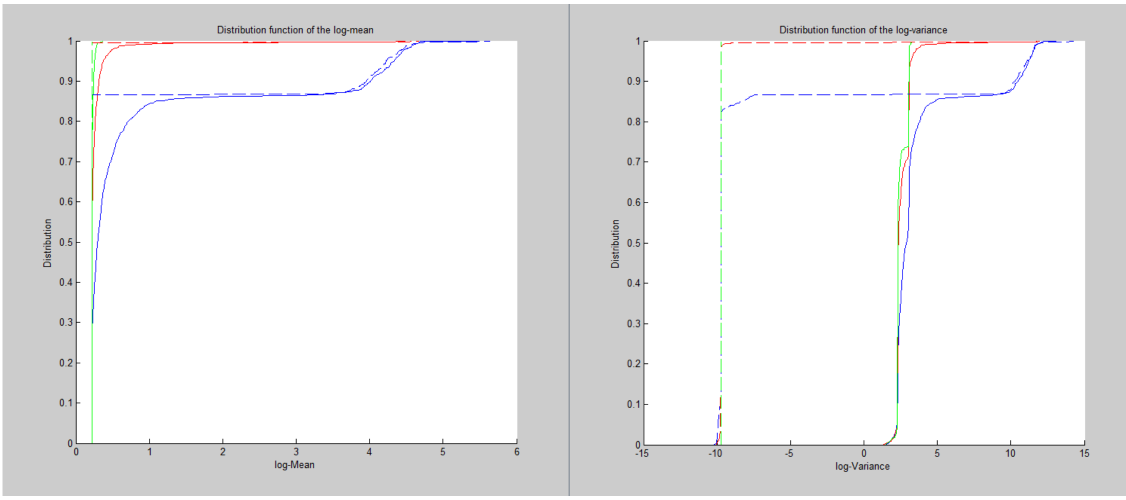

Figure 1: Distribution of log-mean and log-variance of $\max_{t=0,1,2}(Z_t - \hat{M}_t)$ for an 11-dimensional search space. The solid lines correspond to the solution of problem (3), the dashed lines correspond to (9). The color is determined by the number of samples N. Here blue, red and green denote the values 50,100, and 200, respectively.

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A Convex analysis

In this section we introduce some notions from convex analysis and conic optimization. The dual of the real vector space $\mathbb{R}^n$ will be denoted by $\mathbb{R}_n$, and the scalar product between $y \in \mathbb{R}_n$ and $x \in \mathbb{R}^n$ will be denoted by $\langle y, x \rangle$. Let $\mathbf{1}^n, \mathbf{1}_n = (1, \dots, 1)^T$ be the all-ones vector in $\mathbb{R}^n$ and in the dual space $\mathbb{R}_n$, respectively.

A.1 Conic programs

Conic programming is a generalization of linear programming where the ordinary inequality constraints are replaced by a more general notion of inequality defined by a convex cone.

A *conic program* over a closed convex cone $\mathcal{K} \subset \mathbb{R}^n$ is an optimization problem of the form

$$\inf_{x \in \mathcal{K}} \langle c, x \rangle \ : \ Ax = b, \tag{25}$$

where $c \in \mathbb{R}_n$ is a vector defining the linear cost function of the problem, A is an $m \times n$ matrix, and $b \in \mathbb{R}^m$. Here A, b define the linear constraints of the problem.

The availability of algorithms for solving a conic program depends on the nature of the cone $\mathcal{K}$. For example, if $\mathcal{K}$ is the positive orthant $\mathbb{R}^n_+$, then (25) is a linear program which can be easily solved. Efficient solution algorithms are also available if $\mathcal{K}$ is a second order cone, in which case the program is called *conic quadratic program*.

A.2 Exposed and extreme points

In this subsection we introduce the notions of exposed and extreme points of a closed convex set and consider the relations between them.

Definition A.1 *[9, p.162] Let $C \subset \mathbb{R}^n$ be a closed convex set. A point $x \in C$ is called* extreme point *if there does not exist an open line segment $L \subset C$ such that $x \in L$.*

Lemma A.2 *[9, Corollary 18.5.1] A closed bounded convex set is the convex hull of its extreme points.*

Definition A.3 *[9, pp.162–163] Let $C \subset \mathbb{R}^n$ be a closed convex set. A point $x \in C$ is called an* exposed point *if there exists a supporting affine hyperplane $H \subset \mathbb{R}^n$ to C such that $H \cap C = \{x\}$.*

Lemma A.4 *[9, Theorem 18.6] Let $C \subset \mathbb{R}^n$ be a closed convex set. Then the set of exposed points of C is dense in the set of extreme points of C.*

Corollary A.5 *Let $C \subset \mathbb{R}^n$ be a bounded closed convex set and E its set of exposed points. Then C is the convex hull of the closure of E.*

Proof. The corollary follows immediately from the two lemmas above. ∎

A.3 Convex functions and subgradients

In this subsection we introduce the notion of a subgradient.

Definition A.6 *[9, pp.214–215] Let $D \subset \mathbb{R}^n$ be a convex set and $f : D \to \mathbb{R}$ a convex function. A* subgradient *of f at $x \in D$ is a dual vector $y \in \mathbb{R}_n$ such that $f(z) \geq f(x) + \langle y, z - x \rangle$ for all $z \in D$. The set of all subgradients at $x \in D$ is called* subdifferential *at $x \in D$ and denoted by $\partial f(x)$.*

The subdifferential is a closed convex set [9, p.215]. If f is differentiable at x, then the gradient $f'(x)$ is the only subgradient [9, p.216].

Lemma A.7 *[5, p.261] Let $D \subset \mathbb{R}^n$ be a convex domain, $F : D \to \mathbb{R}$ a convex function, and $h : D \to \mathbb{R}$ a convex C^1 function. For $x \in D$ and $\lambda \geq 0$ we then have $\partial(\lambda F + h)(x) = \lambda \partial F(x) + h'(x)$.*

Lemma A.8 *[9, Theorem 23.9] Let $D \subset \mathbb{R}^m$ be a convex domain, and $H : \mathbb{R}^m \to \mathbb{R}^n$ an affine map given by $H(x) = A(x) + b$, with A, b the linear part of H and the shift, respectively. Let further $F : H[D] \to \mathbb{R}$ be a convex function. Then for $x \in D$ we have $\partial(F \circ H)(x) = A^*[\partial F(Ax + b)]$, where A^* is the adjoint map of A.*

A.4 Convex sets and polars

In this subsection we introduce the notion of a polar and study its properties.

Definition A.9 *[9, p.125] Let $C \subset \mathbb{R}^n$ be a closed convex set containing the origin of $\mathbb{R}^n$. The set $C^o = \{y \in \mathbb{R}_n \mid \langle y, x \rangle \leq 1 \ \forall \ x \in C\}$ is called the polar of the set C.*

The set C^o is also closed, convex, and contains the origin [9, p.125]. It is bounded if and only if the origin of $\mathbb{R}^n$ is contained in the interior of C [9, Corollary 14.5.1]. Moreover, the polar of C^o is again C [9, Theorem 14.5]. If C, C' are two closed convex sets containing the origin and satisfying $C \subset C'$, then their polars satisfy $(C')^o \subset C^o$ [9, p.125].

Let now $L \subset \mathbb{R}^n$ be a linear subspace, and let $L^\perp \subset \mathbb{R}_n$ be the orthogonal subspace. Then the dual space L^* of L can be identified with the quotient $\mathbb{R}_n/L^\perp$. Let $\Pi : \mathbb{R}_n \to \mathbb{R}_n/L^\perp$ be the corresponding projection. Let $C \subset \mathbb{R}^n$ be a closed convex set containing the origin. Then the intersection $C \cap L \subset L$ is a closed convex set in L, containing the origin of L. The next result gives a convenient description of the polar of $C \cap L$ as a subset of L in terms of the polar C^o.

Lemma A.10 *Assume the notations of the previous paragraph. Then the polar $(C \cap L)^o$ is given by the closure of the projection $\Pi[C^o]$.*

Proof. Let $y \in C^o$ be an arbitrary point in the polar of C and $\Pi(y) = y + L^\perp$ its projection on the quotient $\mathbb{R}_n/L^\perp$. Then we have $\langle y, x \rangle \leq 1$ for all $x \in C$. In particular, we have $\langle y, x \rangle \leq 1$ for all $x \in C \cap L$. Hence $\Pi(y) \in (C \cap L)^o$. It follows that $\Pi[C^o] \subset (C \cap L)^o$. However, $(C \cap L)^o$ is closed, and hence the closure of $\Pi[C^o]$ is also a subset of $(C \cap L)^o$.

Let now $y \in \mathbb{R}_n$ such that $\Pi(y) = y + L^\perp$ is not contained in the closure of $\Pi[C^o]$. Then there exists a hyperplane $H \subset \mathbb{R}_n/L^\perp$ which separates $\Pi(y)$ from $\Pi[C^o]$, and such that $\Pi(y) \notin H$. Then the hyperplane $\Pi^{-1}[H] \subset \mathbb{R}_n$ separates y from C^o, and $y \notin \Pi^{-1}[H]$. Note also that $y \neq 0$. It follows that there exists a vector $z \in L$ such that $\langle y, z \rangle > 1$ and $\langle w, z \rangle \leq 1$ for all $w \in C^o$. Hence $z \in C \cap L$. But then $y + L^\perp \notin (C \cap L)^o$. This proves the converse inclusion and completes the proof. ■

B Convexity of the penalized problem

The following consideration shows that if the condition in Lemma 2.2 is not satisfied, then problem (8) may not be convex at all.

Let $\alpha^* \in \mathbb{R}^K$ be an arbitrary vector, and define the vector c^* by $c_k^* = Z^{(k)}(M(\alpha^*))$. Assume that g is differentiable and g' is not nonnegative at c^*, i.e., there exists an index l such that $\nabla_l g(c^*) < 0$. Suppose further that the maximum $\max_t(Z_t^{(l)} - \sum_{r=1}^K \alpha_r^* M_t^{r,(l)})$ is attained at more than one index t, e.g., at the indices i, j, and suppose that there exists a direction $\delta \in \mathbb{R}^K$ such that $\sum_{r=1}^K \delta_r M_t^{r,(k)}$ is zero for pairs (k, t) other than (l, i) and (l, j) such that $Z_t^{(k)} - \sum_{r=1}^K \alpha_r^* M_t^{r,(k)} = Z^{(k)}(M(\alpha^*))$, and $\sum_{r=1}^K \delta_r M_i^{r,(l)} \neq \sum_{r=1}^K \delta_r M_j^{r,(l)}$. Then problem (8) is not convex.

Indeed, for real ε define $\alpha_\varepsilon = \alpha^* + \varepsilon\delta$ and a vector $c(\varepsilon)$ by $c_k(\varepsilon) = Z^{(k)}(M(\alpha_\varepsilon))$. Let without loss of generality $\sum_{r=1}^K \delta_r M_i^{r,(l)} < \sum_{r=1}^K \delta_r M_j^{r,(l)}$. For $\varepsilon > 0$ small enough we then have $c_k(\pm\varepsilon) = c_k^*$ for all $k \neq l$, $c_l(\varepsilon) = c_l^* - \varepsilon \sum_{r=1}^K \delta_r M_i^{r,(l)}$, and $c_l(-\varepsilon) = c_l^* + \varepsilon \sum_{r=1}^K \delta_r M_j^{r,(l)}$. The cost function of problem (8) is given by $g(c(\varepsilon))$ for $\alpha = \alpha_\varepsilon$. We have $\frac{d}{d\varepsilon} \frac{g(c(\varepsilon)) + g(c(-\varepsilon))}{2}|_{\varepsilon=0} = \nabla_l g(c^*) \frac{\sum_{r=1}^K \delta_r M_j^{r,(l)} - \sum_{r=1}^K \delta_r M_i^{r,(l)}}{2} < 0$, and the cost function is not convex.

If K is not too small, then the above conditions are in general verified for some value of α. Hence it is reasonable to demand the condition given in Lemma 2.2.

C Justification of condition (12)

We need to prove the equivalence of the following two conditions.

(i) For every $x \in \mathbb{R}^n$ there exists a subgradient $y \in \partial g(x)$ whose elements are all nonnegative.

(ii) The set $\frac{1}{n}\mathbf{1}_n + \lambda \Pi^*[F_1^o]$ is contained in the nonnegative orthant.

We shall prove the two directions of the equivalence relation separately.

(ii) ⇒ (i). First we consider condition (i) for the case $\Pi x = 0$. We shall show that the dual vector $y = \frac{1}{n}\mathbf{1}_n \geq 0$ is a subgradient of g at x. We have $F(\Pi x) = 0$ and hence $g(x) = \frac{1}{n}\langle \mathbf{1}_n, x\rangle$. For every $z \in \mathbb{R}^n$ we have $F(\Pi z) \geq 0$ by assumption (7) on F and hence $g(z) \geq \frac{1}{n}\langle \mathbf{1}_n, z\rangle = g(x) + \langle \frac{1}{n}\mathbf{1}_n, z - x\rangle$. This proves $\frac{1}{n}\mathbf{1}_n \in \partial g(x)$.

Now let $x \in \mathbb{R}^n$ be such that $L \ni \Pi x \neq 0$. Then $F(\Pi x) > 0$, and we can define $\tilde{x} = \frac{\Pi x}{F(\Pi x)} \in L$. By definition, we have $F(\tilde{x}) = 1$. It follows that $\tilde{x}$ is on the boundary of the set F_1. Hence there exists an element $w \in F_1^o$ such that $\langle \tilde{w}, x\rangle = 1$, and hence $\langle w, \Pi x\rangle = F(\Pi x)$. By assumption we have $y = \frac{1}{n}\mathbf{1}_n + \lambda \Pi^* w \geq 0$. We shall show that $y \in \partial g(x)$.

Indeed, let $z \in \mathbb{R}^n$. Then we have $g(z) - g(x) - \langle y, z - x\rangle = \lambda(F(\Pi z) - F(\Pi x) - \langle \Pi^* w, z - x\rangle) = \lambda(F(\Pi z) - \langle w, \Pi z\rangle)$. If $\Pi z = 0$, then $F(\Pi z) - \langle w, \Pi z\rangle = 0$. Let us assume that $\Pi z \neq 0$. Then $F(\Pi z) > 0$, and we may define $\tilde{z} = \frac{\Pi z}{F(\Pi z)}$. We get $F(\tilde{z}) = 1$, and $\tilde{z} \in F_1$. It follows that $\langle w, \tilde{z}\rangle \leq 1$, because $w \in F_1^o$. But then $\langle w, \Pi z\rangle \leq F(\Pi z)$, which proves $g(z) - g(x) - \langle y, z - x\rangle \geq 0$. Hence $y \in \partial g(x)$, which yields (i).

(i) ⇒ (ii). First we shall prove an auxiliary result.

Lemma C.1 *Let $\tilde{F}_1 = F_1 \cap L$. Then the polar $\tilde{F}_1^o$ of $\tilde{F}_1$ in L is given by the projection $\Pi^*[F_1^o]$.*
Proof. By Lemma A.10 the polar $\tilde{F}_1^o$ is given by the closure of $\Pi^*[F_1^o]$. It remains to show that $\Pi^*[F_1^o]$ is closed. We have $F(0) = 0$, and hence F_1 contains a ball around the origin with positive radius r. It follows that the polar F_1^o is contained in a ball with radius r^{-1}, and is hence compact. But projections of compact sets are compact, and in particular closed. ■

We now come to the implication (i) ⇒ (ii). Assume (i) and consider first an exposed point $w \in \tilde{F}_1^o$. Our aim is to show that $\frac{1}{n}\mathbf{1}_n + \lambda w \geq 0$. By definition, there exists $x \in \tilde{F}_1$ such that $\langle w, x\rangle = 1$, $\langle v, x\rangle \leq 1$ for all $v \in \tilde{F}_1^o$, and $\{v \in \tilde{F}_1^o \,|\, \langle v, x\rangle = 1\} = \{w\}$. Note that $x \neq 0$, hence $F(x) > 0$, and $\tilde{x} = \frac{x}{F(x)} \in \tilde{F}_1$. Therefore $\langle w, \tilde{x}\rangle \leq 1$ and $1 = \langle w, x\rangle \leq F(x)$. It follows that $F(x) = 1$.

Let $y \geq 0$ be a subgradient of g at x. By Lemmas A.7 and A.8 there exists $v \in \partial F(x)$ such that $y = \frac{1}{n}\mathbf{1}_n + \lambda \Pi^* v$. By definition, for all z we have $F(z) - F(x) - \langle v, z - x\rangle \geq 0$. Inserting $z = \alpha x$ for $\alpha \geq 0$, we obtain $(\alpha - 1)F(x) \geq (\alpha - 1)\langle v, x\rangle$. Since $\alpha - 1$ assumes positive as well as negative values for $\alpha \geq 0$, it follows that $1 = F(x) = \langle v, x\rangle = \langle v, \Pi x\rangle = \langle \Pi^* v, x\rangle$. Thus we get for all z that $F(z) - \langle v, z\rangle \geq 0$. In particular, for $z \in \tilde{F}_1$ we have $1 \geq F(z) \geq \langle v, z\rangle = \langle \Pi^* v, z\rangle$, and $\Pi^* v \in \tilde{F}_1^o$. From $\langle \Pi^* v, x\rangle = 1$ it follows that $\Pi^* v = w$, and $y = \frac{1}{n}\mathbf{1}_n + \lambda w \geq 0$.

Thus $\frac{1}{n}\mathbf{1}_n + \lambda w \geq 0$ for all exposed points $w \in \tilde{F}_1^o$. By Corollary A.5 we get that $\frac{1}{n}\mathbf{1}_n + \lambda w \geq 0$ for all $w \in \tilde{F}_1^o = \Pi^*[F_1^o]$. This shows (ii).

D Proof of Theorem 3.1

We first need the following Lemma.

Lemma D.1 *Let $K, N \in \mathbb{N}_+$ and $\beta \in \mathbb{R}^K$ be fixed. For a fixed set of N Monte Carlo realizations, let $t_\beta^{(n)}$, $n = 1, ..., N$, be such that*

$$\max_{t=0,\dots,T} \left(Z_t^{(n)} - \sum_{k=1}^K \beta_k M_t^{k,(n)} \right) = Z_{t_\beta^{(n)}}^{(n)} - \sum_{k=1}^K \beta_k M_{t_\beta^{(n)}}^{k,(n)}.$$

If

$$\max_{n=1,\dots,N} \sum_{k=1}^K \delta_k M_{t_\beta^{(n)}}^{k,(n)} \geq 0 \text{ for all } \delta \in \mathbb{R}^K \tag{26}$$

then it holds that

$$\min_{n=1,\dots,N}\left(Z^{(n)}_{t^{(n)}_\beta}-\sum_{k=1}^K\beta_k M^{k,(n)}_{t^{(n)}_\beta}\right)$$
$$\leq \inf_\alpha \max_{n=1,\dots,N}\max_{t=0,\dots,T}\left(Z^{(n)}_t-\sum_{k=1}^K\alpha_k M^{k,(n)}_t\right)$$
$$\leq \max_{n=1,\dots,N}\max_{t=0,\dots,T}\left(Z^{(n)}_t-\sum_{k=1}^K\beta_k M^{k,(n)}_t\right).$$

Proof. With $\alpha=\beta-\delta$ for $\delta\in\mathbb{R}^K$ we have on the one hand

$$\inf_\alpha \max_{n=1,\dots,N}\max_{t=0,\dots,T}\left(Z^{(n)}_t-\sum_{k=1}^K\alpha_k M^{k,(n)}_t\right)$$
$$=\inf_\delta \max_{n=1,\dots,N}\max_{t=0,\dots,T}\left(Z^{(n)}_t-\sum_{k=1}^K\beta_k M^{k,(n)}_t+\sum_{k=1}^K\delta_k M^{k,(n)}_t\right)$$
$$\leq \max_{n=1,\dots,N}\max_{t=0,\dots,T}\left(Z^{(n)}_t-\sum_{k=1}^K\beta_k M^{k,(n)}_t\right),$$

and on the other hand

$$\inf_\alpha \max_{n=1,\dots,N}\max_{t=0,\dots,T}\left(Z^{(n)}_t-\sum_{k=1}^K\alpha_k M^{k,(n)}_t\right)$$
$$\geq \inf_\delta \max_{n=1,\dots,N}\left(Z^{(n)}_{t^{(n)}_\beta}-\sum_{k=1}^K\beta_k M^{k,(n)}_{t^{(n)}_\beta}+\sum_{k=1}^K\delta_k M^{k,(n)}_{t^{(n)}_\beta}\right)$$
$$\geq \inf_\delta \left(\max_{n=1,\dots,N}\left(\min_{n'=1,\dots,N}\left(Z^{(n')}_{t^{(n')}_\beta}-\sum_{k=1}^K\beta_k M^{k,(n')}_{t^{(n')}_\beta}\right)+\sum_{k=1}^K\delta_k M^{k,(n)}_{t^{(n)}_\beta}\right)\right)$$
$$=\inf_\delta \left(\min_{n'=1,\dots,N}\left(Z^{(n')}_{t^{(n')}_\beta}-\sum_{k=1}^K\beta_k M^{k,(n')}_{t^{(n')}_\beta}\right)+\max_{n=1,\dots,N}\sum_{k=1}^K\delta_k M^{k,(n)}_{t^{(n)}_\beta}\right)$$
$$\geq \min_{n=1,\dots,N}\left(Z^{(n)}_{t^{(n)}_\beta}-\sum_{k=1}^K\beta_k M^{k,(n)}_{t^{(n)}_\beta}\right),$$

by using (26). ■

Corollary D.2 *Suppose that for a fixed $K\in\mathbb{N}_+$ there exists an $\alpha^*\in\mathbb{R}^K$ such that*

$$M^*:=\sum_{k=1}^K\alpha^*_k M^k_t \tag{27}$$

is surely optimal in the sense of [10]. That is

$$Y^*=\max_{t=0,\dots,T}\left(Z_t-\sum_{k=1}^K\alpha^*_k M^k_t\right)\qquad \textit{almost surely,}$$

and so we have

$$Y^*=\max_{t=0,\dots,T}\left(Z^{(n)}_t-\sum_{k=1}^K\alpha^*_k M^{k,(n)}_t\right),\quad n=1,\dots,N.$$

Let $t^{(n)}_*$, $n=1,\dots,N$, be such that

$$Y^*=\max_{t=0,\dots,T}\left(Z^{(n)}_t-\sum_{k=1}^K\alpha^*_k M^{k,(n)}_t\right)=Z^{(n)}_{t^{(n)}_*}-\sum_{k=1}^K\alpha^*_k M^{k,(n)}_{t^{(n)}_*}$$

for each n. By virtue of Lemma D.1 we then obtain for $\beta = \alpha^*$

$$Y^* = \inf_{\alpha} \max_{n=1,\ldots,N} \max_{t=0,\ldots,T} \left(Z_t^{(n)} - \sum_{k=1}^{K} \alpha_k M_t^{k,(n)} \right),$$

provided that (26) holds for $\beta = \alpha^*$.

Proposition D.3 *Let us assume M^* as in (27) in Corollary D.2 and that*

$$\max_{n=1,\ldots,N} \sum_{k=1}^{K} \delta_k M_{t_*^{(n)}}^{k,(n)} \geq c \|\delta\| \quad \textit{for all} \quad \delta \in \mathbb{R}^K \quad \textit{and some} \quad c > 0, \tag{28}$$

that is, a stronger version of (26) holds. If now

$$\alpha^\circ = \arg\inf_{\alpha} \max_{n=1,\ldots,N} \max_{t=0,\ldots,T} \left(Z_t^{(n)} - \sum_{k=1}^{K} \alpha_k M_t^{k,(n)} \right),$$

then it follows that $\alpha^\circ = \alpha^*$.

Proof. Let us define

$$F(\alpha) = \max_{n=1,\ldots,N} \max_{t=0,\ldots,T} \left(Z_t^{(n)} - \sum_{k=1}^{K} \alpha_k M_t^{k,(n)} \right).$$

So by Corollary D.2, $F(\alpha^\circ) = F(\alpha^*) = Y^*$, and for any $\delta \in \mathbb{R}^K$ we have

$$\begin{aligned}
F(\alpha^* - \delta) &= \max_{n=1,\ldots,N} \max_{t=0,\ldots,T} \left(Z_t^{(n)} - \sum_{k=1}^{K} \alpha_k^* M_t^{k,(n)} + \sum_{k=1}^{K} \delta_k M_t^{k,(n)} \right) \\
&\geq \max_{n=1,\ldots,N} \left(Z_{t_*^{(n)}}^{(n)} - \sum_{k=1}^{K} \alpha_k^* M_{t_*^{(n)}}^{k,(n)} + \sum_{k=1}^{K} \delta_k M_{t_*^{(n)}}^{k,(n)} \right) \\
&= Y^* + \max_{n=1,\ldots,N} \sum_{k=1}^{K} \delta_k M_{t_*^{(n)}}^{k,(n)} \geq c \|\delta\|,
\end{aligned}$$

hence α^* is a strict local minimum of F. Since F is convex, α^* is also a unique strict global minimum. Thus, it must hold that $\alpha^\circ = \alpha^*$. ■

We next suppose that an almost surely optimal martingale M^* satisfies

$$M^* := \sum_{k=1}^{\infty} \alpha_k^* M_t^k$$

where the convergence is understood almost surely (and if it is needed to be in an L_p sense for some $p \geq 1$). Let us introduce two convex functions

$$G_{K,N}(\alpha) = \max_{n=1,\ldots,N} \max_{t=0,\ldots,T} \left(Z_t^{(n)} - \sum_{k=1}^{K} \alpha_k M_t^{k,(n)} - \sum_{k=K+1}^{\infty} \alpha_k^* M_t^{k,(n)} \right)$$

and

$$F_{K,N}(\alpha) = \max_{n=1,\ldots,N} \max_{t=0,\ldots,T} \left(Z_t^{(n)} - \sum_{k=1}^{K} \alpha_k M_t^{k,(n)} \right).$$

It then holds that

$$\sup_{\alpha} |F_{K,N}(\alpha) - G_{K,N}(\alpha)| \leq \max_{n=1,\ldots,N} \max_{t=0,\ldots,T} \left| \sum_{k=K+1}^{\infty} \alpha_k^* M_t^{k,(n)} \right|.$$

Indeed, for fixed α, n^* and $t_*^{n^*}$ such that

$$F_{K,N}(\alpha) = Z^{(n^*)}_{t_*^{n^*}} - \sum_{k=1}^{K} \alpha_k M^{k,(n^*)}_{t_*^{n^*}}$$

we have on the one hand

$$\begin{aligned}
&F_{K,N}(\alpha) - G_{K,N}(\alpha) \\
&\leq Z^{(n^*)}_{t_*^{n^*}} - \sum_{k=1}^{K} \alpha_k M^{k,(n^*)}_{t_*^{n^*}} - \left(Z^{(n^*)}_{t_*^{n^*}} - \sum_{k=1}^{K} \alpha_k M^{k,(n^*)}_{t_*^{n^*}} - \sum_{k=K+1}^{\infty} \alpha_k^* M^{k,(n^*)}_{t_*^{n^*}} \right) \\
&= \sum_{k=K+1}^{\infty} \alpha_k^* M^{k,(n^*)}_{t_*^{n^*}} \leq \max_{n=1,\ldots,N} \max_{t=0,\ldots,T} \left| \sum_{k=K+1}^{\infty} \alpha_k^* M_t^{k,(n)} \right|,
\end{aligned}$$

and on the other hand, with n° and $t_\circ^{n^\circ}$ such that

$$G_{K,N}(\alpha) = Z^{(n^\circ)}_{t_\circ^{n^\circ}} - \sum_{k=1}^{K} \alpha_k M^{k,(n^\circ)}_{t_\circ^{n^\circ}} - \sum_{k=K+1}^{\infty} \alpha_k^* M^{k,(n^\circ)}_{t_\circ^{n^\circ}}$$

$$\begin{aligned}
&G_{K,N}(\alpha) - F_{K,N}(\alpha) \\
&\leq Z^{(n^\circ)}_{t_\circ^{n^\circ}} - \sum_{k=1}^{K} \alpha_k M^{k,(n^\circ)}_{t_\circ^{n^\circ}} - \sum_{k=K+1}^{\infty} \alpha_k^* M^{k,(n^\circ)}_{t_\circ^{n^\circ}} - \left(Z^{(n^\circ)}_{t_\circ^{n^\circ}} - \sum_{k=1}^{K} \alpha_k M^{k,(n^\circ)}_{t_\circ^{n^\circ}} \right) \\
&= - \sum_{k=K+1}^{\infty} \alpha_k^* M^{k,(n^\circ)}_{t_\circ^{K,n^\circ}} \leq \max_{n=1,\ldots,N} \max_{t=0,\ldots,T} \left| \sum_{k=K+1}^{\infty} \alpha_k^* M_t^{k,(n)} \right|.
\end{aligned}$$

Now let $t_*^{(n)}$, $n = 1, ..., N$, be defined such that for $\alpha^* := (\alpha_1^*, \ldots, \alpha_K^*)$,

$$G_{K,N}(\alpha^*) = Z^{(n)}_{t_*^{(n)}} - \sum_{k=1}^{K} \alpha_k^* M^{k,(n)}_{t_*^{(n)}} - \sum_{k=K+1}^{\infty} \alpha_k^* M^{k,(n)}_{t_*^{(n)}} = Y^*$$

for each n, and assume that

$$\max_{n=1,\ldots,N} \sum_{k=1}^{K} \delta_k M^{k,(n)}_{t_*^{(n)}} \geq c \, \|\delta\| \quad \text{for all} \quad \delta \in \mathbb{R}^K \quad \text{and some} \quad c > 0. \tag{29}$$

By applying Proposition D.3 to the cash-flow

$$Z_t - \sum_{k=K+1}^{\infty} \alpha_k^* M_t^k$$

it thus follows that

$$\arg\inf_{\alpha \in \mathbb{R}^K} G_{K,N}(\alpha) = (\alpha_1^*, \ldots, \alpha_K^*)$$

on $\mathcal{E}_{c,K,N}$. Then, using the Markov and Doob inequalities, we get

$$\begin{aligned}
\mathbb{P}\left(\sup_{\alpha} |F_{K,N}(\alpha) - G_{K,N}(\alpha)| \geq \varepsilon \right) &\leq \mathbb{P}\left(\max_{n} \max_{t=0,\ldots,T} \left| \sum_{k=K+1}^{\infty} \alpha_k^* M_t^{k,(n)} \right| \geq \varepsilon \right) \\
&= 1 - \mathbb{P}\left(\max_{n} \max_{t=0,\ldots,T} \left| \sum_{k=K+1}^{\infty} \alpha_k^* M_t^{k,(n)} \right| < \varepsilon \right) \\
&= 1 - \left(\mathbb{P}\left(\max_{t=0,\ldots,T} \left| \sum_{k=K+1}^{\infty} \alpha_k^* M_t^{k,(n)} \right| < \varepsilon \right) \right)^N \\
&\leq 1 - (1 - A_p \, \eta \, \varepsilon^{-p} \, K^{-\rho})^N \leq A_p \, \eta N \varepsilon^{-p} \, K^{-\rho}
\end{aligned} \tag{30}$$

for $K > K_0$ and some constant A_p depending on p. Now consider K and N to be fixed and let

$$\alpha_{\inf}^F := (\alpha_{\inf,1}^F, ..., \alpha_{\inf,K}^F) := \underset{\alpha \in \mathbb{R}^K}{\arg\inf} F_{K,N}(\alpha).$$

Due to

$$\begin{aligned} G_{K,N}(\alpha_{\inf}^F) &= G_{K,N}\left(\alpha^* - \left(\alpha^* - \alpha_{\inf}^F\right)\right) \\ &= \max_{n=1,\dots,N} \max_{t=0,\dots,T} \left(Z_t^{(n)} - \sum_{k=1}^K \alpha_k^* M_t^{k,(n)} - \sum_{k=K+1}^\infty \alpha_k^* M_t^{k,(n)} + \sum_{k=1}^K \left(\alpha_k^* - \alpha_{\inf,k}^F\right) M_t^{k,(n)} \right) \\ &\geq \max_{n=1,\dots,N} \left(Z_{t_*^{(n)}}^{(n)} - \sum_{k=1}^K \alpha_k^* M_{t_*^{(n)}}^{k,(n)} - \sum_{k=K+1}^\infty \alpha_k^* M_{t_*^{(n)}}^{k,(n)} + \sum_{k=1}^K \left(\alpha_k^* - \alpha_{\inf,k}^F\right) M_{t_*^{(n)}}^{k,(n)} \right) \\ &= Y^* + \max_{n=1,\dots,N} \sum_{k=1}^K \left(\alpha_k^* - \alpha_{\inf,k}^F\right) M_{t_*^{(n)}}^{k,(n)} \\ &\geq Y^* + c \left\| \alpha^* - \alpha_{\inf}^F \right\|, \end{aligned}$$

by virtue of (29), it holds that

$$\begin{gathered} \left\| \alpha^* - \alpha_{\inf}^F \right\| \leq \frac{1}{c}\left(G_{K,N}(\alpha_{\inf}^F) - G_{K,N}(\alpha^*) \right) \\ \leq \frac{1}{c}\left| G_{K,N}(\alpha_{\inf}^F) - F_{K,N}(\alpha_{\inf}^F) \right| + \frac{1}{c}\left(F_{K,N}(\alpha_{\inf}^F) - F_{K,N}(\alpha^*) \right) + \frac{1}{c}\left| F_{K,N}(\alpha^*) - G_{K,N}(\alpha^*) \right| \\ \leq \frac{2}{c} \sup_\alpha \left| F_{K,N}(\alpha) - G_{K,N}(\alpha) \right|. \end{gathered}$$

So we have

$$\mathbb{P}\left(\left\{ \| \alpha^{*,K} - \alpha_{\inf}^F \| \geq \varepsilon \right\} \cap \mathcal{E}_{c,K,N} \right) \leq \mathbb{P}\left(\frac{2}{c} \sup_\alpha \left| F_{K,N}(\alpha) - G_{K,N}(\alpha) \right| \geq \varepsilon \right) \leq A_p\, \eta 2^p\, N (c\varepsilon)^{-p}\, K^{-\rho}$$

by (30).

E Proof of Theorem 3.2

The proof goes along the lines in [7], where a similar result was proven for the covering of the sphere by random spherical caps.

Let $z \in \mathsf{S}^{K-1}$ be a point such that $z \notin \bigcup_{k=1}^N S^{(n)}$. Then the spherical cap $B(z, 2\delta)$ centered on z and with opening angle 2δ is contained in the complement of the union $\bigcup_{k=1}^N S^{\delta,(n)}$, where $S^{\delta,(n)}$ is the realization of the random subset $S^\delta = \bigcap_{t=1,\dots,T} S_t^\delta$ corresponding to the realization $M_t^{k,(n)}$.

In particular, the fraction u_δ of points of the sphere S^{K-1} which is not covered by the union $\bigcup_{k=1}^N S^{\delta,(n)}$ is not smaller than

$$\frac{|B(z,2\delta)|}{|\mathsf{S}^{K-1}|} = \frac{\frac{2\pi^{(K-1)/2}}{\Gamma((K-1)/2)} \int_0^\delta (\sin\varphi)^{K-2} d\varphi}{\frac{2\pi^{K/2}}{\Gamma(K/2)}} = \frac{\Gamma(K/2) \int_0^\delta (\sin\varphi)^{K-2} d\varphi}{\sqrt{\pi}\,\Gamma((K-1)/2)}.$$

Hence

$$\mathbb{E} u_\delta \geq \mathbb{P}\left\{ \bigcup_{n=1}^N S^{(n)} \neq \mathsf{S}^{K-1} \right\} \cdot \frac{\Gamma(K/2) \int_0^\delta (\sin\varphi)^{K-2} d\varphi}{\sqrt{\pi}\,\Gamma((K-1)/2)}. \tag{31}$$

The expectation of u_δ can now be computed as in [7]. By the independence of the $S^{\delta,(n)}$ we have that

$$\mathbb{P}\left\{ z \notin \bigcup_{n=1}^N S^{\delta,(n)} \right\} = \left(1 - \mathbb{P}\left\{ z \in S^\delta \right\}\right)^N$$

and by Robbins' theorem [8]

$$\mathbb{E}u_\delta = \int_{\mathsf{S}^{K-1}} (1 - \mathbb{P}\left\{z \in S^\delta\right\})^N d\mu(z) \le (1 - \min_{z\in \mathsf{S}^{K-1}} \mathbb{P}\left\{z \in S^\delta\right\})^N$$

with μ the canonical measure on the sphere summing to 1.

We therefore obtain by the assumption of Theorem 3.2 that

$$\mathbb{P}\left\{\bigcup_{n=1}^N S^{(n)} \neq \mathsf{S}^{K-1}\right\} \le \frac{(1-\pi_0+\pi_1\delta)^N\sqrt{\pi}\Gamma((K-1)/2)}{\Gamma(K/2)\int_0^\delta (\sin\varphi)^{K-2}d\varphi} \le \frac{\pi^{K-3/2}\sqrt{K}(1-\pi_0+\pi_1\delta)^N}{2^{K-5/2}\delta^{K-1}}.$$

Here we have used that $\frac{\Gamma(\frac{K-1}{2})}{\Gamma(\frac{K}{2})} \le \frac{\sqrt{2K}}{K-1}$, and $\int_0^\delta (\sin\varphi)^{K-2}d\varphi \ge (\frac{2}{\pi})^{K-2}\frac{\delta^{K-1}}{K-1}$ for $\delta \le \frac{\pi}{2}$. With $\delta = \frac{(1-\pi_0)(K-1)}{\pi_1(N-K+1)}$ we then get

$$\mathbb{P}\left\{\bigcup_{k=1}^N S_{X_k} \neq \mathsf{S}^{K-1}\right\} \le 2\sqrt{\frac{2K}{\pi}}(1-\pi_0)^{N-K+1}\left(\frac{\pi\pi_1 Ne}{2(K-1)}\right)^{K-1},$$

where we used $(\frac{N}{N-K+1})^{N-K+1} \le e^{K-1}$.

This completes the proof of Theorem 3.2.