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Uniform approximation of the CIR process via exact simulation at random times

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Abstract

In this paper we uniformly approximate the trajectories of the Cox-Ingersoll-Ross (CIR) process. At a sequence of random times the approximate trajectories will be even exact. In between, the approximation will be uniformly close to the exact trajectory. From a conceptual point of view the proposed method gives a better quality of approximation in a path-wise sense than standard, or even exact simulation of the CIR dynamics at some deterministic time grid.

1 Introduction

The Cox-Ingersoll-Ross (CIR) process $X(s) = X_{t,x}(s)$ is determined by the following stochastic differential equation (SDE)

$$dX(s) = k(\lambda - X(s))ds + \sigma\sqrt{X}dw(s),\ X(t) = x,\ s \geq t \geq 0, \tag{1}$$

where $k,\ \lambda,\ \sigma$ are positive constants, and w is a scalar Brownian motion. Due to [6] this process has become very popular in financial mathematical applications. The CIR process is used in particular as volatility process in the Heston model [12]. It is known ([13]) that for $x > 0$ there exists a unique strong solution $X_{t,x}(s)$ of (1) for all $s \geq t \geq 0$. The CIR process $X(s) = X_{t,x}(s)$ is positive in the case $2k\lambda \geq \sigma^2$ and nonnegative in the case $2k\lambda < \sigma^2$. Moreover, in the last case the origin is a reflecting boundary.

As a matter of fact, (1) does not satisfy the global Lipschitz assumption. The difficulties arising in a usual simulation method, such as the Euler method for example, for (1) are connected with this fact and with the natural requirement of preserving nonnegative approximations. A lot of approximation methods for the CIR processes are proposed. For an extensive list of articles on this subject we refer to [3] and [7]. Besides [3] and [7] we also refer to [1, 2, 10, 11], where a number of discretization schemes for the CIR process can be found. Further we note that in [20] a weakly convergent fully implicit method is implemented for the Heston model. Exact simulation of (1) at some deterministic time grid is considered in [5, 8] (see [3] as well).

In [17], we have considered uniform path-wise approximation of $X(s)$ on an interval $[t, t+T]$ using the Doss-Sussmann transformation ([22]) which allows for expressing any trajectory of $X(s)$ by the solution of some ordinary differential equation that depends on the realization of $w(s)$. The approximation $\overline{X}(s)$ will be uniform in the sense that the path-wise error will be uniformly bounded, i.e.

$$\sup_{t \leq s \leq t+T} \left| \overline{X}(s) - X(s) \right| \leq r \quad \text{almost surely,} \tag{2}$$

where $r > 0$ is fixed in advance.

Let us consider the uniform pathwise approximation for a Wiener process $W(t)$. First consider simulating W on a fixed time grid

$$t_0, t_1, ..., t_n = T.$$

Although W may be even exactly simulated at the grid points, the usual piecewise linear interpolation

$$\overline{W}(t) = \frac{t_{i+1} - t}{t_{i+1} - t_i} W(t_i) + \frac{t - t_i}{t_{i+1} - t_i} W(t_{i+1}) \tag{3}$$

is not uniform in the sense of (2). Put differently, for any (large) positive number $A,$ there is always a positive probability that

$$\sup_{t_0 \le t \le t_0 + T} \left| \overline{W}(t) - W(t) \right| > A.$$

Therefore, for path dependent applications for instance, such a standard, even exact, simulation method may be not desirable and a uniform method preserving (2) may be preferred.

To uniformly approximate $W(t),\ t \ge t_0,$ (where $W(t_0)$ is known) we simulate the points $(t_m + \theta_m, W(t_m + \theta_m) - W(t_m)),\ \ m = 0,\ 1,\ 2,\ ...,$ by simulating θ_m as being the first-passage (stopping) time of the Wiener process $W(t) - W(t_m), t \ge t_m,$ to the boundary of the interval $[-r, r]$. So, $|W(t) - W(t_m)| \le r$ for $t_m \le t \le t_m + \theta_m$ and, moreover, the random variable $W(t_m + \theta_m) - W(t_m),$ which equals to random variable r_m taking the values $-r$ or $+r$ with probability $1/2,$ is independent of the stopping time θ_m. The values $W(t_0), ..., W(t_m), ...,$ where t_m is the random time: $t_m = t_0 + \theta_0 + ... + \theta_{m-1}$ and $W(t_m) = W(t_{m-1}) + r_{m-1},$ are exactly simulated values of the Wiener process $W(t)$ at random times t_m. Clearly, the piecewise linear interpolation (3) satisfies

$$\sup_{s \ge t_0} \left| \overline{W}(s) - W(s) \right| \le 2r \quad \text{almost surely,} \tag{4}$$

i.e., the uniform path-wise approximation for a Wiener process $W(t)$ is achieved.

In [17], by simulating the first-passage times of the increments of the Wiener process to the boundary of an interval and solving the ordinary differential equation after using the Doss-Sussmann transformation, we approximately construct a generic trajectory of $X(s)$. Such kind of simulation is more simple than the one proposed in [5] and moreover has the advantage of uniform nature. Apart from application, uniform simulation of trajectories of an SDE in the sense of (2) may be considered as an interesting mathematical problem in its own right. Let us note that the uniform approximation is connected with simulation of space-time bounded diffusions (see [18] and Ch. 5 of [19]). However, the results of [17] are obtained under the restriction $4k\lambda > \sigma^2$. If $4k\lambda \le \sigma^2$, we could not extend the results of [17] to this case.

Let $\Delta > 0$ be a small number, $x > \Delta$, and $\tau(x)$ be the first-passage time of the trajectory $X_{0,x}(s)$ to the boundary to the band $(x - \Delta, x + \Delta)$. If $x \le \Delta$, we denote by $\tau(x)$ the first-passage time of $X_{0,x}(s)$ to the upper bound of $[0, 2\Delta)$. Clearly, for any Markov moment τ the line segment between the points (τ, x) and $(\tau + \tau(x), X_{\tau,x}(\tau + \tau(x))$ uniformly (with exactness 2Δ) approximates the trajectory $X_{\tau,x}(s),\ \tau \le s \le \tau + \tau(x)$. To simulate $\tau(x)$ we solve a parabolic boundary value problem for the distribution function of $\tau(x)$ by separation of variables. The corresponding Sturm-Liouville problem in the region $x > \Delta$ is regular. The case $0 < x \le \Delta$ is more complicated. If $2k\lambda/\sigma^2 \ge 1$ then the point $x = 0$ is not attainable in contrary to $2k\lambda/\sigma^2 < 1$ when $x = 0$ is attainable. These distinctions result in different boundary value problems. In the next section we construct the distributions needed in terms of solutions of the confluent hypergeometric equation. There the simulated random values of $X_{0,x}(\tau(x))$ belong to a fixed space discretization grid $0 = x_0 < x_1 < x_2 < \cdots < x_n < \cdots$. In Section 3, we develop uniform approximation of the CIR process using the squared Bessel processes. We obtain there the required distributions in terms of Bessel functions. However, in contrast to Section 2, the simulated values of $X_{0,x}(\tau(x))$ do not belong to a fixed space discretization grid anymore, while they are still exact. In comparison with [17] the methods developed here can be applied for any set of positive parameters k, λ, σ of the CIR process. Moreover, we here simulate exact values of the CIR process at random exactly simulated times.

2 Distribution functions for first-passage times of CIR trajectories to boundaries of narrow bands

2.1 The main construction

The space domain for the equation (1) is the real semi-axis $[0, \infty)$ as $X_{t,x}(s) \geq 0$ for any $s \geq t \geq 0,\ x \geq 0$. Consider a space discretization

$$0 = x_0 < x_1 < x_2 < \cdots < x_n < \cdots, \tag{5}$$

where we assume for simplicity that $x_{i+1} - x_i = \Delta,\ i = 0, 1, ...$.

Let the initial value x for the solution $X_{0,x}(s),\ s \geq 0$, be equal to x_n for some $n \geq 2$. We set $X^0 = x = x_n$. Let $\tau^0 = 0,\ \tau^1 = \tau(X^0) = \tau(x_n)$ be the first-passage time of the trajectory $X_{0,X^0}(s)$ to the boundary of the band (x_{n-1}, x_{n+1}), i.e., $X_{0,X^0}(\tau^1)$ is equal either to x_{n-1} or to x_{n+1}, and $x_{n-1} < X_{0,X^0}(s) < x_{n+1}$ for $0 \leq s < \tau^1$. Then we set $X^1 = X_{0,X^0}(\tau^1)$. If the initial value $x = X^0$ is equal to x_1 then $X_{0,x_1}(s)$ with probability 1 attains x_2 for some time $\tau^1 = \tau(X^0) = \tau(x_1)$ which is the first-passage time of the trajectory $X_{0,X^0}(s)$ to the upper bound of the band $[0, x_2)$, i.e., $X_{0,X^0}(\tau^1)$ is equal to x_2, and $0 \leq X_{0,X^0}(s) < x_2$ for $0 \leq s < \tau(x_1)$. We denote $X_{0,X^0}(\tau^1) = X_{0,x_1}(\tau(x_1))$ again by X^1. So, for any X^0 from the set $\{x_1, ..., x_n, ...\}$ we get $X^1 = X_{0,X^0}(\tau^1)$ belonging to the same set. By the same way one can get $X^2 = X_{0,X^1}(\tau^2)$. Due to autonomy of equation (1), we have $X^2 = X_{0,X^1}(\tau^2) = X_{\tau^1,X^1}(\tau^1 + \tau^2) = X_{0,X^0}(\tau^1 + \tau^2)$. Continuing we obtain the sequence $X^m = X_{0,X^{m-1}}(\tau^m) = X_{\tau^0+...+\tau^{m-1},X^{m-1}}(\tau^0 + ... + \tau^m) = X_{0,X^0}(\tau^0 + ... + \tau^m)$. The points $(0, X^0),\ (\tau^1, X^1),\ ...,\ (\tau^0 + ... + \tau^m, X^m)$ belong to the trajectory $(s, X_{0,X^0}(s))$.

If the initial value x is not equal to x_n, we first model X^1 to be equal to one of nodes and then repeat the previous construction. If $0 \leq x = X^0 < x_1 + \Delta/2$ then X^1 is equal to $X_{0,x}(\tau^1)$ where τ^1 is the first-passage time of the trajectory $X_{0,x}(s)$ to the upper bound of the band $[0, x_2)$, i.e., $X_{0,x}(\tau^1)$ is equal to x_2, and $0 \leq X_{0,x}(s) < x_2$ for $0 \leq s < \tau^1$. If $x_n - \Delta/2 \leq x = X^0 < x_n + \Delta/2,\ n = 2, 3, ...,$ then $X^1 = X_{0,x}(\tau^1)$ where τ^1 is the first-passage time of the trajectory $X_{0,x}(s)$ to the boundary of the band (x_{n-1}, x_{n+1}), i.e., $X_{0,x}(\tau^1)$ is equal either to x_{n-1} or to x_{n+1}, and $x_{n-1} < X_{0,x}(s) < x_{n+1}$ for $0 \leq s < \tau^1$.

Suppose the sequence $(0, X^0),\ (\tau^1, X^1),\ ...,\ (\tau^0 + ... + \tau^m, X^m)$ is constructed. As an approximation trajectory $\overline{X}_{0,x}(s)$, we introduce the polygonal line which passes through the points of this sequence:

$$\overline{X}_{0,x}(s) = X^{i-1} + \frac{X^i - X^{i-1}}{\tau^i}(s - (\tau^0 + ... + \tau^{i-1})), \tag{6}$$
$$\tau^0 + ... + \tau^{i-1} \leq s \leq \tau^0 + ... + \tau^i,\ i = 1, 2, ...$$

Because $X^i = \overline{X}_{0,x}(\tau^0 + ... + \tau^i) = X_{0,x}(\tau^0 + ... + \tau^i)$ and both the trajectory $X_{0,x}(s)$ and the line segment (6) of the polygonal line connecting the points $(\tau^0 + ... + \tau^{i-1}, X^{i-1})$ and $(\tau^0 + ... + \tau^i, X^i)$ belong to a band of the width 2Δ, we have obtained the following proposition.

Proposition 1 *Approximation (6) satisfies*

$$\sup_{0 \leq s < \infty} \left| \overline{X}_{0,x}(s) - X_{0,x}(s) \right| \leq 2\Delta, \tag{7}$$

i.e., this approximation is uniform.

2.2 Probabilities connected with attainability of boundaries and boundary value problems for the probabilities

If $0 \le x < x_1 + \Delta/2$ then $X_{0,x}(s)$ with probability 1 attains x_2 for some time $\tau(x)$ which is the first-passage time of $X_{0,x}(s)$ to the upper bound of the band $[0, x_2)$. If $x_n - \Delta/2 \le x < x_n + \Delta/2,\ n = 2, 3, ...,$ then $X_{0,x}(\tau(x))$, where $\tau(x)$ is the first-passage time of the trajectory $X_{0,x}(s)$ to the boundary of the band (x_{n-1}, x_{n+1}), attains either x_{n-1} or x_{n+1} with probability 1. Let $p_l(x)$ be the probability $P(X_{0,x}(\tau(x)) = x_{n-1})$ and $p_r(x) := P(X_{0,x}(\tau(x)) = x_{n+1})$. Clearly, $p_l(x) + p_r(x) = 1$. Though we need $p_l(x)$ and $p_r(x)$ for $x_n - \Delta/2 \le x < x_n + \Delta/2$ only, we shall consider these functions for $x_{n-1} \le x < x_{n+1}$. The probability $p_l(x)$ satisfies the one-dimensional Dirichlet problem for elliptic equation ([19], Ch. 6, Sec. 3).

$$\frac{1}{2}\sigma^2 x \frac{\partial^2 p}{\partial x^2} + k(\lambda - x)\frac{\partial p}{\partial x} = 0, \tag{8}$$

$$p_l(x_{n-1}) = 1,\ p_l(x_{n+1}) = 0. \tag{9}$$

From (8)-(9) (in particular, for $x_n - \Delta/2 \le x < x_n + \Delta/2,\ n = 2, 3, ...$)

$$p_l(x) = \frac{\int_x^{x_{n+1}} \xi^{-\frac{2k\lambda}{\sigma^2}} e^{\frac{2k}{\sigma^2}\xi} d\xi}{\int_{x_{n-1}}^{x_{n+1}} \xi^{-\frac{2k\lambda}{\sigma^2}} e^{\frac{2k}{\sigma^2}\xi} d\xi}. \tag{10}$$

Hence

$$p_r(x) = 1 - p_l(x) = \frac{\int_{x_{n-1}}^{x} \xi^{-\frac{2k\lambda}{\sigma^2}} e^{\frac{2k}{\sigma^2}\xi} d\xi}{\int_{x_{n-1}}^{x_{n+1}} \xi^{-\frac{2k\lambda}{\sigma^2}} e^{\frac{2k}{\sigma^2}\xi} d\xi}. \tag{11}$$

For simulating $\tau(x)$ and $X_{0,x}(\tau(x))$ we need the probabilities

$$u(t,x) := P(\tau(x) < t),\ \text{for } 0 \le x < x_1 + \Delta/2, \tag{12}$$

and

$$\begin{aligned} u_l(t,x) &:= P(\tau(x) < t,\ X_{0,x}(\tau(x)) = x_{n-1}), \\ u_r(t,x) &:= P(\tau(x) < t,\ X_{0,x}(\tau(x)) = x_{n+1}), \\ \text{for } x_n - \frac{\Delta}{2} &\le x < x_n + \frac{\Delta}{2},\ n = 2, 3, ... \end{aligned} \tag{13}$$

2.2.1 The region $x_n - \Delta/2 \le x < x_n + \Delta/2,\ n = 2, 3, ...,$

If $x_n - \Delta/2 \le x < x_n + \Delta/2,\ n = 2, 3, ...,$ we use (13) in the following way. First we simulate $X_{0,x}(\tau(x))$ according to probabilities (10)-(11). If we get $X_{0,x}(\tau(x)) = x_{n-1}$ then for simulating $\tau(x)$ we use the conditional probability

$$P(\tau(x) < t \mid\ X_{0,x}(\tau(x)) = x_{n-1}) = \frac{u_l(t,x)}{p_l(x)},$$

and if $X_{0,x}(\tau(x)) = x_{n+1}$, we use

$$P(\tau(x) < t \mid\ X_{0,x}(\tau(x)) = x_{n+1}) = \frac{u_r(t,x)}{p_r(x)}.$$

The functions $u_l(t,x)$ and $u_r(t,x)$ satisfy the equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\sigma^2 x\frac{\partial^2 u}{\partial x^2} + k(\lambda - x)\frac{\partial u}{\partial x},\ t>0,\ x_{n-1}<x<x_{n+1},\ n=2,3,..., \tag{14}$$

The function $u_l(t,x)$ satisfies the initial condition

$$u_l(0,x)=0, \tag{15}$$

and the boundary conditions

$$u_l(t,x_{n-1})=1,\ u_l(t,x_{n+1})=0. \tag{16}$$

The function $u_r(t,x)$ satisfies the initial condition

$$u_r(0,x)=0, \tag{17}$$

and the boundary conditions

$$u_r(t,x_{n-1})=0,\ u_r(t,x_{n+1})=1. \tag{18}$$

To get homogeneous boundary conditions for the problem (14)-(16) we introduce

$$v_l = u_l - \frac{x_{n+1}-x}{x_{n+1}-x_{n-1}} \tag{19}$$

and for the problem (14), (17)-(18)

$$v_r = u_r - \frac{x-x_{n-1}}{x_{n+1}-x_{n-1}}. \tag{20}$$

The function v_l satisfies the equation (for the corresponding $n=2,3,...$)

$$\frac{\partial v_l}{\partial t} = \frac{1}{2}\sigma^2 x\frac{\partial^2 v_l}{\partial x^2} + k(\lambda - x)[\frac{\partial v_l}{\partial x} - \frac{1}{x_{n+1}-x_{n-1}}],\ t>0,\ x_{n-1}<x<x_{n+1}, \tag{21}$$

with the initial condition

$$v_l(0,x) = -\frac{x_{n+1}-x}{x_{n+1}-x_{n-1}} \tag{22}$$

and the homogeneous boundary conditions

$$v_l(t,x_{n-1})=0,\ v_l(t,x_{n+1})=0. \tag{23}$$

The function v_r satisfies the equation (for the corresponding $n=2,3,...$)

$$\frac{\partial v_r}{\partial t} = \frac{1}{2}\sigma^2 x\frac{\partial^2 v_r}{\partial x^2} + k(\lambda - x)[\frac{\partial v_r}{\partial x} + \frac{1}{x_{n+1}-x_{n-1}}],\ t>0,\ x_{n-1}<x<x_{n+1}, \tag{24}$$

with the initial condition

$$v_r(0,x) = -\frac{x-x_{n-1}}{x_{n+1}-x_{n-1}} \tag{25}$$

and the homogeneous boundary conditions of the form (23).

By separation of variables we get $\mathcal{T}(t)\mathcal{X}(x)$ as elementary independent solutions to the homogeneous equation corresponding to (21) satisfying (23), where

$$\mathcal{T}'(t)+\mu\mathcal{T}(t)=0,\ \text{i.e.,}\ \mathcal{T}(t)=\mathcal{T}_0 e^{-\mu t},\ \mu>0, \tag{26}$$

and

$$\frac{1}{2}\sigma^2 x\mathcal{X}'' + k(\lambda - x)\mathcal{X}' + \mu\mathcal{X} = 0 \tag{27}$$

with the homogeneous boundary conditions

$$\mathcal{X}(x_{n-1}) = \mathcal{X}(x_{n+1}) = 0. \tag{28}$$

Introduce

$$p(x) := \exp(-\frac{2k}{\sigma^2}x) \cdot x^{\frac{2k\lambda}{\sigma^2}},\ q(x) := \frac{2}{\sigma^2 x}p(x),\ x_{n-1} < x < x_{n+1},\ n = 2, 3, ...$$

Then (27) can be expressed in the self-adjoint form

$$(p(x)\mathcal{X}')' + \mu q(x)\mathcal{X} = 0,\ \mathcal{X}(x_{n-1}) = \mathcal{X}(x_{n+1}) = 0. \tag{29}$$

On the intervals $(x_{n-1}, x_{n+1}),\ n = 2, 3, ...$, we have $p(x) > 0,\ q(x) > 0$, i.e., the Sturm-Liouville problem (29) is regular. Therefore all the eigenvalues $\mu_j,\ j = 1, 2, ...$, of problem (29) (hence (27)-(28)) are positive. Let $\mathcal{X}_j,\ j = 1, 2, ...$, be the corresponding eigenfunctions which are orthogonal w.r.t. the scalar product

$$\langle f, g\rangle := \int_{x_{n-1}}^{x_{n+1}} f(y)g(y)q(y)dy.$$

It is well known that the solution of the problem (21)-(23) is equal to

$$v_l(t, x) = \int_{x_{n-1}}^{x_{n+1}} G(x, \xi, t)q(\xi)v_l(0, \xi)d\xi \tag{30}$$

$$+ \int_0^t \int_{x_{n-1}}^{x_{n+1}} G(x, \xi, t - s)q(\xi)[-k(\lambda - \xi)\frac{1}{x_{n+1} - x_{n-1}}]d\xi ds, \tag{31}$$

where the Green function

$$G(x, \xi, t) = \sum_{j=1}^{\infty} e^{-\mu_j t}\frac{\mathcal{X}_j(x)\mathcal{X}_j(\xi)}{\|\mathcal{X}_j\|^2},\ \|\mathcal{X}_j\|^2 = \int_{x_{n-1}}^{x_{n+1}} q(\xi)\mathcal{X}_j^2(\xi)d\xi. \tag{32}$$

The function $v_r(t, x)$ is found analogously.

The eigenvalues μ_j and eigenfunctions $\mathcal{X}_j$ can be found in terms of the solutions of the confluent hypergeometric equation (the Kummer equation). Indeed, the general solution of the linear equation (27) is given by the formula

$$\mathcal{X}(x) = C_1\Phi(b, c; \zeta) + C_2\Psi(b, c; \zeta), \tag{33}$$

where C_1 and C_2 are arbitrary constants,

$$b = \frac{2k\lambda}{\sigma^2} + \frac{\mu}{k},\ c = \frac{2k\lambda}{\sigma^2};\ \zeta = -\frac{2k}{\sigma^2}x$$

and $\Phi(b, c; \zeta),\ \Psi(b, c; \zeta)$ are the known linear independent solutions of the confluent hypergeometric equation

$$\zeta y''_{\zeta\zeta} + (c - \zeta)y'_{\zeta} - by = 0 \tag{34}$$

(see [4], Sec. 6.2). The problem (24)-(25) is solved analogously.

2.2.2 The region $0 \leq x < x_1 + \Delta/2$

If $0 \leq x < x_1 + \Delta/2$ then $X_{0,x}(\tau(x)) = x_2$ with probability 1 and for simulating $\tau(x)$ we use the probability (12). Here we do not give a method for computing the probability $u(t,x)$ in (12) in the spirit of Section 2.2.1. As an alternative, such a method will be presented in the next section in the context of another, computationally more tractable approach. On the other hand, from a practical point of view, one could apply the following approximate result derived in [17],

$$u(t,x) \approx 1 - 2x^{\gamma}(2\Delta)^{-\gamma} \sum_{m=1}^{\infty} \frac{J_{-2\gamma}\left(\pi_{-2\gamma,m}\sqrt{\frac{x}{2\Delta}}\right)}{\pi_{-2\gamma,m} J_{-2\gamma+1}(\pi_{-2\gamma,m})} \exp\left[-\frac{\sigma^2 \pi_{-2\gamma,m}^2}{16\Delta} t\right], \quad 0 \leq x \leq 2\Delta,$$

where $\pi_{-2\gamma,m}, m = 1, 2, ...$ are the positive zeros of $J_{-2\gamma}$.

From a theoretical point of view the developed approach can be applied for uniform approximation of the solutions of a lot of other SDEs. However as a rule we shall not get a sufficiently constructive method for the probabilities $u_l(t,x)$ and $u_r(t,x)$ in such a way. Here we find them in terms of solutions of the Kummer equation. In the next section we develop uniform approximation of the CIR process using the squared Bessel process.

3 Using squared Bessel processes

Due to [9], the solution $X(s) = X_{t,x}(s)$ of (1) has the representation

$$X(s) = e^{-k(s-t)} Y\left(\frac{\sigma^2}{4k}(e^{k(s-t)} - 1)\right), \; s \geq t, \tag{35}$$

where $Y(s) = Y_{t,x}(s)$ denotes a squared Bessel process with dimension $\delta = 4k\lambda/\sigma^2$ starting at x, i.e., $Y(s)$ satisfies the equation

$$dY(s) = \delta ds + 2\sqrt{Y(s)}dw(s), \; Y(t) = X(t) = x, \tag{36}$$

see also [21].

3.1 Method

Due to autonomy of (1) and (35), one can start at $t = 0$. Let $x > \Delta$. Let $\theta = \theta(x)$ be the first-passage time of the trajectory $Y_{0,x}(\vartheta)$ to the boundary of the band $(x - \Delta, x + \Delta)$, i.e., $Y_{0,x}(\theta(x))$ is equal either $x - \Delta$ or $x + \Delta$ and $x - \Delta < Y_{0,x}(\vartheta) < x + \Delta$ for $0 \leq \vartheta < \theta(x)$. If $x \leq \Delta$, we denote by $\theta(x)$ the first-passage time of the trajectory $Y_{0,x}(s)$ to the upper bound $[0, 2\Delta)$, i.e., $Y_{0,x}(\theta(x)) = 2\Delta$ and $0 \leq Y_{0,x}(s) < 2\Delta$ for $0 \leq s < \theta(x)$.

Due to (35) the solution $X_{0,x}(s)$ of (1) is equal to

$$X_{0,x}(s) = e^{-ks} Y_{0,x}\left(\frac{\sigma^2}{4k}(e^{ks} - 1)\right), \; s \geq 0. \tag{37}$$

Let us introduce

$$\tau(x) := \frac{1}{k}\ln(1 + \frac{4k}{\sigma^2}\theta(x)). \tag{38}$$

For $0 \le s \le \tau(x)$ we have $\frac{\sigma^2}{4k}(e^{ks}-1) \le \theta(x)$. Hence for these s we have

$$x - \Delta \le Y_{0,x}\left(\frac{\sigma^2}{4k}(e^{ks}-1)\right) \le x + \Delta,\ x > \Delta, \tag{39}$$
$$Y_{0,x}\left(\frac{\sigma^2}{4k}(e^{ks}-1)\right) \le 2\Delta,\ x \le \Delta.$$

Therefore

$$(x-\Delta)\, e^{-ks} \le X_{0,x}(s) \le (x+\Delta)\, e^{-ks},\ x > \Delta,\ 0 \le s \le \tau(x), \tag{40}$$
$$0 \le X_{0,x}(s) \le 2\Delta e^{-ks},\ x \le \Delta,\ 0 \le s \le \tau(x).$$

Let us introduce the interpolation

$$\overline{X}_{0,x}(s) := xe^{-ks} + \frac{s}{\tau(x)}\left(X_{0,x}(\tau(x))e^{k\tau(x)} - x\right)e^{-ks}, \quad 0 \le s \le \tau(x). \tag{41}$$

For $x > \Delta$ we then have by (40),

$$(x-\Delta)\, e^{-ks} \le xe^{-ks} - \frac{s}{\tau(x)}\Delta e^{-ks} \le \overline{X}_{0,x}(s) \le xe^{-ks} + \frac{s}{\tau(x)}\Delta e^{-ks} \le (x+\Delta)\, e^{-ks},$$

and by using (40) again,

$$\left|\overline{X}_{0,x}(s) - X_{0,x}(s)\right| \le 2\Delta e^{-ks}. \tag{42}$$

For $x \le \Delta$ we have by (40)

$$0 \le xe^{-ks} - \frac{s}{\tau(x)}xe^{-ks} \le \overline{X}_{0,x}(s) \le xe^{-ks} + \frac{s}{\tau(x)}\left(2\Delta - x\right)e^{-ks} \le 2\Delta e^{-ks}$$

yielding (42) for $x \le \Delta$ also.

Denote $X^0 := x$ and set

$$\theta^0 = 0,\ \theta^1 = \theta(X^0),\ \tau^0 = 0,\ \tau^1 = \frac{1}{k}\ln(1 + \frac{4k}{\sigma^2}\theta^1), \tag{43}$$
$$X^1 = X_{0,X^0}(\tau^1) = e^{-k\tau^1}Y_{0,X^0}(\theta^1),$$

where $Y_{0,X^0}(\theta^1) = X^0 \pm \Delta$ if $X^0 > \Delta$ and $Y_{0,X^0}(\theta^1) = 2\Delta$ if $X^0 \le \Delta$, and construct the interpolation (41) for $\tau^0 \le s \le \tau^1$.

Then we set

$$\theta^2 = \theta(X^1),\ \tau^2 = \frac{1}{k}\ln(1 + \frac{4k}{\sigma^2}\theta^2), \tag{44}$$
$$X^2 = X_{0,X^1}(\tau^2) = X_{\tau^1,X^1}(\tau^1+\tau^2) = X_{0,X^0}(\tau^1+\tau^2) = e^{-k\tau^2}Y_{0,X^1}(\theta^2),$$

where $Y_{0,X^1}(\theta^2) = X^1 \pm \Delta$ if $X^1 > \Delta$ and $Y_{0,X^1}(\theta^2) = 2\Delta$ if $X^1 \le \Delta$, and construct the interpolation (41) for $\tau^1 \le s \le \tau^2$.

Continuing we obtain the sequence

$$\theta^m = \theta(X^{m-1}),\ \tau^m = \frac{1}{k}\ln(1+\frac{4k}{\sigma^2}\theta^m), \tag{45}$$
$$X^m = X_{0,X^{m-1}}(\tau^m) = X_{\tau^0+...+\tau^{m-1},X^{m-1}}(\tau^0+...+\tau^m) =$$
$$X_{0,X^0}(\tau^0+...+\tau^m) = e^{-k\tau^m}Y_{0,X^{m-1}}(\theta^m),\ m=1,2,....$$

and a piecewise interpolated trajectory

$$\overline{X}_{0,x}(s) = \left(X^{i-1} + \frac{s-(\tau^0+...+\tau^{i-1})}{\tau^i}\left(X^i e^{k\tau^i} - X^{i-1}\right)\right)e^{-k\left(s-(\tau^0+...+\tau^{i-1})\right)}, \tag{46}$$
$$\tau^0+...+\tau^{i-1} \le s \le \tau^0+...+\tau^i,\ i=1,2,... .$$

The points $(0,X^0),\ (\tau^1,X^1),\ ...,\ (\tau^1+...+\tau^m,X^m),...$ belong to the trajectory $(s,X_{0,x}(s))$. Unlike to modeling in Section 2, the difference between X^{m-1} and X^m is not a multiple of Δ here because of presence of the random factor $e^{-k\tau^m}$. Also, the X^m generally do not jump over a pre-fixed grid like in Section 2. Now, obviously, for the present method we have the following proposition analogue to Proposition 1.

Proposition 2 *Approximation (46) is uniform and satisfies*

$$\sup_{0\le s<\infty}\left|\overline{X}_{0,x}(s) - X_{0,x}(s)\right| \le 2\Delta.$$

3.2 Simulating $\theta(x)$ and $Y_{0,x}(\theta(x))$

In Section 2 we have developed a method of simulating the first-passage time $\tau(x)$ of the solution $X_{0,x}(s)$ of (1). Here we develop analogous methods for simulating $\theta(x)$ and $Y_{0,x}(\theta(x))$ and then use algorithm (43)-(46) for uniform approximation of solutions of (1). Due to simplicity of (36) in comparison with (1), such an approach is more effective than the direct one.

3.2.1 The region $x>\Delta$

The time $\theta(y)$ is the first-passage time of the solution $Y_{0,y}(s)$ of (36) to the boundary of the band $(x-\Delta,x+\Delta),\ x-\Delta\le y\le x+\Delta$. Let $p_l(y)$ be the probability $P(Y_{0,y}(\theta(y))=x-\Delta)$ and $p_r(y)=P(Y_{0,y}(\theta(y))=x+\Delta),\ x-\Delta\le y\le x+\Delta$. Clearly, $p_l(y)+p_r(y)=1$. The probability $p_l(y)$ satisfies the one-dimensional Dirichlet problem for elliptic equation ([19], Ch. 6, Sec. 3).

$$2y\frac{\partial^2 p_l}{\partial y^2} + \frac{4k\lambda}{\sigma^2}\frac{\partial p_l}{\partial y} = 0,\ x-\Delta<y<x+\Delta, \tag{47}$$

$$p_l(x-\Delta)=1,\ p_l(x+\Delta)=0. \tag{48}$$

The solution $p_l(y)$ of problem (47)-(48) is equal to

$$p_l(y) = \begin{cases} \frac{y^{\frac{-2k\lambda}{\sigma^2}+1} - (x+\Delta)^{\frac{-2k\lambda}{\sigma^2}+1}}{(x-\Delta)^{\frac{-2k\lambda}{\sigma^2}+1} - (x+\Delta)^{\frac{-2k\lambda}{\sigma^2}+1}},\ \frac{2k\lambda}{\sigma^2}\ne 1, \\ \frac{\ln\frac{y}{x+\Delta}}{\ln\frac{x-\Delta}{x+\Delta}},\ \frac{2k\lambda}{\sigma^2}=1. \end{cases}$$

Hence the probability

$$p_l(x) = P(Y_{0,x}(\theta(x)) = x-\Delta) = \begin{cases} \frac{x^{\frac{-2k\lambda}{\sigma^2}+1} - (x+\Delta)^{\frac{-2k\lambda}{\sigma^2}+1}}{(x-\Delta)^{\frac{-2k\lambda}{\sigma^2}+1} - (x+\Delta)^{\frac{-2k\lambda}{\sigma^2}+1}}, & \frac{2k\lambda}{\sigma^2} \neq 1, \\ \frac{\ln\frac{x}{x+\Delta}}{\ln\frac{x-\Delta}{x+\Delta}}, & \frac{2k\lambda}{\sigma^2} = 1, \end{cases} \tag{49}$$

and $p_r(x) = 1 - p_l(x)$.

For simulating $\theta(x)$ and $Y_{0,x}(\theta(x))$ we need the probabilities

$$u(t, y) = P(\theta(y) < t),\ x-\Delta \leq y \leq x+\Delta, \tag{50}$$

and

$$\begin{aligned} u_l(t,y) &= P\left(\theta(y) < t, Y_{0,y}(\theta(y)) = x-\Delta\right), \\ u_r(t,y) &= P\left(\theta(y) < t, Y_{0,y}(\theta(y)) = x+\Delta\right), \\ &\text{for } x-\Delta \leq y \leq x+\Delta. \end{aligned} \tag{51}$$

We use (51) in the following way. First we simulate $Y_{0,x}(\theta(x))$ according to probabilities $q_l(x)$ and $q_r(x)$. If we get $Y_{0,x}(\theta(x)) = x-\Delta$ then for simulating $\theta(x)$ we use the conditional probability

$$P(\theta(x) < t \mid Y_{0,x}(\theta(x)) = x-\Delta\ ) = \frac{u_l(t,x)}{p_l(x)} \tag{52}$$

and if $Y_{0,x}(\theta(x)) = x+\Delta$, we use

$$P(\theta(x) < t \mid Y_{0,x}(\theta(x)) = x+\Delta\ ) = \frac{u_r(t,x)}{p_r(x)}. \tag{53}$$

The functions $u_l(t,y)$ and $u_r(t,y)$ are the solutions of the first boundary value problem of parabolic type ([19], Ch. 5, Sec. 3)

$$\frac{\partial u}{\partial t} = 2y\frac{\partial^2 u}{\partial y^2} + \frac{4k\lambda}{\sigma^2}\frac{\partial u}{\partial y} = 0,\ t > 0,\ x-\Delta < y < x+\Delta. \tag{54}$$

The function $u_l(t,y)$ satisfies the initial condition

$$u_l(0,y) = 0, \tag{55}$$

and the boundary conditions

$$u_l(t, x-\Delta) = 1,\ u_l(t, x+\Delta) = 0. \tag{56}$$

To get homogeneous boundary conditions for problem (54)-(56) we introduce

$$v_l(t,y) = u_l(t,y) - \frac{x+\Delta-y}{2\Delta}. \tag{57}$$

The function $v_l(t,y)$ satisfies the equation

$$\frac{\partial v_l}{\partial t} = 2y\frac{\partial^2 v_l}{\partial y^2} + \frac{4k\lambda}{\sigma^2}[\frac{\partial v_l}{\partial y} - \frac{1}{2\Delta}] = 0,\ t > 0,\ x-\Delta < y < x+\Delta, \tag{58}$$

with the initial condition

$$v_l(0,y) = -\frac{x+\Delta-y}{2\Delta} \tag{59}$$

and the homogeneous boundary conditions

$$v_l(t,x-\Delta) = 0,\ v_l(t,x+\Delta) = 0. \tag{60}$$

Analogous equations can be written out for $u_r(t,y)$.

In connection with the problem (58)-(60), we use the method of separation of variables to the homogeneous equation

$$\frac{\partial v}{\partial t} = 2y\frac{\partial^2 v}{\partial y^2} + \frac{4k\lambda}{\sigma^2}\frac{\partial v}{\partial y} = 0$$

with the homogeneous boundary conditions

$$v(t,x-\Delta) = 0,\ v(t,x+\Delta) = 0. \tag{61}$$

For elementary independent solutions $\mathcal{T}(t)\mathcal{Y}(y)$ we so have

$$\frac{\mathcal{T}'}{\mathcal{T}} = \frac{2y\mathcal{Y}'' + \delta\mathcal{Y}'}{\mathcal{Y}} =: -\mu = \text{const},$$

and for $\mathcal{Y}(y)$ we then get the corresponding Sturm-Liouville problem

$$2y\mathcal{Y}'' + \delta\mathcal{Y}' + \mu\mathcal{Y} = 0, \tag{62}$$

$$\mathcal{Y}(x-\Delta) = 0,\ \mathcal{Y}(x+\Delta) = 0, \tag{63}$$

along with

$$\mathcal{T}(t) = \mathcal{T}_0 e^{-\mu t}.$$

It can be straightforwardly checked that elementary solutions of (62) are given in terms of Bessel functions by

$$\mathcal{Y}_1(y) = y^\gamma J_{-2\gamma}\left(\sqrt{2\mu y}\right),\quad \mathcal{Y}_2(y) = y^\gamma J_{2\gamma}\left(\sqrt{2\mu y}\right) \tag{64}$$

$$\text{with}\quad \gamma := \frac{1}{2} - \frac{k\lambda}{\sigma^2} = \frac{1}{2} - \frac{\delta}{4} \tag{65}$$

(cf. [17]). If 2γ is not an integer, $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are independent. If 2γ is an integer, i.e. when

$$\frac{2k\lambda}{\sigma^2} = 1, 2, ... \tag{66}$$

these solutions are dependent however. In this case we may take as second independent solution

$$\mathcal{Y}_2(y) = y^\gamma Y_{2\gamma}\left(\sqrt{2\mu y}\right), \tag{67}$$

where $Y_{2\gamma}$ is a Bessel function of the second kind. Note that for (66) we have that $\sigma^2 \le 2k\lambda$, i.e. the boundary 0 is not attainable. We omit the analysis connected with (66) since it is similar to the derivations below.

Due to the boundary condition (61), the eigenvalues of the problem (62) follow by requiring that the system

$$C_1 J_{2\gamma}\left(\sqrt{2\mu(x+\Delta)}\right) + C_2 J_{-2\gamma}\left(\sqrt{2\mu(x+\Delta)}\right) = 0$$
$$C_1 J_{2\gamma}\left(\sqrt{2\mu(x-\Delta)}\right) + C_2 J_{-2\gamma}\left(\sqrt{2\mu(x-\Delta)}\right) = 0$$

has a non-trivial solution. Thus we must have

$$J_{2\gamma}\left(\sqrt{2\mu(x+\Delta)}\right)J_{-2\gamma}\left(\sqrt{2\mu(x-\Delta)}\right) \qquad (68)$$
$$-J_{2\gamma}\left(\sqrt{2\mu(x-\Delta)}\right)J_{-2\gamma}\left(\sqrt{2\mu(x+\Delta)}\right)=0$$

Let us denote the solutions with $0<\mu_1<\mu_2<\cdots$, and the respective eigenfunctions by

$$\mathcal{Y}_j(y)=J_{-2\gamma}\left(\sqrt{2\mu_j(x+\Delta)}\right)y^{\gamma}J_{2\gamma}\left(\sqrt{2\mu_j y}\right) \qquad (69)$$
$$-J_{2\gamma}\left(\sqrt{8\mu_j(x+\Delta)}\right)y^{\gamma}J_{-2\gamma}\left(\sqrt{2\mu_j y}\right).$$

We note that the equation (62) can be written in the selfadjoint form

$$(p(y)\mathcal{Y}')'+\mu q(y)\mathcal{Y}=0,\ p(y)=y^{\delta/2},\ q(y)=\frac{1}{2}y^{\delta/2-1},$$

i.e. eigenfunctions corresponding to different eigenvalues are orthogonal w.r.t. the scalar product

$$\langle f,g\rangle:=\int_{x-\Delta}^{x+\Delta}f(y)g(y)q(y)dy.$$

Thus the Green function of the considered problem is given by

$$G(y,\eta,t)=\sum_{j=1}^{\infty}e^{-\mu_j t}\frac{\mathcal{Y}_j(y)\mathcal{Y}_j(\eta)}{\|\mathcal{Y}_j\|^2}, \qquad (70)$$
$$\|\mathcal{Y}_j\|^2=\int_{x-\Delta}^{x+\Delta}q(\xi)\mathcal{Y}_j^2(\xi)d\xi,$$

and the solution to (58) is equal to

$$v_l(t,y)=\int_{x-\Delta}^{x+\Delta}G(y,\eta,t)q(\eta)v_l(0,\eta)d\eta$$
$$+\int_0^t\int_{x-\Delta}^{x+\Delta}G(y,\eta,t-s)q(\eta)[-\frac{4k\lambda}{\sigma^2}\frac{1}{2\Delta}]d\eta ds. \qquad (71)$$

Example 3 *Let us illustrate the method for $2k\lambda/\sigma^2=1/2$. Hence $\gamma=1/4$. We then have*

$$J_{1/2}(z)=\sqrt{\frac{2}{\pi z}}\sin z,\ J_{-1/2}(z)=\sqrt{\frac{2}{\pi z}}\cos z$$

and then (68) implies,

$$\sin\left(\sqrt{2\mu(x+\Delta)}\right)\cos\left(\sqrt{2\mu(x-\Delta)}\right)$$
$$-\sin\left(\sqrt{2\mu(x-\Delta)}\right)\cos\left(\sqrt{2\mu(x+\Delta)}\right)=0,\quad \textit{hence}$$
$$\sin\left(\sqrt{2\mu(x+\Delta)}-\sqrt{2\mu(x-\Delta)}\right)=0,\quad \textit{i.e.}$$
$$\mu_j=\left[\frac{j\pi\left(\sqrt{2\,(x+\Delta)}+\sqrt{2\,(x-\Delta)}\right)}{4\Delta}\right]^2.$$

Thus, as eigenfunctions for (69) we may take

$$\mathcal{Y}_j(y) = \sin\left(\sqrt{2\mu_j y} - \sqrt{2\mu_j(x-\Delta)}\right)$$

while

$$\begin{aligned}\|\mathcal{Y}_j\|^2 &= \int_{x-\Delta}^{x+\Delta} \frac{1}{2}\xi^{-1/2}\sin^2\left(\sqrt{2\mu_j\xi} - \sqrt{2\mu_j(x-\Delta)}\right)d\xi \\ &= \frac{\Delta}{\sqrt{x+\Delta}+\sqrt{x-\Delta}}.\end{aligned}$$

The solution is then found by (70) and (71).

3.2.2 The region $x \leq \Delta$

Let us recall that the scale density $s(y)$ and the speed density $m(y)$ of the process (36) determined via the relation

$$\frac{1}{2}\frac{1}{m(y)}\frac{d}{dy}\left(\frac{1}{s(y)}\frac{d}{dy}\right) = \delta\frac{d}{dy} + 2y\frac{d^2}{dy^2},$$

where the r.h.s. is the generator of the process (36) (see for example, [14], Ch. 4, and [15], Ch. 6). We thus obtain straightforwardly,

$$s(y) = Cy^{-\delta/2} \quad \text{and} \quad m(y) = \frac{1}{4C}y^{\delta/2-1} \quad \text{for arbitrary} \quad C > 0.$$

Case I: $\delta/2 = 2k\lambda/\sigma^2 \geq 1$. In this case we have for any $r > 0$,

$$\begin{aligned} S(0,r] &:= \int_0^r s(y)dy = \infty, && (72)\\ M(0,r] &:= \int_0^r m(y)dy < \infty, \\ \Sigma(0,r] &:= \int_0^r S(0,h]m(h)dh = \infty, \\ N(0,r] &:= \int_0^r m(\eta)d\eta \int_\eta^r s(y)dy < \infty. && (73)\end{aligned}$$

As a consequence of (72), for the process Y in (36) the boundary 0 is unattainable if it starts somewhere in $Y(0) > 0$. Therefore, the state space of Y is considered to be $(0,\infty)$ in this case. For details see for example [15].

Case II: $\delta/2 = 2k\lambda/\sigma^2 < 1$. In this case we have for any $r > 0$,

$$S(0,r] := \int_0^r s(y)dy < \infty, \tag{74}$$

$$M(0,r] := \int_0^r m(y)dy < \infty, \tag{75}$$

$$\Sigma(0,r] := \int_0^r S(0,h]m(h)dh < \infty,$$

$$N(0,r] := \int_0^r m(\eta)d\eta \int_\eta^r s(y)dy < \infty. \tag{76}$$

As a consequence of (74) and (75), the point 0 is a regular boundary point of Y in (36) (Karlin and Taylor (1981)). That is, 0 is attainable for Y from any starting point $Y(0) > 0$, and the process starts afresh after reaching 0 (strong Markov property), and reaches any positive level in finite time due to (76). Since no atomic speed mass at the boundary is imposed, the boundary 0 is *reflecting*.

Let $\theta(y)$ be the first-passage time of the solution $Y_{0,y}(s)$ to (36) of the level $2\Delta,\ 0 \le y \le 2\Delta$, and let

$$q(t,y) := P(\theta(y) \ge t). \tag{77}$$

Though we need $q(t,y)$ for $0 \le y \le \Delta$ only, we shall consider boundary value problems for q with $0 \le y \le 2\Delta$.

Proposition 4 ***(Case I)*** *If $2k\lambda/\sigma^2 \ge 1$, the probability q in (77) satisfies and is uniquely determined as a bounded solution of the following mixed initial-boundary value problem,*

$$2y\frac{\partial^2 q}{\partial y^2} + \delta\frac{\partial q}{\partial y} = \frac{\partial q}{\partial t},\ 0 < y < 2\Delta, \tag{78}$$

$$q(0,y) = 1, \tag{79}$$

$$q(t,2\Delta) = 0,\ q(t,0) \text{ is bounded.} \tag{80}$$

Proof. A bounded solution q (with bounded $\partial q/\partial y$) in the considered case can be constructed by separation of variables (see Proposition 6). Due to the boundedness of q, we may take the Laplace transform

$$\widehat{q}(\alpha,y) := \int_0^\infty e^{-\alpha t}q(t,y)dt, \tag{81}$$

and then take the Laplace transform of (78)-(80) w.r.t. t, yielding the system

$$2y\frac{\partial^2 \widehat{q}}{\partial y^2} + \delta\frac{\partial \widehat{q}}{\partial y} = \alpha\widehat{q}(\alpha,y) - 1, \tag{82}$$

$$\widehat{q}(\alpha,2\Delta) = 0, \quad \widehat{q}(\alpha,0) \text{ is bounded.} \tag{83}$$

Then by setting $\widehat{q} =: (1-\widetilde{q})/\alpha$ we obtain,

$$2y\frac{\partial^2 \widetilde{q}}{\partial y^2} + \delta\frac{\partial \widetilde{q}}{\partial y} = \alpha\widetilde{q}, \tag{84}$$

$$\widetilde{q}(\alpha,2\Delta) = 1, \quad \widetilde{q}(\alpha,0) \text{ is bounded.} \tag{85}$$

Since the boundary 0 is not attainable in this case, we may apply the Itô formula to

$$Q\left(s,Y(s)\right):=e^{-\alpha s}\widetilde{q}\left(\alpha,Y(s)\right),$$

where $Y(s)=Y_{0,y}(s)$ is the solution of (36). By using (84) we then get

$$dQ=e^{-\alpha s}\widetilde{q}_y\left(\alpha,Y(s)\right)2\sqrt{Y(s)}dw(s),$$

and so we have

$$e^{-\alpha\theta(y)}\widetilde{q}\left(\alpha,Y(\theta(y))\right)-\widetilde{q}\left(\alpha,y\right)=\int_0^{\theta(y)}e^{-\alpha s}2\sqrt{Y(s)}\widetilde{q}_y\left(\alpha,Y(s)\right)dw(s).$$

By now taking expectations and taking into account (85) it follows that

$$\widetilde{q}\left(\alpha,y\right)=\mathbb{E}\left[e^{-\alpha\theta(y)}\right].$$

We thus have

$$\widetilde{q}(\alpha,y)=\mathbb{E}\left[e^{-\alpha\theta(y)}\right]=-\int_0^\infty e^{-\alpha t}dP(\theta(y)\geq t) \tag{86}$$

$$=1-\alpha\int_0^\infty P(\theta(y)\geq t)e^{-\alpha t}dt, \tag{87}$$

whence

$$\widehat{q}(\alpha,y)=\int_0^\infty P(\theta(y)\geq t)e^{-\alpha t}dt, \tag{88}$$

and so

$$q(t,y)=P(\theta(y)\geq t) \tag{89}$$

by uniqueness of the Laplace transform. ■

Proposition 5 ***(Case II)*** *Let $2k\lambda/\sigma^2<1.$ If $q(t,y)$ is a bounded solution of the mixed initial-boundary value problem consisting of (78)-(80), and the additional boundary condition*

$$\lim_{y\downarrow 0}\frac{q_y(t,y)}{s(y)}=\lim_{y\downarrow 0}q_y(t,y)y^{2k\lambda/\sigma^2}=0\quad \textit{uniformly in } 0<t<\infty, \tag{90}$$

then (77) holds, and so in particular the solution of (78)-(80), and (90), is unique. The existence of $q(t,y)$ follows by construction using the method of separation of variables, see Proposition 6.

Proof. Let $q(t,y)$ be a solution as stated. Due to the boundedness of q the Laplace transform (81) exists as above, and by taking the Laplace transform of (78)-(80), and (90), w.r.t. t we obtain the system consisting of (82)-(83) and, additionally,

$$\lim_{y\downarrow 0}\frac{\widehat{q}_y(\alpha,y)}{s(y)}=\lim_{y\downarrow 0}\widehat{q}_y(\alpha,y)y^{2k\lambda/\sigma^2}=0.$$

Now by setting $\widehat{q}=:\left(1-\widetilde{q}\right)/\alpha$ we obtain the system consisting of (84)-(85), supplemented with

$$\lim_{y\downarrow 0}\frac{\widetilde{q}_y(\alpha,y)}{s(y)}=\lim_{y\downarrow 0}\widetilde{q}_y(\alpha,y)y^{2k\lambda/\sigma^2}=0.$$

The results in [14], Sec. 4.5, 4.6, (see also [16]) then imply that

$$\widetilde{q}(\alpha, y) = \mathbb{E}\left[e^{-\alpha\theta(y)}\right],$$

and finally we obtain

$$q(t, y) = P(\theta(y) \geq t)$$

analogue to (86)-(89). ■

Remarkably, by the next proposition, (77) can be represented by one and the same expression for both Case I and Case II.

Proposition 6 *For both Case I and Case II, the probability $q(t, y)$ in (77) satisfies,*

$$q(t, y) = 2y^{\gamma}(2\Delta)^{-\gamma} \sum_{m=1}^{\infty} \frac{J_{-2\gamma}\left(\pi_{-2\gamma,m}\sqrt{\frac{y}{2\Delta}}\right)}{\pi_{-2\gamma,m} J_{-2\gamma+1}(\pi_{-2\gamma,m})} \exp\left[-\frac{\pi^2_{-2\gamma,m}}{4\Delta} t\right], \quad 0 \leq y \leq 2\Delta, \tag{91}$$

where with γ as in (65), $J_{-2\gamma}$ is the Bessel function of the first kind with parameter -2γ, and $\pi_{-2\gamma,m}$, $m = 1, 2, ...$ is the increasing sequence of positive zeros of $J_{-2\gamma}$.

Proof. We apply the method of separation of variables. Let us seek for elementary solutions $\mathcal{T}(t)\mathcal{Y}(y)$ satisfying (78), hence

$$2y\mathcal{Y}''\mathcal{T} + \delta\mathcal{Y}'\mathcal{T} = \mathcal{Y}\mathcal{T}'.$$

We so may set

$$\frac{\mathcal{T}'}{\mathcal{T}} = \frac{2y\mathcal{Y}'' + \delta\mathcal{Y}'}{\mathcal{Y}} =: -\mu = \text{const.}$$

and get the system

$$\mathcal{T}(t) = \mathcal{T}_0 e^{-\mu t}, \quad 2y\mathcal{Y}'' + \delta\mathcal{Y}' + \mu\mathcal{Y} = 0. \tag{92}$$

We recall that elementary independent solutions of (62) are given in terms of Bessel functions cf. (64)-(67).

i) In Case I, where $2k\lambda/\sigma^2 \geq 1$, hence $\gamma \leq 0$, the only feasible elementary solutions are $\mathcal{T}(t)\mathcal{Y}(y)$ where $\mathcal{Y}$ is of type

$$\mathcal{Y}_1(y) = y^{\gamma} J_{-2\gamma}\left(\sqrt{2\mu y}\right) = \text{entire function of } y \text{ not vanishing at } y = 0. \tag{93}$$

Indeed, if 2γ is not an integer we have in particular that $2\gamma < 0$, and then the second independent solution is of type

$$\mathcal{Y}_2(y) = y^{\gamma} J_{2\gamma}\left(\sqrt{2\mu y}\right) = y^{2\gamma} \times \text{entire function of } y \text{ not vanishing at } y = 0, \tag{94}$$

which is unbounded for $y \downarrow 0$. On the other hand, if $2\gamma = 0, -1, -2, ...$, the second independent solution is of type

$$\mathcal{Y}_2(y) = y^{\gamma} Y_{2\gamma}\left(\sqrt{2\mu y}\right)$$

(see (67)), which is also unbounded for $y \downarrow 0$.

ii) In Case II, where $2k\lambda/\sigma^2 < 1$, we have that $\gamma > 0$ and in particular that 2γ is not an integer. Then both solutions (93) and (94) are bounded for $y \downarrow 0$. However, the solution (94), which is by (65) of type

$$y^{1-2k\lambda/\sigma^2} \times \text{entire function of } y \text{ not vanishing at } y = 0,$$

yields an elementary solution $\mathcal{T}(t)\mathcal{Y}(y)$ that clearly violates the boundary condition (90), while (90) is obviously satisfied for elementary solutions $\mathcal{T}(t)\mathcal{Y}(y)$ with $\mathcal{Y}$ of type (93).

As a result, for both Case I and Case II, solutions of type (93) are feasible only. That is, we consider

$$\mathcal{Y}_\gamma(y) := \mathcal{Y}(y) = y^\gamma J_{-2\gamma}\left(\sqrt{2\mu y}\right). \tag{95}$$

In view of boundary condition (80) we next require for both cases $\mathcal{Y}_\gamma(2\Delta) = 0$, leading to the eigenvalues

$$\mu_m := \frac{\pi_{-2\gamma,m}^2}{4\Delta},$$

and the elementary solutions $\mathcal{T}(t)\mathcal{Y}_{\gamma,m}(y)$ with

$$\mathcal{Y}_{\gamma,m}(y) := y^\gamma J_{-2\gamma}\left(\sqrt{2\mu_m y}\right) = y^\gamma J_{-2\gamma}\left(\pi_{-2\gamma,m}\sqrt{\frac{y}{2\Delta}}\right), \tag{96}$$

$m = 1, 2, ...$ Now, as solution candidate for (77), we consider the Fourier-Bessel series

$$q(t,y) = \sum_{m=1}^\infty \beta_m e^{-\frac{\pi_{-2\gamma,m}^2}{4\Delta}t}\mathcal{Y}_{\gamma,m}(y), \quad 0 \le y \le 2\Delta, \tag{97}$$

by (92). The initial condition (79) then yields

$$1 = \sum_{m=1}^\infty \beta_m \mathcal{Y}_{\gamma,m}(y),$$

from which the coefficients $(\beta_m)_{m=1,2,...}$ may be solved straightforwardly by a well known orthogonality relation for Bessel functions as in Appendix C of [17]. Let us recall it for completeness: The well-known relation

$$\int_0^1 zJ_{-2\gamma}(\pi_{-2\gamma,k}z)J_{-2\gamma}(\pi_{-2\gamma,k'}z)dz = \frac{\delta_{k,k'}}{2}J_{-2\gamma+1}^2(\pi_{-2\gamma,k})$$

straightforwardly implies that

$$\int_0^{2\Delta} \mathcal{Y}_{\gamma,m}(y)\mathcal{Y}_{\gamma,m'}(y)y^{-2\gamma}dy = 2\Delta\delta_{m,m'}J_{-2\gamma+1}^2(\pi_{-2\gamma,m}).$$

Further we have that

$$\begin{aligned}\int_0^{2\Delta} \mathcal{Y}_{\gamma,m}(y)y^{-2\gamma}dy &= \int_0^{2\Delta} y^{-\gamma}J_{-2\gamma}\left(\pi_{-2\gamma,m}\sqrt{\frac{y}{2\Delta}}\right)dy \\ &= 2\left(2\Delta\right)^{-\gamma+1}\int_0^1 z^{-2\gamma+1}J_{-2\gamma}\left(\pi_{-2\gamma,m}z\right)dz \\ &= 2\left(2\Delta\right)^{-\gamma+1}\frac{J_{-2\gamma+1}\left(\pi_{-2\gamma,m}\right)}{\pi_{-2\gamma,m}},\end{aligned}$$

and so we get

$$\beta_m = \frac{2\left(2\Delta\right)^{-\gamma}}{\pi_{-2\gamma,m}J_{-2\gamma+1}(\pi_{-2\gamma,m})},$$

from which with (96) and (97) expression (91) follows.

Finally, since the series (91) convergence point-wise and uniformly on any compact subset of $\mathbb{R}_{>0} \times (0, 2\Delta)$ it is straightforward to check that (91) is a solution of the mixed initial-boundary value problem of Proposition 4 in Case I, and of the mixed initial-boundary value problem of Proposition 5 in Case II. In particular, (91) represents (77) in both cases. ■

Remark 7 *It should be noted that in [17] the boundary condition (90), necessary for the case*

$$\frac{2k\lambda}{\sigma^2} < 1, \tag{98}$$

i.e. Case II in the present setting, was not considered there in fact. As such the related proof there was incomplete. However, the above analysis shows that in both Case I and Case II only solutions of type (95) are feasible. Therefore, the results regarding (77) in [17] go through for (98) also.

Example 8 *The case* $2k\lambda/\sigma^2 = 1/2$. *In this case (a subcase of Case II),* $\gamma = 1/4$ *in (91) and thus (91) simplifies to*

$$\begin{aligned} q(t,y) &= 2y^{1/4}\,(2\Delta)^{-1/4}\sum_{m=1}^{\infty}\frac{J_{-1/2}\left(\pi_{-1/2,m}\sqrt{\frac{y}{2\Delta}}\right)}{\pi_{-1/2,m}J_{1/2}(\pi_{-1/2,m})}\exp\left[-\frac{\pi^2_{-1/2,m}}{4\Delta}t\right] \\ &= \frac{4}{\pi}\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{2m-1}\cos\left((2m-1)\,\pi\sqrt{\frac{y}{8\Delta}}\right)\exp\left[-\frac{(2m-1)^2\,\pi^2}{16\Delta}t\right], \quad 0 \le y \le 2\Delta. \end{aligned}$$

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