9. Assignment „Numerische Mathematik für Ingenieure II“

http://www.moses.tu-berlin.de/Mathematik/

Quadratic finite elements

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1. **Exercise:** Quadratic elements in 1D **8 points**

Consider the elliptic problem

$$-\big(a(x)u'(x)\big)' + cu(x) = f(x)$$

for $x \in \Omega = (0, L)$ and $c \geq 0$. We want to modify the solution of assignment 7, so that we solve the problem with piecewise quadratic polynomials as basis functions. Therefore let $0 = x_0 < x_1 < ... < x_N = L$ represent a decomposition of $\Omega = \cup_{k=1}^{N}\Omega_k$ with $\Omega_k = (x_{k-1}, x_k)$ and $F_k(p) = x_{k-1} + (x_k - x_{k-1})p$ the map $F_k : \Omega_{\text{ref}} \to \Omega_k$. Furthermore we add the middle points $x_{N+k} = (x_k + x_{k-1})/2$ for $k = 1 \ldots N$. Basis functions are defined using $w_i(x_j) = \delta_{ij}$ and piecewise quadratic and continuous.

(a) What is the dimension of $V_h = \text{span}\{w_i\}$ if we have homogeneous Dirichlet or homogeneous Neumann boundary conditions for a given N?

(b) Consider the reference element $\Omega_{\text{ref}} = (0, 1)$. For $p_1 = 0, p_2 = 1, p_3 = 1/2$ determine the unique quadratic polynomial ϕ_n such that $\phi_n(p_m) = \delta_{nm}$ (shape function).

(c) Recompute the 3×3 matrices

$$S_{nm} = \int_{\Omega_{\text{ref}}} \phi_n'(p)\phi_m'(p)\,\mathrm{d}p \qquad \text{and} \qquad M_{nm} = \int_{\Omega_{\text{ref}}} \phi_n(p)\phi_m(p)\,\mathrm{d}p,$$

for quadratic shape functions. Use these to determine

$$\bar{S}_{ij} = \int_{\Omega_k} w_i'(x)w_j'(x)\,\mathrm{d}x \qquad \text{and} \qquad \bar{M}_{ij} = \int_{\Omega_k} w_i(x)w_j(x)\,\mathrm{d}x.$$

(d) How do you need to modify the vector `e2p` such that we have the identity

$$w_i(F_k(p)) = \phi_n(p), \qquad p \in \Omega_{\text{ref}}$$

for $i =$ `e2p`(k, n) and all $k = 1 \ldots$ `nelement`, $n = 1 \ldots$ `nphi`? What is the value of `nelement,nphi`? **Remark:** You may assume Neumann boundary conditions.

2. **Programming exercise:** Modify MATLAB program for linear elements **8 points**

(a) Modify your/the solution from assignment 7, exercise 2 as follows:
  - Modify the `function [nelement,npoint,e2p]=generateelements(x)` so that you return the modified `e2p` you derived in exercise (1d).
  - Also modify the function `generatetransformation.m` so that it works with the modified `e2p`.
  - Modify the functions `localstiff` and `localmass`, so that they return the modified 3×3 matrices $\bar{M}$ and $\bar{S}$ you computed before.
  - Modify the main program `elliptic1d.m` by changing `nphi` and the generation of the arrays `ii,jj` in order to adapt for the modified `e2p`.

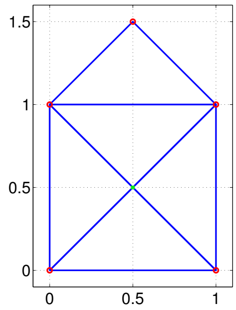
(b) Compute the solution with $f = 1$, $a = 1$, $c = 0$ and homogeneous Dirichlet boundary conditions and compare with exact solution.

(c) Implement a as in assignment 6, exercise 3. Compute the numerical solution and compare with the exact solution.

Remark: For b,c) use $x_i = ih$, $h = 1/N$, $i = 0 \ldots N = 100$. Sample solutions for assignment 7 are available in the supplemental material on ISIS2 (please do not distribute).

3. Exercise: Quadratic elements in 2D **4 points**

Consider the following decomposition of the domain $\Omega = \{\text{Das Haus vom Nikolaus}\}$.



consisting of 6 vertices, 5 elements and 10 edges. Assume you want to construct FEM basis functions which are piecewise quadratic polynomials with Neumann boundary conditions.

(a) What is the dimension of V_h with Dirichlet conditions or Neumann conditions?

(b) What is the number `nphi` of shape functions?

(c) Construct (by hand) the variable `e2p` which would be suitable in this case.

Hint: You may proceed as follows: Associate the basis functions with points $\mathbf{x}_i$ on vertices/edges and enumerate them with numbers $1 .. \dim V_h$. Separately enumerate the elements from 1...`nelement`. Then use `e2p(k,n)` to associate shape functions ϕ_n with the basis functions such that $w_i(F_k(s,t)) = \phi_n(s,t)$ for $\mathtt{i} = \mathtt{e2p}(k,n)$. Perhaps a drawing with different colors would be good.

Remark: You do not need to construct ϕ_n or w_i.

Lesson: This assignment shows you one possiblity to use the variable `e2p` to keep track of the unknowns and extend a code with linear elements to quadratic elements. The generalization to higher order polynomials for Lagrange finite elements is analogous.

total sum: 20 points

As usual, use sparse matrices.