6. Assignment „Numerische Mathematik für Ingenieure II“

http://www.moses.tu-berlin.de/Mathematik/

CFL condition and some finite elements in 1D

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1. **Programming exercise:** The CFL condition for the θ scheme **10 points**

For the heat equation $\partial_t u(t,x) - \partial_x^2 u(t,x) = f(t,x)$ consider the θ-scheme

$$\frac{u^{k+1}-u^k}{\tau} + \theta L^h u^{k+1} + (1-\theta)L^h u^k = f$$

for a given discrete elliptic operator L^h and $u^k = (u(t_k,x_0),...,u(t_k,x_N))^T \in \mathbb{R}^{N+1}$ for the discrete domain $\Omega^h = \{ih : 0 \le i \le N\}$ and $h = 1/N$. Consider the problem with $L^h = -\Delta^h$ the standard 3-point stencil and $f \equiv 0$. The problem is to be solved with Dirichlet boundary conditions $u_0^k = 0$ and $u_N^k = 1$ and initial conditions

(i1)
$$u_i^0 = \begin{cases} 0 & ih \le 1/2, \\ 1 & \text{otherwise}, \end{cases}$$

or with boundary conditions $u_0^k = u_N^k = 0$ and initial conditions

(i2)
$$u_i^0 = \sin(i\pi h).$$

(a) Use Fourier transformation and a separation ansatz to compute exact solutions for (i1) and (i2). For (i1) you might find the representation

$$u^0(x) = x - \frac{1}{\pi}\left(\frac{\sin a}{1} - \frac{\sin 2a}{2} + \frac{\sin 3a}{3} - ....\right)$$

with $a = 2\pi x$ helpful (it suffices to use 40 terms in the expansion).

(b) Write a `function [xh,Ah,Bh,Mii]=a06ex01getLh(p,tau,theta)` which returns matrices `Ah,Bh` that the problem above is equivalent to $A^h u^{k+1} = B^h u^k$ on a uniform mesh `xh` with $N = 2^p - 1$. Suppose $B_h = \mathbb{I} - M_h$, then return `Mii`$= \max_i M_{ii}$.

(c) Solve the problem with initial conditions (i1) and (i2) for $p = 5$ and $\tau = 0.01$. Plot the solution after one time-step to `a06ex01sol1.pdf` for i) $\theta = 0$, ii) $\theta = 1/4$, iii) $\theta = 1/2$ and iv) $\theta = 1$ and compute the corresponding norms $\|u^k\|_\infty$ and $\|u^k\|_{2,h}$ for $k = 0, 1$ for i)-iv) and compare with the value `Mii`. Discuss your observations.

(d) Repeat the previous step, but now plot the solution `a06ex01sol2.pdf` after 10 time-steps with $\tau = 0.001$ (do not plot the explicit solution). Discuss the differences.

(e) Use $\tau = \{0.01, 0.001, 0.0001\}$ to compute the numerical solution at $T = 0.01$ for θ i)-iv), provided the method gives a reasonable result for that value of θ. Compare the errors in the norms $\max_k \|u^k - r_h u\|_{2,h}$ and $\max_k \|u^k - r_h u\|_\infty$.

- Which method returns the smallest error (in most cases)?
- Which method is most reliable, i.e. works so you do not worry about convergence?
- Which method is the fastest? **Remark:** Do not solve a linear equation for $\theta = 0$! Use $p = 7$ and $\tau = 10^{-5}$, $T = 0.01$ for comparison with `tic,toc`.

Lesson: Here you further improve your understanding of stability in different norms and the practical value of different discretization schemes.

2. **Exercise:** Equivalence to a minimization problem **3 points**

Let $A \in \mathbb{R}^{n\times n}$ a symmetric matrix with positive eigenvalues and $b \in \mathbb{R}^n$ arbitrary. Show that $Ax = b$ is equivalent to x minimizing the expression $\frac{1}{2}x^T Ax - bx$.

Lesson: A different approach to state certain linear equations, which will be useful later.

3. **Exercise:** Weak form of elliptic equation **6 points**

Consider the problem for the electric potential

$$\begin{aligned} -\big(\varepsilon(x)\phi'(x)\big)' &= \rho(x) && \text{in } (0,1),\\ \phi &= 0 && \text{at } \{0,1\}, \end{aligned}$$

where the relative permittivity is

$$\varepsilon(x) = \begin{cases} 1 & \text{for } 0 < x < 1/2 \\ \bar{\varepsilon} & \text{for } 1/2 \le x < 1 \end{cases}$$

and with given charge density $\rho(x)$. The interpretation is that we have different materials in $0 < x < 1/2$ and in $1/2 < x < 1$, so that the permittivity might jump.

(a) Show that the general requirement

$$\lim_{\delta\to 0} \int_{1/2-\delta}^{1/2+\delta} \rho(x)\,\mathrm{d}x = 0$$

i.e., the interface carries no extra charges, leads to the transmission condition

$$\lim_{\delta\to 0} \Big[\varepsilon(1/2+\delta)\phi'(1/2+\delta) - \varepsilon(1/2-\delta)\phi'(1/2-\delta)\Big] = 0$$

(b) State the weak form of the problem. **Hint:** Assume the test functions fulfill the boundary conditions and ϕ can be integrated by parts separately on $(0, 1/2)$ and $(1/2, 1)$. The solution and the test functions are continuous.

(c) Find the weak solution for $\bar{\varepsilon} = 2$, $\rho(x) = -1$. **Hint:** Try to combine polynomials on $(0, 1/2)$ and on $(1/2, 1)$ with proper continuity of $\phi, \varepsilon\phi'$.

(d) Is the solution also a classical solution?

Lesson: A very practical example for a elliptic equation, where solutions have low regularity.

4. **Exercise:** Local matrices for a 1D finite element formulation **4 points**

Let $\Omega = (0, 1)$ and $\phi_1(x) = 1 - x$, $\phi_2(x) = x$.

- Compute the following two 2×2 matrices

$$\begin{aligned} M_{ij} &= \int_\Omega \phi_i(x)\phi_j(x)\,\mathrm{d}x \\ S_{ij} &= \int_\Omega \phi_i'(x)\phi_j'(x)\,\mathrm{d}x \end{aligned}$$

- For $a > 0$ let $F(x) = ax + b$ and define $\bar{\phi}_i(F(x)) = \phi_i(x)$. Compute the matrices

$$\begin{aligned} \bar{M}_{ij} &= \int_{F(\Omega)} \bar{\phi}_i(x)\bar{\phi}_j(x)\,\mathrm{d}x \\ \bar{S}_{ij} &= \int_{F(\Omega)} \bar{\phi}_i'(x)\bar{\phi}_j'(x)\,\mathrm{d}x \end{aligned}$$

explicitly using a, b. **Hint:** Use integration by substitution.

Lesson: This is groundwork for later assignments.

total sum: 23 points