Technische Universität Berlin **WS 2013/2014**
Fakultät II – Institut f. Mathematik
Instructor: Peschka Handed out: 19.11.
Tutor: Marschall Return during lecture: 26.11.

5. Assignment „Numerische Mathematik für Ingenieure II“

http://www.moses.tu-berlin.de/Mathematik/

Periodic boundary conditions and the 1D heat equation

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1. Exercise: Periodic boundary conditions **8 points**

Let $f : [0, 1) \to \mathbb{R}$ be sufficiently smooth and 1-periodic. We want to find 1-periodic solutions $u : [0, 1) \to \mathbb{R}$ of $-u''(x) = f(x)$. Using the 3-point stencil the discrete problem is

$$(1) \qquad L^h u^h = \frac{1}{h^2} \begin{pmatrix} 2 & -1 & 0 & & 0 & 0 & -1 \\ -1 & 2 & -1 & & 0 & 0 & 0 \\ & & \ddots & \ddots & \ddots & & \\ 0 & 0 & 0 & & -1 & 2 & -1 \\ -1 & 0 & 0 & & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} u_0^h \\ u_1^h \\ \vdots \\ u_{N-2}^h \\ u_{N-1}^h \end{pmatrix} = \begin{pmatrix} f_0^h \\ f_1^h \\ \vdots \\ f_{N-2}^h \\ f_{N-1}^h \end{pmatrix}$$

where $u_i^h = u^h(x_i)$, $f_i^h = f(x_i)$, $x_i = ih$ $(i = 0, .., N-1)$ with $h = 1/N$.

(a) Assume you have a solution for the discrete problem u^h or the continuous problem u. Show that $\bar{u}^h = u^h + C$ or $\bar{u} = u + C$ are also a solutions for any given $C \in \mathbb{R}$.

(b) What is the solvability condition in the continuous and in the discrete case?

(c) For $b \in \mathbb{R}^N$ formulate an extended equation

$$\begin{pmatrix} L^h & b \\ b^T & 0 \end{pmatrix} \begin{pmatrix} u^h \\ \rho \end{pmatrix} = \begin{pmatrix} f^h \\ 0 \end{pmatrix}$$

which takes care of the solvability condition. Discuss the cases $\rho = 0$ and $\rho \neq 0$.

(d) Rewrite (1) using the fourth-order consistent central difference scheme

$$(2) \qquad -u''(x) = \tfrac{1}{12h^2} \left(u_{i-2} - 16u_{i-1} + 30u_i - 16u_{i+1} + u_{i+2} \right) + R.$$

Lesson: Here you learn how to use periodicity conditions.

2. Exercise: A slightly different problem statement **4 points**

Let L^h as in eq. (1) and for $c > 0$ consider the problem

$$(\mathbb{I} + c\,L^h\,)u^h = f^h$$

with $\mathbb{I}$ being the $N \times N$ identity matrix.

(a) Is the discrete problem uniquely solvable?

(b) Compute $h \sum_{i=0}^{N-1} u_i^h$ just using f^h. What does this imply for the continuous solution?

Hint: You might use that for any given real matrix A

$$\big(v, Au\big) = \big(A^T v, u\big)$$

for all u, v with Euclidean scalar product $(u, v) := h \sum_{i=0}^{N-1} u_i v_i$ and $(A^T)_{ij} = A_{ji}$.

Remark: The factor h in the scalar product is so that $(u, u) = \|u\|_{2,h}^2$. For the current assignment this is of no importance.

Lesson: This shows you how much simpler a slightly modified problem might be. The result can be applied to time-dependent problems, where $c = \tau\theta$.

3. Programming exercise: Parabolic θ scheme **10 points**

For given 1-periodic $f(x)$ consider the problem

$$\partial_t u + Lu = f, \qquad u(0,x) = u_0(x),$$

where $u(t,x)$ is 1-periodic[1] in x. The discrete θ scheme for $u^{k+1} \in \mathbb{R}^N$ is

$$\frac{u^{k+1} - u^k}{\tau} + \theta L^h u^{k+1} + (1-\theta) L^h u^k = f.$$

(a) Write a `function [xh,Lh]=a05ex03getLh(p,flag)` which for `flag=0` returns the matrix L^h in eq. (1), for `flag=1` returns the corresponding matrix with 5-point stencil from eq. (2). **Note**: The grid has $N = 2^p$ points.

(b) Use `a05ex03getLh` to write a `function [xh,Ah,Mh]=a05ex03getTheta(p,theta,tau,flag)` which returns the grid `xh`, the matrices `Ah,Mh` for given parameters `p,theta,tau` and `flag` (as before), so that the θ-scheme can be written

$$A^h u^{k+1} = M^h u^k + \tau f.$$

(c) What can you say about

$$h \sum_{i=0}^{N-1} u_i^{k+1} \qquad ?$$

(d) For $f = 0$ and $u_0(x) = 2 + \cos(2\pi x) + \sin(6\pi x)/2$ the smooth exact solution is given by

$$u(t,x) = 2 + \exp(-(2\pi)^2 t)\cos(2\pi x) + \tfrac{1}{2}\exp(-(6\pi)^2)\sin(6\pi x).$$

Write a `function [error,xh,uh]=a05ex03solveThetaStep(p,M,theta,tau,flag)` which performs M steps $t \in (0, \tau, 2\tau, ..., M\tau = T)$ and solves the problem with discrete initial $u^0 = r_h u_0(x)$ and returns the `error` $= \max_k \max_i |u_i^k - u(k\tau, ih)|$, the grid `xh`, and `uh` $= u^M$.

(e) Make case studies for $T = 1$ with the following parameters:

i. $p = 5$, $M = 300$, $\theta = \{0, 1/2, 1\}$, `flag=0`,
ii. $p = 5$, $M = 300$, $\theta = \{0, 1/2, 1\}$, `flag=1`,
iii. $p = 10$, $M = 300$, $\theta = \{0, 1/2, 1\}$, `flag=0`,
iv. $p = 10$, $M = 300$, $\theta = \{0, 1/2, 1\}$, `flag=1`,
v. $p = 5$, $M = 3000$, $\theta = \{0, 1/2, 1\}$, `flag=0`,
vi. $p = 5$, $M = 3000$, $\theta = \{0, 1/2, 1\}$, `flag=1`.

If the error is unreasonably large, e.g. `error>1`, then return `error=Inf`. What is the error and the theoretical stability criterion from the lecture in each case (for both norms)?

(f) What order of convergence do you expect for different θ and `flag=0,1`? What are advantages and disadvantages of different θ and `flag`? For small τ check the EOC[2] as $h \to 0$ and for small h check the EOC for $\tau \to 0$ for $\theta = 1/2$ and `flag=1`.

Lesson: Here you learn about the practical advantages and disadvantages of various numerical schemes with different consistency orders in space and time.

4. Programming exercise: Parabolic problem with Dirichlet data **3 points**

How do you need to modify `Ah,Mh` above to account for homogeneous Dirichlet boundary conditions instead of periodic boundary conditions? Be specific and write `Ah,Mh` in a schematic form similar to the matrix eq. (1). You do not need to write a MATLAB program.

Note: The initial condition u^0 satisfies the boundary conditions.

Lesson: You learn how to implement boundary conditions for time-dependent problems.

total sum: 25 points

[1] As in the lecture this means $u(t, x+1) = u(t,x)$ for all x.
[2] EOC = experimental order of convergence