4. Assignment „Numerische Mathematik für Ingenieure II“

http://www.moses.tu-berlin.de/Mathematik/

Eigenvalues and boundary conditions

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1. **Exercise:** Eigenvalues and eigenfunctions of $-\Delta$ on a disc **8 points**

Consider the disc of radius 1 in two dimensions

$$B_1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\} \tag{1}$$

and the eigenvalue problem

$$-\Delta u = \lambda u, \qquad \text{in } B_1, \tag{2a}$$

$$u = 0, \qquad \text{on } \Gamma = \partial B_1. \tag{2b}$$

(a) Make an ansatz $u_c = a(r\sqrt{\lambda})\cos(n\phi)$ for $n \geq 0$ and $u_s = b(r\sqrt{\lambda})\sin(n\phi)$ for $n \geq 1$ in polar coordinates. The Laplace operator in polar coordinates is

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2}$$

Verify that the corresponding ODE for $a(x)$, $b(x)$ is Bessel's differential equation

$$x^2 a'' + xa' + (x^2 - n^2)a = 0, \qquad x^2 b'' + xb' + (x^2 - n^2)b = 0.$$

where $x = \sqrt{\lambda} r$. Solutions are given by Bessel functions of the first kind $J_n(x)$.

(b) How can one use the zeros[1] of J_n and eq. (2b) to determine λ?

(c) Determine the five smallest eigenvalues of (2) and their multiplicity. Plot the corresponding eigenfunctions (the script `polarplot.m` on the ISIS2 page might be helpful).

(d) Check the validity of the lower bound $\lambda \geq C_\Omega$, which we derived in the lecture.

Note: Here you lean how to solve a PDE by transforming the coordinates first.

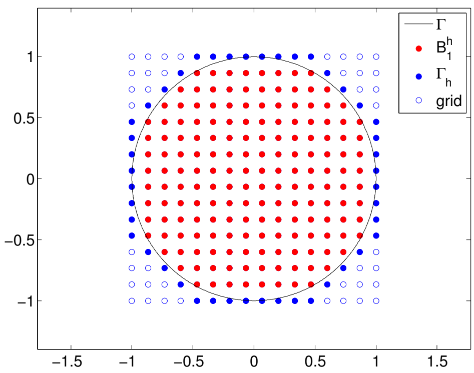
2. **Programming exercise:** Eigenvalues and eigenfunctions of $-\Delta_h$ on a disc **8 points**

Consider discrete eigenvalue problem for the Laplacian

$$\frac{-u_{(i-1),j} - u_{i+1,j} + 4u_{i,j} - u_{i,j-1} - u_{i,j+1}}{h^2} = \lambda u_{i,j}, \qquad x_{i,j} \in B_1^h \tag{3a}$$

$$u_{i,j} = 0, \qquad x_{i,j} \in \Gamma_h \tag{3b}$$

where $x_{i,j} = (ih, jh) - (1, 1)$, $h = 2/N$. Using the box $\bar{\Omega}^h = \{x_{i,j} : 0 \leq i, j \leq N\}$ we get a disc-like domain with B_1 from eq. (1) by defining $B_1^h = \bar{\Omega}^h \cap B_1$. For a 5-point stencil the boundary of B_1^h is $\Gamma^h = \{(x, y) \in \bar{\Omega}^h \setminus B_1^h : (x \pm h, y) \in B_1^h \text{ or } (x, y \pm h) \in B_1^h\}$ (see picture below for $N = 2^p - 1, p = 4$).



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[1] ger: "Nullstellen", values available for instance at `http://mathworld.wolfram.com/BesselFunctionZeros.html`

To solve this problem we will use a little trick, so that we can recycle solution `a02ex05getPDE5` from assignment 2.

(a) Write a `function [xh,yh,inB1,Ahr]=a04ex02LaplaceDisc(p)` which

- uses the function `a02ex05getPDE5` to create the matrix `Ah` and vectors `xh,yh` with $L = 2$ and shift the coordinates `xh,yh`, so that both run from -1 to $+1$,
- creates a boolean vector `inB1` $\in \{\text{true}, \text{false}\}^{(N+1)^2}$ which is `true` for the point `xh,yh` $\in \mathbb{R}^{(N+1)^2}$ being in B_1^h, and false otherwise,
- uses `Ahr = Ah(inB1,inB1)` to define the reduced Laplacian on B_1^h,

and returns these quantities as output. Explain how and why this is $-\Delta_h$ on B_1^h.

(b) Write a script `a04ex02solve.m` which solves the eigenvalue problem for `Ahr` using MATLAB's sparse eigenproblem solver `eigs`. Compare with the eigenvalues you computed in exercise 1. Therefore make a `loglog`-plot of the error $e_\lambda^h := |\lambda_h^k - \lambda^k|$ for $p = \{3, .., 10\}$ versus h. The plot should include a legend which identifies the eigenvalue. Export the plot into a file `a04ex02lambda.pdf`. Which eigenvalue is approximated best and what is the order of convergence of e_λ^h?

(c) Write a script `a04ex02plot.m`, which plots the eigenvectors corresponding to the smallest 7 eigenvalues for $p = 10$. **Hint:** If `vr` is an eigenvector of `Ahr`, then it can be extended to the full domain by

```
vext = zeros((N+1)^2,1); vext(inB1) = vr; vext = reshape(vext,[N+1 N+1]);
```

and then plotted using `surf(xh,yh,vext)` and compare the plot with 1(c). You might need to use `reshape` to adjust the dimension of `xh,yh` as well. Use `subplot` and export the plot to `a04ex02eigenfunc.jpg`.

Hint: What does the MATLAB command `inB1=((xh(:).^2+yh(:).^2)<1-eps)` do?

Note: Such a trick works well with Dirichlet condition on general domains, but the order of convergence is not so good. That is because generally we don't hit the boundary exactly but miss it with a distance $\sim h$. Double counting of multiple eigenvalues is ok in exercise 2.

3. Programming exercise: Boundary conditions **7 points**

Consider the problem

$$-u''(x) = f(x), \qquad \text{for } x \in (0, 1)$$

and its discrete version.

(a) Write a `function [xh,Ah]=a04ex03getPDE(p,flag0,flag1)` which returns the corresponding grid `xh` with $h = 1/(2^p - 1)$ and matrix `Ah` with standard 3-point stencil. The character `flag0,flag1` specifies the boundary condition at `x=0` and `x=1`. We use `'D'` for Dirichlet boundary conditions, `'N'` for Neumann boundary conditions.

(b) Solve the discrete problem with $u(0) = 0, u(1) = 0, f(x) = x$,

(c) Solve the discrete problem with $u(0) = 0, u'(1) = 0, f(x) = x$,

(d) Find $\rho \in \mathbb{R}$ such that there exists a solution with $u'(0) = 0, u'(1) = 0, f(x) = x - \rho$.

(e) Assume $\sum_{i=1}^{N-1} u_i = 0$. Find the unique solution with $u'(0) = 0, u'(1) = 0, f(x) = x - \rho$ using the extended equation

$$\begin{pmatrix} \mathtt{Ah} & b^t \\ b & 0 \end{pmatrix} \begin{pmatrix} u_h \\ \rho \end{pmatrix} = \begin{pmatrix} \tilde{f} \\ 0 \end{pmatrix}$$

where $b^t = (0, 1, 1, ..., 1, 1, 0)^t \in \mathbb{R}^{N+1}$ and $\tilde{f} = (0, f(h), f(2h), ..., f(1-h), 0)^t$. Write a separate `function [xh,Ah]=a04ex03getPDEext(p)` for the extended system with Neumann conditions at $x = \{0, 1\}$.

Plot all three solutions from (b,c,e) for $p = 10$ and interprete ρ in (d,e). For (b,c,e) write a script `a04ex03solve.m`.

Hint: For $u'(0)$ use D^+u and $u'(1)$ use D^-u as in the lecture.

Note: You see how serious the impact of boundary conditions is.

total sum: 23 points