Technische Universität Berlin **WS 2013/2014**
Fakultät II – Institut f. Mathematik
Instructor: Peschka Handed out: 5.11.
Tutor: Marschall Return during lecture: 13.11.

3. Assignment „Numerische Mathematik für Ingenieure II“

`http://www.moses.tu-berlin.de/Mathematik/`

A problem where finite differences on a nonuniform grid are helpful

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1. **Exercise:** Update of difference formula for nonuniform grid **6 points**

   Let $\bar{\Omega} = [0, 1]$ and the grid $\bar{\Omega}^h = \{x_0, ..., x_N\}$ where $0 = x_0 < x_1 < ... < x_N = 1$. Now fix $i = 1, .., N - 1$ and use $h_i := x_i - x_{i-1}$ to define the adapted differences

$$(D^- u)_i := \frac{u_i - u_{i-1}}{x_i - x_{i-1}} = \frac{u_i - u_{i-1}}{h_i},$$
$$(D^+ u)_i := \frac{u_{i+1} - u_i}{x_{i+1} - x_i} = \frac{u_{i+1} - u_i}{h_{i+1}},$$
$$(D^0 u)_i := \frac{u_{i+1} - u_{i-1}}{x_{i+1} - x_{i-1}} = \frac{u_{i+1} - u_{i-1}}{h_i + h_{i+1}}.$$

   where $u_i = u(x_i)$ and $u \in C^4(\bar{\Omega})$. For h_i = constant for all i such a $\bar{\Omega}^h$ is called *uniform grid*, otherwise it is called *nonuniform grid.*

   (a) Derive a formula for the second derivative of $u''(x_i)$ which uses $\{u_{i-1}, u_i, u_{i+1}\}$, i.e.

$$u''(x_i) = \alpha_i u_{i-1} + \beta_i u_i + \gamma_i u_{i+1} + R$$

   where $\alpha_i, \beta_i, \gamma_i$ depend on h_i, h_{i+1}.

   (b) What is the order p of the remainder for general h_i, h_{i+1}, i.e., $|R| < C_1 h^p$?

   (c) Under which special conditions on h_i, h_{i+1} is the order of the remainder better?

   Hint: Use Taylor's series as in the lecture.

2. **Programming exercise:** Elliptic boundary value problem **6 points**

   Implement a finite difference scheme for the problem

$$\text{(1)} \qquad -au''(x) + bu'(x) + cu(x) = f(x), \qquad x \in (0, 1)$$
$$u(0) = \alpha, \qquad u(1) = \beta,$$

   for a given arbitrary nonuniform mesh `xh=` $\bar{\Omega}^h$ with constants a, b, c, α, β.

   (a) Specify the resulting set of linear equations `Lh*uh=fh` using D^+ or D^- or D^0 as an approximation for the first order derivative in a compact form.

   (b) Write a MATLAB `function [Lh,fh] = a03ex02getPDE(xh,f,consts,flag)` where the model parameters are contained in `consts=[a b c alpha beta]` and $f_i = f(x_i)$ for $i = 0..N$. The character `flag` selects the approximation for u': `'-'` for D^-, `'+'` for D^+ and `'0'` for D^0.

   Assume `N=length(xh)-1`, then the function should either return a sparse $\mathbb{R}^{N+1\times N+1}$ or a sparse $\mathbb{R}^{N-1\times N-1}$ matrix `Lh` depending on whether you choose `Lh,fh,uh` to represent the full or reduced problem (your choice).

3. **Programming exercise:** Singularly perturbed boundary-value problem **8 points**

   Explanation: Consider the elliptic boundary value problem (1) with $a = \varepsilon, b = 1, c = 0, \alpha = \beta = 0, f(x) = 1$, where we are interested in small but positive values of ε. Simply setting

$a = 0$ does not help, because the solution (if exists) is not close to the one for $0 < a = \varepsilon \ll 1$ and it changes the order of the PDE. This basically defines a singularly perturbed problem. The exact solution for this problem

$$u_\varepsilon(x) = x - \frac{e^{-(1-x)/\varepsilon} - e^{-1/\varepsilon}}{1 - e^{-1/\varepsilon}},$$

is shown in the figure below. As we see $u_0(x) = \lim_{\varepsilon\to 0} u_\varepsilon(x) = x$ for $0 \le x < 1$, but $u_0(x)$ does not satisfy the boundary conditions at $x = 1$ in a smooth way. Thereby the solution creates a thin region near $x = 1$ (also known as boundary layer) where u_ε changes rapidly. The width of the region depends on ε and thereby derivatives of u_ε become large as $x \to 1$ and $\varepsilon \to 0$. That means that the constant $u^{(k)}(\xi)h^{k-2}$ $(k > 2)$ in the remainder of the difference quotient for u'' is big. In order to make the remainder small we need to make h very very small.

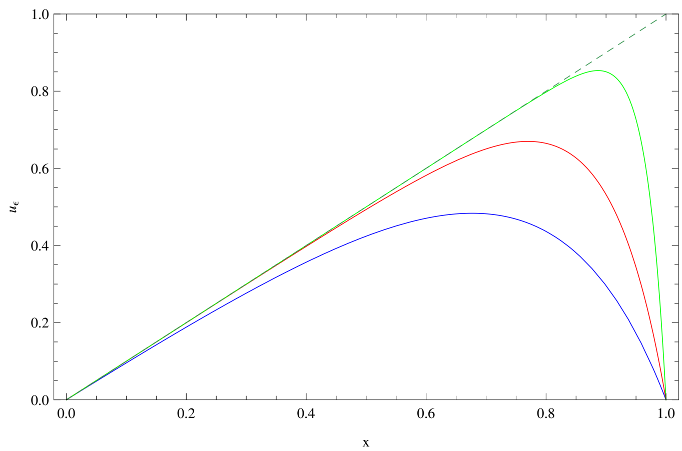

Figure: $u_\varepsilon(x)$ for $\varepsilon = \{1/5, 1/10, 1/30\}$ (blue,red,green) and the function $u_0(x) = x$ (dashed)

(a) Use the function from the previous programming exercise and write a `function uh=a03ex03solve(eps,xh,flag)` which returns the solution of the singularly perturbed problem for given `eps`$=\varepsilon$, grid `xh`, and approximation for the first derivative selected by `flag` as in the previous exercise.

(b) Write a `function [err,uex] = a03ex03error(eps,xh,uh)` that returns the error `err` between `uh` and the restricted exact solution, which is also to be returned as `uex`.

(c) Write a `function xh = a03ex03shishkin(k,sigma)` that generates a column vector of size 2k+1 describing a "Shishkin" grid `xh` that is defined by

$$\texttt{xh(i)} = \begin{cases} (\texttt{i}-1)H & \text{for} \quad \texttt{i} = 1,\dots,\texttt{k} \\ (1-\texttt{sigma}) + (\texttt{i}-\texttt{k}-1)h & \text{for} \quad \texttt{i} = \texttt{k}+1,\dots,2\texttt{k}+1 \end{cases}$$

where $H = (1-\texttt{sigma})/\texttt{k}$ and $h = \texttt{sigma}/\texttt{k}$.

(d) Write a script `a03ex03experiment.m` that plots the exact and approximated solution for `eps`=0.001 and all $k \in \{5, 50, 500, 5000\}$ on a

i. uniform grid with $h = 1/(2k)$ and forward difference operator D^+ (`flag='+'`).
ii. uniform grid with $h = 1/(2k)$ and central difference operator D^0 (`flag='0'`).
iii. uniform grid with $h = 1/(2k)$ and backward difference operator D^- (`flag='-'`).
iv. non-uniform Shishkin grid with `sigma=4*eps*log(2*k)` and central difference operator D^0 (`flag='0'`).

Use the command `figure(1),figure(2),figure(3),figure(4)` to group plots corresponding to i,ii,iii,iv in plot 1,2,3,4. Results with different k but same operator should go into the same figure. Use the command `title` to add information about the error for different k and the used operator.

Remark: If we write h in this exercise, we mean $h = \max_{\{i-1,i,i+1\}} h_i$.

total sum: 20 points