10. Assignment „Numerische Mathematik für Ingenieure II“

http://www.moses.tu-berlin.de/Mathematik/

Quadrature, nonsymmetric bilinear form and instationary problem.

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1. **Exercise:** Elliptic problem with convection term **8 points**

Consider the elliptic problem

$$(1) \qquad \begin{aligned} -(au')' + bu' + cu &= f, \qquad \text{in } (0,1) \\ u(0) = u(1) &= 0 \end{aligned}$$

with given functions a, b, c. Assume $0 < m \leq a \leq M$, $c \geq 0$, $|b| < \mu$ and $c - \frac{1}{2}b' \geq 0$.

(a) Derive a weak form of (1) and construct the corresponding bilinear form.

(b) As in the lecture show that the assumptions of the Lax-Milgram theorem hold.

(c) With basis w_i rewrite the problem using the matrices

$$S_{ij} = \int_{(0,1)} a\, w_i' w_j' \,\mathrm{d}x, \qquad M_{ij} = \int_{(0,1)} c\, w_i w_j \,\mathrm{d}x, \qquad C_{ij} = \int_{(0,1)} b\, w_i' w_j \,\mathrm{d}x,$$

Check the symmetry properties of S, M, C.

(d) Consider the weak form of (1) with constant b and integrate the convection term $\int bu'v \,\mathrm{d}x$ by parts. Set the occuring boundary terms to zero. Instead of pure Dirichlet conditions consider $u(0) = 0$ but natural boundary conditions at $x = 1$. What is the boundary condition now at $x = 1$ explicitly?

2. **Programming exercise:** Elliptic problem with convection term **8 points**

Consider the weak form of (1) derived in the previous exercise with constant coefficents $b, c \in \mathbb{R}$. As usual let w_i the standard finite element basis of piecewise linear functions on elements $\Omega_k = (x_{k-1}, x_k)$ for $k = 1 \ldots N$.

(a) Consider the matrix element corresponding to a convection term

$$C_{ij} = \int_{\Omega_k} w_i' w_j \mathrm{d}x.$$

Rewrite C as in integral in the reference configuration and transform the integral using integration by substitution.

(b) The functions `localstiff, localmass` are already available from previous assignments (in 1D with linear elements). In addition write a `function c=localconv(edet,dFinv)` which returns the local 2×2 matrix corresponding to C.

(c) Build and solve the problem with $a = 1/4, b = 1, c = 0, f = 1$ and compare with the exact solution (assignment 3, exercise 3). Modify `elliptic1d.m` from assignment 7.

(d) Now a is supposed to depend on x. Obviously the S_{ij} looks like

$$S_{ij} = \sum_k \sigma_{ij}^k \int_{\Omega_k} a(x) \,\mathrm{d}x$$

where σ_{ij}^k is known and constant matrix. Therefore we need to be able to compute the integral of $a(x)$ in Ω_k which we do using Gauss quadrature. Therefore write a `function qa = gaussquad(k,e2p,x,fun)` which accepts the element `k`, the decomposition `x,e2p` and a function handle `fun` to a `function y=fun(x)` and returns the integral of the function over the kth element. Use a 5-point Gauss-Legendre quadrature (e.g. see `http://en.wikipedia.org/wiki/Gaussian_quadrature`).

Now modify `elliptic1d.m` from assignment 7 by changing the 2×2 matrix `sloc` for the $k$the element with the result of the integration before assigning it to the global matrix ‘`a(k,:)=sloc(:)`’.

(e) Solve the problem

$$-\left(\frac{1}{1+\sqrt{x}}u'\right)' = 1, \qquad \text{in } (0,1)$$

with homogeneous Dirichlet boundary conditions. Explicitly check that the solution is

$$u(x) = \frac{x}{50}\left(27 + 18\sqrt{x} - 25x - 20x^{3/2}\right),$$

compare with the numerical solution, and create a plot.

(f) Is the solution in $H^1(0,1)$ and/or in $H^2(0,1)$ and/or in $C^2\overline{(0,1)}$?

3. Programming exercise: Heat equation with FEM **6 points**

Consider the heat-equation

$$\partial_t u - \partial_{xx} u = 0,$$

on $(0,1)$ with homogeneous Neumann boundary conditions and initial data u_0. Let $a(u,v)$ the bilinear form corresponding to the elliptic problem with Neumann conditions and $\langle u, v\rangle = \int_0^1 uv \, dx$ the scalar product in $L^2(0,1)$.

(a) Show the weak space-discrete form of the heat-equation is given by

$$\frac{d}{dt}\langle u_h(t), v_h\rangle + a(u_h, v_h) = 0$$

for all $v_h \in V_h \subset H^1(0,1)$. Furthermore show that after discretization in time by implicit Euler the equation becomes

$$(M + \tau A)\alpha^{k+1} = M\alpha^k.$$

where $u_h(t_k) = \sum_i \alpha_i^k w_i \in V_h$ for $t_k = k\tau$. What are M, A?

(b) Use the solutions of assignment 7 to modify `elliptic1d.m` into `parabolic1d.m` and construct $M + \tau A$ and M. Use $u_0 = 2 + \cos(\pi x)$, compute a solution on a sufficiently fine decomposition and τ and compare with the exact solution for $0 \le t \le 1$. As initial data for the discrete problem use $\alpha_i^0 = u_0(x_i)$.

total sum: 22 points

As usual, use sparse matrices.