1. Assignment „Numerische Mathematik für Ingenieure II“

http://www.moses.tu-berlin.de/Mathematik/

Some linear algebra, exact solutions of PDE, MATLAB warm-up

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1. Exercise: Some linear algebra **4 points**

This exercise is a little warm-up in linear algebra. Let $K_4 \in \mathbb{R}^{4\times 4}$ the following matrix

$$K_4 = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

which we will encounter in $1D$ elliptic boundary value problems. Note: For general $n \in \mathbb{N}$ the matrix has entries

$$(K_n)_{ij} = \begin{cases} 2 & 1 \le i = j \le n, \\ -1 & |i-j| = 1, 1 \le i, j \le n, \\ 0 & \text{otherwise} \end{cases}$$

(a) Show that K_4 is positive definite, i.e., $x^T K_4 x > 0$ for all $x \in \mathbb{R}^4 \setminus \{0\}$.

(b) Show that K_4 is invertible and the inverse is also symmetric, positive definite.

(c) Prove for general $n \in \mathbb{N}$ that $\det(K_n) = n + 1$. Hint: Use Laplace's formula and a proof by induction.

2. Exercise: Solutions of the transport equation **4 points**

For $\Omega = \mathbb{R}$ and $u : [0, \infty) \times \Omega \to \mathbb{R}$ consider the PDE

$$u_t + b\, u_x = 0$$

with initial data $u(0, x) = g(x)$ at time $t = 0$.

(a) Fix any point $(t, x) \in [0, \infty) \times \Omega$ and define $z(s) := u(t + s, x + sb)$. Using the PDE calculate z' and write down the general solution.

(b) Using $g \in C^1(\Omega)$ write down the general solution $u(t, x)$ in terms of g.

(c) Let $\Omega = \mathbb{R}^d$, $b \in \mathbb{R}^d$ and $u : [0, \infty) \times \Omega \to \mathbb{R}$. What is the general solution of the PDE

$$u_t + b \cdot \nabla u = 0$$

with initial data $g(x)$?

3. Exercise: Solutions of the heat equation **4 points**

For $\Omega = [0, \pi]$ find solutions $u : [0, \infty) \times \Omega \to \mathbb{R}$ of the PDE

$$u_t - u_{xx} = 0$$

with boundary conditions

$$u_x(t, 0) = u_x(t, \pi) = 0$$

and initial conditions.

$$u(0, x) = \cos^2 x.$$

Hint: separation of variables, superposition of Fourier modes, $\cos^2 x = \frac{1}{2}(\cos(2x) + 1)$.

4. Programming exercise: Getting started with MATLAB **3 points**

If necessary familiarize yourself with the MATLAB development environment in the UNIX pool. Study the tutorials on the ISIS website and learn using the help (using the commands `help` and `doc`). For this assignment you might want to study commands for generating sparse matrices such as `sparse`, `speye`.

(a) Write a MATLAB function `Kn = a01e04del2(n)` which generates the sparse matrix K_n.

(b) What are the advantages of working with sparse matrices?

5. Programming exercise: Inversion of K_n **8 points**

Consider the matrix K_n defined as in exercise 1.

(a) We want to compute solutions $x \in \mathbb{R}^n$ of $K_n x = b$ for any given $b \in \mathbb{R}^n$. Therefore study the LU (or Cholesky) decomposition of M_n using the MATLAB function `[L,U]=lu(Kn)` (or `C=chol(Kn)`) for various small n. Study the nonzero matrix components $(L)_{ij}$, $(U)_{ij}$ (or $(C)_{ij}$) and guess their values for general n, i.e. you dont have to prove that here. Using these write a function `x = a01e05invKn(b)` which returns the solution of $K_n x = b$ using forward and backward substitution.

(b) Use the commands `tic,toc` and the functions created before to test if you can beat the runtime of MATLAB's own

`Kn\b` or `full(Kn)\b` or `inv(full(Kn))*b`

using random vectors `b` of corresponding size for $n = 10, 100, 1000$. How do these four algorithms compare concerning the residuum $||K_n x - b||_2$ (MATLAB command `norm`)?

(c) What is the computational effort (runtime) of you `a01e05invKn` as $n \to \infty$

total sum: 23 points

Important note: The assignments are handed out in the lecture or available on the ISIS 2 webpage. The exercises are solved in **fixed groups of 3 students** (in special cases 2 or 4 is ok) and returned as stated on the assignment. Please **upload programming exercises** to the ISIS 2 webpage of the course (`https://www.isis.tu-berlin.de/2.0/course/view.php?id=620`) **or send by e-mail** to `peschka@wias-berlin.de`. Please use the name conventions e.g. `x = a01e05invMn(b)` and the corresponding filename `a01e05invMn.m` as specified in each programming exercise and state the members in your student group in the header of the file.

E.g. in the file `a01e05invMn.m` you find a MATLAB function starting with

```
function x = a01e05invMn(b)
% Assignment 01, Programming exercise 05, by Student 1, Student 2, Student 3
%
% Below you would find well documented MATLAB code
%
```