Possible exam questions for the course

“Numerische Mathematik 2 für Ingenieure”

Winter Semester 2013/2014 Dirk Peschka

Exams will take 20-30 min and will be conducted in English or German. Topics covered in the lecture are relevant, in particular the topics which were covered in assignments. These topics where:

1) General theory of PDEs
2) Finite Difference Methods
3) Finite Element Methods
4) Iterative Methods

The student may choose to start the oral exam with a presentation (topic is chosen by the student) not longer than 7 minutes. The presentation may use blackboard/whiteboard/paper; it may cover general PDE theory or aspects of finite elements or finite differences. Students may also choose to compactly present a solution to the optional problems. Students should expect a few follow-up questions following the presentation. Please do not choose topic 4) for a presentation, since 1,2,3) are more important in this course.

Afterward the exam will continue with various questions regarding the topics 1-4) above. Expect that each topic is covered at least with a few questions. Below you find a list of typical exam questions. Questions in the actual exam might be phrased differently; there may be additional or follow-up questions to an answer you provide.

General theory:

- What is the general form of a linear PDE of second order?
- What are the three different standard types of PDEs and how are they determined from the standard form?
- What types of boundary conditions are discussed in the course?
- What is the general form of a linear elliptic PDE?
- What are classical solutions for an elliptic problem with Dirichlet boundary conditions?
- When do we call a PDE well-posed/ill-posed? Can you give examples?
- Can you give examples for exact solutions of elliptic/parabolic equations (1,2 dimension, on the square, on a disc, on the whole space)?
- How can one solve the eigenvalue problem for the Laplace operator on a disc in two dimensions?
- What can one use Fourier series for?
- What is the statement of the discrete maximum principle and what is it used for?

Finite differences:

- What is the advantage of working with sparse matrices?
- How are finite difference methods derived?
- What are one-sided and central differences?
- How can one construct the global Laplace operator, e.g. with homogeneous Dirichlet conditions, from the difference stencil? What is lexicographical ordering?
- What are properties of the resulting linear algebraic systems; in which cases is the matrix symmetric?
- What are convergence and consistency orders of finite difference methods?
- How do we usually derive consistency orders and what are typical assumptions?
- What does stability mean for a finite difference method?
- When do we have convergence for a finite difference method?
- What is the classical 3-point stencil (1D), 5-point stencil (2D)?
- How can one derive finite difference approximations on nonuniform grids? For what type of problems are nonuniform grids useful?
- What is the classical stencil for the Laplace operator in n-dimensions?
- How can one implement homogeneous & inhomogeneous Dirichlet boundary conditions?

- How can one implement homogeneous & inhomogeneous Neumann boundary conditions?
- How do we solve the problem of the Poisson problem with homogeneous Neumann conditions not having a unique solution? Under what conditions (compatibility condition) do we have existence of solutions?
- How can one implement periodic boundary conditions?
- What are typical consistency/convergence orders of the Poisson equation?
- How can we discretize parabolic problems in space and time?
- Which advantages and disadvantages do implicit Euler/explicit Euler/Crank-Nicolson have?
- What is their consistency order and what are the assumptions?
- How can we show convergence for time-discretizations of parabolic problems in the L2 norm?
- What does the CFL condition state?
- What are advantages and disadvantages of finite difference methods?
- What are eigenfunctions/eigenvalues of the discrete Laplace operator constructed by the 3-point stencil on the unit interval?
- * How can we show stability?
- * How can we prove lower bounds for eigenvalues of the Laplace operator?

Finite elements:

- What is a weak solution? How does it compare to classical solutions? How is it derived?
- When is a weak solution also a classical solution? When is a classical solution also a weak solution?
- What the variational form of the Poisson equation (linear elliptic) PDE?
- How do we construct the Galerkin approximation? Show that there is an equivalent minimization problem.
- What are the ideas of Galerkin approximation with finite elements and what are essential differences to the finite difference method?
- How can one state the variational form in abstract form using bilinear and linear forms?
- What is an admissible decomposition?
- How are Dirichlet and Neumann boundary conditions treated in the finite element method?
- Why are they called essential and natural boundary conditions?
- What are examples of finite elements (decompositions and functions) treated in the lecture?
- What function spaces are typically used in the statement/analysis of variational forms? Why?
- How are the linear algebraic systems (Galerkin matrix) assembled?
- What is the idea behind shape functions and numerical quadrature?

- Can you tell/discuss the dimension of the discrete finite element space, provided you are given a triangulation and know that you have piecewise linear/quadratic basis functions? What happens if you have essential boundary conditions on some part of the domain?
- When is a bilinear form/Galerkin matrix symmetric, when is it positive definite?
- What is the content of the Theorem of Lax & Milgram and of Cea's Lemma?
- What is meant by the term "Galerkin orthogonality"?
- State variational forms satisfying the conditions of Lax & Milgram (and not satisfying them).
- State a problem where the bilinear form is nonsymmetric and discuss various possible natural boundary conditions.
- Under which conditions can we prove convergence of finite element methods?
- In which cases is it useful to consider basis functions which are piecewise polynomials of a higher degree?
- Can you give a simple example for an elliptic problem, which has a weak solution but no classical solution? Can you give the solution explicitly and discuss its properties?

Iterative methods:

- What is the energy norm of the error?
- How can we write a variational form as a minimization problem and are these two equivalent?
- How can we write a quadratic minimization problem with linear constraints as a linear equation?
- What is the idea behind the gradient descent (GD) method?
- How fast does the GD method converge? When does it converge fast, when slow?
- Derive the algorithm, based on the assumption that the direction is the residual?
- What are the ideas behind the CG method? What are Krylov spaces, in which sense is the CG optimal?
- In which norm do we compute/estimate the error for GD/CG and why?
- Describe the main ingredients of the CG method conjugate direction, Gram-Schmidt conjugation and how they lead to the efficient CG method.
- Let A be n x n matrix. Why does CG terminate in a finite number of n steps; in which cases can it converge faster?
- How fast does CG converge in general (in terms of the condition number of A)?
- What is the idea behind preconditioned CG and how is it implemented? Why does it work with matrices C=inv(B)*A, where A,B are SPD, but C is not symmetric?

Typical questions regarding the assignments/practical implementations might be stated as follows:

FD:

- How would you implement a Poisson equation with homogeneous Dirichlet boundary conditions in one dimension using finite differences?
- How would you implement Dirichlet/Neumann boundary conditions in the finite difference method (even in higher dimension)?
- What are periodic boundary conditions? What is their interpretation, when are they useful and how would you implement them?
- How would you solve a discrete Poisson problem/eigenvalue problem on a disc using finite differences? Which eigenvectors/eigenvalues are approximated well?
- Assume you have a matrix A corresponding to an elliptic operator L. How can you use it to solve a parabolic problem $u' + Lu=0$? (u' = derivative with respect to time)
- Discuss various reasons why to consider/not to consider
  i) higher order space discretization, ii) higher order time discretization.
  Give explicit examples!
- Show the equivalence of $Ax=b$ and $\min (x,Ax)/2-(b,x)$.

FEM:

- Derive a weak form of the Poisson problem with homogeneous Dirichlet boundary conditions.

- … with inhomogeneous Neumann boundary conditions on some part of the boundary, homogeneous Dirichlet on some other part of the boundary.
- How do you compute stiffness matrices and mass matrices (1D/2D)?
- How do we account for Dirichlet conditions in the FEM in practice?
- Which computations are necessary to compute the Galerkin matrix for a 2D Poisson problem? Be ready to explain *element generation, computation of transformation, computation of local matrices, and construction of the global matrix* in a little more detail (assignment 7/8).
- What is a major difference between the computation of the stiffness matrix in 1D and in 2D? (Linear elements, linear transformation)
- How can we compute the stiffness matrix on a single element using shape functions?
(Linear elements, linear transformation, 1D & 2D)
- How do we treat Neumann conditions and Dirichlet conditions?
- How are piecewise quadratic elements treated in the finite element method? How can we ensure continuity of the basis functions (by construction from shape functions)?
- How many quadratic shape functions do we have in 1D, 2D, 3D on the reference interval, triangle, and tetrahedron?
- What is the role of the variable e2p (1D, 2D, linear elements, quadratic elements)?
- How do we define basis functions from shape functions (using e2p)?

- How can we treat convection terms, space dependent coefficients (say space dependent diffusion constant)?
- How can we treat parabolic problems using finite elements?
- Are the piecewise linear basis functions (1D) in H^1, H^2, C^0, C^1, C^2 on (0,1)?

Iterative methods:

- Show that the extended system (corresponding to minimization with constraint) is invertible, if the constraint has full rank.
- Explain how you can use this constraint to enforce:
  i) Dirichlet boundary conditions, ii) the integral over the solution to vanish, to get solutions with homogeneous Neumann boundary conditions.
- Derive the gradient descent method
- Derive the CG method
- Derive the PCG method