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Existence results for a contact problem with varying friction coefficient and nonlinear forces

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Abstract

We consider the rate-independent problem of a particle moving in a three - dimensional half space subject to a time-dependent nonlinear restoring force having a convex potential and to Coulomb friction along the flat boundary of the half space, where the friction coefficient may vary along the boundary. Our existence result allows for solutions that may switch arbitrarily often between unconstrained motion in the interior and contact where the solutions may switch between sticking and frictional sliding. However, our existence result is local and guarantees continuous solutions only as long as the convexity of the potential is strong enough to compensate the variation of the friction coefficient times the contact pressure. By simple examples we show that our sufficient conditions are also necessary.

Our method is based on the energetic formulation of rate-independent systems as developed in [MTL02, MT04]. We generalize the time-incremental minimization procedure of [MR06] for the present situation of a non-associative flow rule.

1 Introduction

The mathematical work on friction problems falls into two categories. In the first area one is dealing with an elastic body that may come into unilateral contact with a given surface. Thus, this area deals with partial differential equations formulated in Hilbert spaces. In the second area one is interested in a finite-dimensional system that models one or several rigid bodies that are driven by external and internal forces and may have unilateral contact with given surfaces or amongst each other. We call the first area the "continuous case" and the second the "discrete case". The latter case also appears when the first one is spatially discretized for numerical purposes.

A further criterion to distinguish frictional problems are the forces which are considered in the models. In *dynamic problems* all possible forces (i.e. inertial, viscous, elastic and frictional forces) are modeled. In *quasi-static problems* the inertial forces are neglected. If additionally no viscous forces are considered the problem turns out to be *rate-independent*. In *static* problems all data is assumed to be constant in time. We give a short overview of the corresponding formulas:

$$\begin{array}{ll}
\text{dynamic } (m>0,\ \nu\geq 0): & 0 \in \underbrace{m\ddot{z}}_{\text{inertial force}} + \underbrace{\nu\dot{z}}_{\text{viscous force}} + \underbrace{r(t,z,\dot{z})}_{\text{frictional forces}} - \underbrace{F(t,z)}_{\text{restoring force}} \\
\text{quasi-static } (\nu\geq 0): & 0 \in \nu\dot{z} + r(t,z,\dot{z}) - F(t,z) \\
\text{rate independent:} & 0 \in r(t,z,\dot{z}) - F(t,z) \\
\text{static:} & 0 \in r(z,0) - F(z)
\end{array}$$

Most articles in literature dealing with quasi-static problems assume $\nu = 0$. We prefer to call them rate-independent.

For the *continuous case* Signorini [Sig59] was the first to formulate the *static problem* of a linearly elastic body submitted to frictionless unilateral contact with a rigid obstacle. The problem was solved by Fichera [Fic72]. Then, Duvaut and Lions [DL72, DL76] gave the first proper formulation of unilateral contact with friction. The first existence results for this static friction problem were obtained by Nečas, Jarušek and Haslinger [NJH80] using shifting techniques and fixed point arguments. Eck and Jarušek [EJ00] improved the result using a penalization method for the unilateral boundary conditions.

The *quasi-static friction* problem on a continuous level was first solved by Klarbring, Mikelić and Shillor [KMS88, KMS89, KMS91]. In their models they still had to regularize the boundary conditions and the friction using a so called non-local Coulomb law. The problem was solved without any regularization and with a local friction law in the work of Andersson [And00] or in the works of Rocca and Cocou [Roc99, Roc01]. An extension to nonlinear elasticity but non-local friction laws can already be found in [TM05].

In all the continuous cases the analytical results assumes small displacements. This makes the unilateral constraint easier since the tangential displacement is neglected. For instance, if the admissible domain $\mathcal{A}$ is given in the form $\{z \in \mathbb{R}^d \,:\, \Phi(z) \leq 0\}$ with the friction surface $\partial\mathcal{A}$, then the correct, geometrically exact constraint reads $\Phi(x+u(t,x)) \leq 0$ for all points in the body $x \in \Omega$, where $u(t,\cdot) : \Omega \to \mathbb{R}^d$ denotes the displacement. As u is assumed to be small this possibly nonconvex constraint is replaced by the simpler convex condition $\Phi(x) + \nabla\Phi(x) \cdot u(t,x) \leq 0$ for all $x \in \partial\Omega$. In the *discrete case* one is interested in rigid bodies or systems of rigid bodies with large displacement. Hence, one always uses the geometrically exact unilateral condition and hence must deal with the arising non-convexity.

For the *discrete case*, Jankovsky [Jan81] was the first to treat the *static friction problem* with unilateral contact. He obtained existence for all friction coefficients and uniqueness for "small" coefficients. Later Alart [Ala93] derived a necessary and sufficient condition for uniqueness. Klarbring [Kla90] provided a *quasi-static* two-degree of freedom model that displays non-uniqueness and non-existence. We recall this model in Section 4.2. The quasi-static frictional contact problem was solved for a linear elastic body with finitely many degrees of freedom and a flat obstacle by Andersson [And99].

Martins et al. [PdCM03, GMM98] investigated quasi-static friction problems with two degrees of freedom with nonlinear elasticity and curved obstacles. They study situations in which discontinuous solutions may appear. For recent results on the *discrete, dynamic* case that includes inertial terms we refer to [MM93, MMMP05, BB05].

The present work is also devoted to the *discrete case*. We approach the general problem of a mass-less particle that is subject to a general restoring force and to Coulomb friction if it hits a unilateral constraint. For simplicity we formulate everything in three dimensions and assume that the admissible set is the upper half space $\mathcal{A} = \{z \in \mathbb{R}^3 \,:\, z_3 \geq 0\}$. In contrast to the usual modeling with a constant coefficient of friction (see the above literature) we allow for a general smooth dependence of the friction coefficient $\mu(z_1, z_2)$ on the contact points $(z_1, z_2) \in \partial\mathcal{A}$. In fact, we will allow for more general friction by introducing a matrix that may model some anisotropy concerning the sliding directions of the point. Together with the nonlinear restoring force this makes our model general enough to extend our existence result to the situation of a curved obstacle by using coordinate transformations that flatten the boundary. The corresponding interplay of the curvature of the boundary and the convexity of the force potential will be studied in subsequent work [Sch07].

Since we neglect inertia and viscosity terms our evolution problem turns out to be rate-independent, i.e. a rescaling of time in the input functionals leads to the same rescaling of time in the solution z. By $z \in \mathcal{A} \subset \mathbb{R}^3$ we denote the position of the particle and by $F(t,z) \in \mathbb{R}^3$ the restoring force, then the contact problem with isotropic friction reads

$$0 \in C(z, \dot{z}) - F(t, z) \quad \subset \mathbb{R}^3,$$

where $C(z, \dot{z})$ is the set of possible contact forces. For $z_3 > 0$ we have no contact and set $C(z, v) = \{0\}$. In case of contact with $z_3 = \dot{z}_3 = 0$ we have the friction cone

$$C(z, \dot{z}) = \Big\{\Sigma \in \mathbb{R}^3 \,:\, (\Sigma_1^2{+}\Sigma_2^2)^{1/2} \leq \mu(z_1, z_2)\Sigma_3,\ \Sigma_1\dot{z}_1 + \Sigma_2\dot{z}_2 = \mu(z_1, z_2)\Sigma_3(\dot{z}_1^2 + \dot{z}_2^2)^{1/2}\Big\}.$$

To explain our approach we reformulate this problem into an *energetic formulation* for rate-independent systems similar to the one introduced in [MT99, MTL02, MT04]. This formulation was originally developed to model shape-memory alloys but is now shown to apply to many different rate-independent material models such as finite-strain elastoplasticity, damage, brittle fracture, delamination and vortex pinning in superconductors (cf. [SM05]). We refer to [Mie05] for a survey.

The energetic formulation is based on a *energy functional* $\mathcal{E}(t, z)$ and a *dissipation functional* $\Psi(t, z, \dot{z})$. The functional $\mathcal{E}$ describes the energy that is stored in the system at time t if the particle is in the position $z \in \mathcal{A}$. In particular we have $F(t, z) = -\mathrm{D}\mathcal{E}(t, z)$. We define the dissipation functional $\Psi : [0, T] \times \mathcal{A} \times \mathbb{R}^3 \to [0, \infty)$ in such a way that it depends on the positive part of the normal pressure $\sigma(t, z)_+ = \max\{0, \langle -\mathrm{D}\mathcal{E}(t, z), (0, 0, -1)\rangle\}$, namely $\Psi(t, z, v) = \sigma(t, z)_+ \, \mu(z_1, z_2)(v_1^2{+}v_2^2)^{1/2}$ for $z_3 = 0$ and $\Psi \equiv 0$ for $z_3 > 0$.

The general theory is based on a purely static stability condition (S) and the energy balance (E). The conditions (S) and (E) have to hold for all $t \in [0, T]$:

$$\mathcal{E}(t, z(t)) \leq \mathcal{E}(t, y) + \Psi(t, z(t), y - z(t)) \quad \text{ for all } y \in \mathcal{A}, \tag{S}$$

$$\mathcal{E}(0, z(0)) + \int_0^t \partial_\tau \mathcal{E}(\tau, z(\tau))\mathrm{d}\tau = \mathcal{E}(t, z(t)) + \int_0^t \Psi(\tau, z(\tau), \dot{z}(\tau))\mathrm{d}\tau. \tag{E}$$

We would like to mention that the stability condition (S) should not be confused with any notion of stability known from the theory of differential equations. It is also different from the global stability condition (S) in the energetic formulation used in [MT04, Mie05, MR06]. Our condition (S) corresponds rather to $(\mathrm{S})_{\mathrm{loc}}$ there.

In Section 2 we introduce our exact modeling including a more general friction law allowing for anisotropy in (2.1). We provide several equivalent and more common formulations for our problem, e.g., the formulation as variational inequality or as differential inclusion. Our main existence result is stated in Theorem 2.2 and the proof is worked out in Section 3. It is based on a semi-implicit time discretization. For a given partition $0 = t_0 < t_1 < \cdots < t_N = T$ we define the incremental minimization problems

$$z_k \in \operatorname{argmin}\,\{\mathcal{E}(t_k, y) + \Psi(t_{k-1}, z_{k-1}, y{-}z_{k-1}) \,:\, y \in \mathcal{A}\}$$

where the initial condition z_0 is given. As in [MR06] the essential step is to prove the estimate

$$\alpha_* \|z_k - z_{k-1}\| \leq \mathrm{c}_* \max\{|t_k - t_{k-1}|, |t_{k-1} - t_{k-2}|\} + \mathrm{q}_* \|z_{k-1} - z_{k-2}\|.$$

Our main assumption is then $\alpha_* > \mathrm{q}_*$ which implies an a priori Lipschitz estimate that is uniform for all equidistant partitions. Here α_* measures the uniform convexity of $\mathcal{E}(t, \cdot)$ and q_* accounts for the sum of two products, see (2.4) and (2.5). The first product involves the normal pressure σ_+ and the derivative of the friction coefficient $\mathrm{D}\mu$ and the second product involves the friction coefficient μ and some off-diagonal terms of the Hessian $\mathrm{D}^2\mathcal{E}(t, \cdot)$.

In Section 4 we illustrate the physical meaning of our assumptions by presenting two example for nonexistence of solutions. In the first example the first product in q_* is large whereas the second examples is from [Kla90] in which the second product in q_* is large.

2 Modeling and existence result

Apart of the usual modeling of the problem we will use, following [MT04], an energetic formulation. Equivalent differential inclusions and variational inequalities to this formulation will be presented later on.

In the following we call $\mathcal{A} := \{z \in \mathbb{R}^3 \,:\, z_3 \geq 0\}$ the *admissible set* of our solution and denote by $\nu := (0,0,-1)^\top$ the outward normal vector of the boundary $\partial\mathcal{A} = \{z \in \mathbb{R}^3 \,:\, z_3 = 0\}$ which presents our obstacle. We assume that from a physical point of view the dependence of the energy of the system on the time t and the position $z \in \mathcal{A}$ is known and we denote it by the *energy functional* $\mathcal{E}$

$$\mathcal{E} : [0,T] \times \mathbb{R}^3 \to \mathbb{R}.$$

Using $\mathcal{E}$, we can describe the *normal forces* σ at the time t to which the body is subject as $\sigma(t,z) := \langle -\mathrm{D}\mathcal{E}(t,z), \nu\rangle$, where $\langle\cdot,\cdot\rangle$ is the standard euclidian scalar product in $\mathbb{R}^3$. In most common models with Coulomb friction the roughness of the surface is modeled by some coefficient of friction $\mu : \partial\mathcal{A} \to [0,\infty)$. We will slightly generalize this description by allowing for some anisotropy depending on the direction in which our particle will slide. For this we will introduce the *matrix of friction* $\mathrm{M} : \partial\mathcal{A} \to \mathbb{R}^{3\times 3}$ that satisfies $\mathrm{M}(z)\nu = 0$ for all $z \in \partial\mathcal{A}$.

The *dissipation potential* $\Psi : [0,T] \times \mathcal{A} \times \mathbb{R}^3 \to [0,\infty)$ is now defined by

$$\Psi(t,z,v) := \begin{cases} \sigma(t,z)_+ \,\|\mathrm{M}(z)v\| & \text{if } z \in \partial\mathcal{A}, \\ 0 & \text{else,} \end{cases} \tag{2.1}$$

with $\sigma(t,z)_+ := \max\{0, \sigma(t,z)\}$ and $\|\cdot\|$ being the usual Euclidian norm. A careful checking of the article shows that all results remain valid if one choses any other norm on $\mathbb{R}^3$. For simplicity of notation we decided to restrict to the Euclidian norm. Note that $\Psi(t,z,v)$ has the physical dimension of a power.Using the homogenuity of degree 1 of $\Psi(t,z,\cdot)$ we may also write $\Psi(t,z,y-z)$ which has the physical dimension of an energy. It is a rough approximation of the energy that is dissipated due to friction if the particle slides instantaneously at time t from the position z to y. For $\mu = \mathrm{M}_{1,1} = \mathrm{M}_{2,2}$ and $\mathrm{M}_{i,j} = 0$ else, we are again in the usual isotropic setting of friction, as used in the introduction. After having introduced the energies $\mathcal{E}$ and Ψ we are now able to formulate our problem, which consist of a *stability condition* (S) and a *energy balance* (E). Note that this problem is rate-independent.

Problem 2.1 *For a given initial value $\tilde{z} \in \mathcal{A}$ and initial time $\tilde{T} \in [0,T)$ find a time span $\Delta \in (0, T-\tilde{T}]$ and a solution $z \in \mathrm{W}^{1,\infty}([\tilde{T},\tilde{T}+\Delta],\mathcal{A})$ such that $z(\tilde{T}) = \tilde{z}$ and for all $t \in [\tilde{T},\tilde{T}+\Delta]$ the following two conditions hold:*

$$\mathcal{E}(t,z(t)) \leq \mathcal{E}(t,y) + \Psi(t,z(t),y-z(t)) \qquad \textit{for all } y \in \mathcal{A}, \tag{S}$$

$$\mathcal{E}(\tilde{T},z(\tilde{T})) + \int_{\tilde{T}}^{t} \partial_\tau\mathcal{E}(\tau,z(\tau))\mathrm{d}\tau = \mathcal{E}(t,z(t)) + \int_{\tilde{T}}^{t} \Psi(\tau,z(\tau),\dot{z}(\tau))\mathrm{d}\tau. \tag{E}$$

Here we denote by $\dot{z} = \frac{\mathrm{d}}{\mathrm{d}t}z$ the derivative with respect to time. In the energy balance law (E) the integral on the left-hand side expresses the work done by external forces while the integral on the right-hand side expresses the total amount of energy that is dissipated due to friction along the path of $z : [0,T] \to \mathcal{A}$.

The stability condition (S) expresses that the amount of energy that we might gain by switching from $z(t)$ to any other admissible position y is less than the energy that has to be paid for this switch due to friction. Since $z(t)$ is a minimizer of the right-hand side in (S) and since $\mathcal{E}(t,\cdot)$ and $\Psi(t,z,\cdot)$ are convex, it is immediate that (S) is equivalent to

$$0 \in \partial_v \Psi(t,z(t),0) + \mathrm{D}\mathcal{E}(t,z(t)) + \partial\mathcal{X}_{\mathcal{A}}(z(t)).$$

Here $\mathcal{X}_{\mathcal{A}}$ is the characteristic function with $\mathcal{X}_{\mathcal{A}}(z) := \begin{cases} 0 & \text{for } z \in \mathcal{A}, \\ +\infty & \text{otherwise,} \end{cases}$ and $\partial\mathcal{X}_{\mathcal{A}}$ its subdifferential whereas $\partial_v\Psi$ denotes the subdifferential of $\Psi(t,z,\cdot)$.

Furthermore, if for some $0 < T_1 < T_2 \leq T$ the functions z_1 and z_2 satisfy (S) and (E) on the corresponding intervals $[0,T_1]$ and $[T_1,T_2]$ and z_2 further satisfies the initial condition $z_2(T_1) = \tilde{z}_2$ with $\tilde{z}_2 := z_1(T_1)$, then the concatenation $z(t) = \begin{cases} z_1(t) & \text{for} \quad t \in [0,T_1], \\ z_2(t) & \text{for} \quad t \in [T_1,T_2], \end{cases}$ satisfies (S) and (E) on the whole interval $[0,T_2]$. Thus, if we assume that a solution exists on the interval $[0,\tilde{T}]$, then Problem 2.1 suggests the existence of a local extension of the solution.

2.1 Equivalent formulations

Since the contact problem with friction is usually described using different formulations we would like to present equivalent and more familiar formulations of the conditions (S) and (E). For details of the proof of the equivalences see [MT04]. Recalling the definition of $\mathcal{X}_{\mathcal{A}}$ we rewrite (S) and (E) equivalently as the following *differential inclusion*:

$$0 \in \underbrace{\partial_v\Psi\big(t,z(t),\dot{z}(t)\big)}_{\text{frictional force}} + \underbrace{\mathrm{D}\mathcal{E}\big(t,z(t)\big)}_{-\text{potential restoring force}} + \underbrace{\partial\mathcal{X}_{\mathcal{A}}\big(z(t)\big)}_{\text{normal contact force}} \quad \text{for a.e. } t \in [\tilde{T},\tilde{T}+\Delta]. \tag{2.2}$$

The above differential inclusion is further equivalent to the following *variational inequality*:

$$0 \leq \langle \mathrm{D}\mathcal{E}(t,z(t)), v - \dot{z}(t)\rangle + \Psi(t,z(t),v) - \Psi(t,z(t),\dot{z}(t)) + \mathcal{X}_{\mathrm{T}_{\mathcal{A}(z)}}(v) - \mathcal{X}_{\mathrm{T}_{\mathcal{A}(z)}}(\dot{z}) \tag{2.3}$$
$$\text{for all } v \in \mathbb{R}^3 \text{ and a.a. } t \in [\tilde{T},\tilde{T}+\Delta],$$

where $\mathrm{T}_{\mathcal{A}}(z)$ the tangential cone $\mathrm{T}_{\mathcal{A}}(z) := \big\{v \in \mathbb{R}^3 \ :\ z + \lambda v \in \mathcal{A} \text{ for some } \lambda > 0\big\}$.

Next we are even more specific and we assume our energy to be quadratic, $\mathcal{E}(t,z) := \frac{1}{2}\langle \mathrm{H}z, z\rangle - \langle f(t), z\rangle$ with $\mathrm{H} \in \mathbb{R}^{3\times 3}$ being the symmetric and positive definite stiffness matrix and $f : [\tilde{T},\tilde{T}+\Delta] \to \mathbb{R}^3$ representing the external forces. Further we assume that we are in the situation of isotropic friction with a scalar coefficient of friction $\mu : \partial\mathcal{A} \to [0,\infty)$. For $z \in \mathrm{M}^3$ we denote by $z_{\mathsf{T}} \in \mathbb{R}^2$ the vector consisting of the first two components and by $z_{\mathsf{N}} = z_3 \in [0,\infty)$, then Problem 2.1 (or (2.2), (2.3)) is equivalent to finding the position $z = (z_{\mathsf{T}}, z_{\mathsf{N}}) : [\tilde{T},\tilde{T}+\Delta] \to \mathcal{A}$ and the reaction forces $r = (r_{\mathsf{T}}, r_{\mathsf{N}}) : [\tilde{T},\tilde{T}+\Delta] \to \mathbb{R}^3$ satisfying the following equations:

$$Hz - f(t) = r \quad \text{for all } t \in [\tilde{T},\tilde{T}+\Delta], \qquad \text{(equation of motion)}$$
$$-\, r_{\mathsf{T}} \in \mu(z_{\mathsf{T}}) r_{\mathsf{N}} \partial\|\cdot\|(\dot{z}_{\mathsf{T}}), \qquad \text{(Coulomb friction law)}$$
$$z_{\mathsf{N}} \geq 0,\ r_{\mathsf{N}} \geq 0,\ z_{\mathsf{N}} r_{\mathsf{N}} = 0. \qquad \text{(unilateral contact condition)}$$

2.2 General assumptions

To avoid disturbing repetitions we will now introduce the assumptions on the energy functional $\mathcal{E}$, the matrix of friction M and the initial condition $\tilde{z}$ in a generic way, so that they can be referenced in each section.

We start with the regularity assumption on the energy. Even if we do not need a second partial derivative in time of $\mathcal{E}$ let us for simplicity assume that

$$\mathcal{E} \in \mathrm{C}^2\big([0,T] \times \mathcal{A}, [0,\infty)\big). \tag{G1}$$

Further, we denote the Hessian matrix of $\mathcal{E}$ with respect to z by $\mathrm{H}(t,z) = \mathrm{D}^2\mathcal{E}(t,z) \in \mathbb{R}^{3\times 3}$. We now assume that $\mathcal{E}$ is α-uniformly elliptic in its second variable, i.e. there exists a positive constant $\alpha_* > 0$ such that the functional $\alpha(t,z) := \min\big\{\langle \mathrm{H}(t,z)v, v\rangle \,:\, v \in \mathbb{R}^3, \|v\| = 1\big\}$ satisfies

$$\alpha(t,z) \geq \alpha_* \qquad \text{for all } (t,z) \in [0,T] \times \mathcal{A}. \tag{G2}$$

For the initial condition $\tilde{z}$ we have to assume that it satisfies (S) and hence is stable at time $t = \tilde{T}$

$$0 \in \mathrm{D}\mathcal{E}(\tilde{T}, \tilde{z}) + \partial_v \Psi(\tilde{T}, \tilde{z}, 0) + \partial \mathcal{X}_{\mathcal{A}}(\tilde{z}). \tag{G3}$$

The next generic assumption we are going to make is about the regularity of the matrix of friction

$$\mathrm{M} \in \mathrm{C}^1\big(\partial\mathcal{A}, \mathbb{R}^{3\times 3}\big) \text{ with } \mathrm{M}(z)\nu = 0 \text{ for all } z \in \partial\mathcal{A}. \tag{G4}$$

Recall that we have defined ν as the unit outward normal vector.

While the above assumptions are somehow classical, the following assumption reveals the nature of our problem and governs the interplay between the different physical data. We introduce the function

$$\mathrm{q}(t,z) := \Big(\|\mathrm{DM}(z)\|\sigma(t,z)_+ + \|\mathrm{M}(z)\|\|(\mathrm{H}_{31}(t,z), \mathrm{H}_{32}(t,z))\|\Big), \tag{2.4}$$

which allows us to formulate the last major condition

$$\mathrm{q}(\tilde{T}, \tilde{z}) < \alpha(\tilde{T}, \tilde{z}). \tag{G5}$$

2.3 Existence result

Before we present our main result we introduce the function

$$\mathrm{c}(t,z) := \big(\|\mathrm{M}(z)\|{+}1\big)\, \|\partial_t \mathrm{D}\mathcal{E}(t,z)\|. \tag{2.5}$$

Theorem 2.2 (Existence of solution) *Let as assume that* (G1)–(G5) *hold, then Problem 2.1 has a solution, i.e. there exists* $\Delta > 0$ *and* $z \in \mathrm{W}^{1,\infty}([\tilde{T}, \tilde{T}{+}\Delta], \mathcal{A})$ *such that* (S), (E) *and the initial condition hold. Furthermore, for each* $\rho > 0$ *there exists a time span* $\Delta(\rho) > 0$ *such that the solution* z *satisfies*

$$\|\dot{z}\|_{\mathrm{L}^\infty([\tilde{T}, \tilde{T}+\Delta(\rho)])} \leq \frac{\mathrm{c}(\tilde{T}, \tilde{z})}{\alpha(\tilde{T}, \tilde{z}) - \mathrm{q}(\tilde{T}, \tilde{z})} + \rho.$$

In Section 4 we present two examples that help to understand the physical meaning of assumption (G5). The examples illustrate that no Lipschitz continuous solution exists in general as soon as (G5) does not hold along the solution path. In Section 4.1 we treat a case where $\|\mathrm{DM}(z)\|\sigma(t,z)_+$ is big while the second term in q vanishes. In Section 4.2 we recall the classical nonexistence example of [Kla90], where M is constant but $(\mathrm{H}_{31}, \mathrm{H}_{32})$ is large.

3 Proof of the existence result

The *basic structure* of the existence proof consists of three steps. In Subsection 3.1 we construct for a given time span $\Delta > 0$ a sequence of approximative solutions $(\hat{z}^l)_{l\in\mathbb{N}} \in \mathrm{W}^{1,\infty}([\tilde{T}, \tilde{T}+\Delta], \mathcal{A})$ using a time discretization technique. We follow the ideas developed in [Mie05], [MT04], [MR06] but need to make suitable adjustments to handle the noncontinuity of the dissipation Ψ, see (3.21).

In Subsection 3.2 we prove that if $\Delta > 0$ is chosen in an appropriate way there exists a global Lipschitz constant for all $\hat{z}^l$ and due to the compactness theorem of Arzela-Ascoli we extract a convergent subsequence $z^{l_k} \to z$ for $k \to \infty$ with some limit function $z \in \mathrm{W}^{1,\infty}([\tilde{T}, \tilde{T}+\Delta], \mathcal{A})$. In Subsection 3.3 we show that the function z represents a solution.

Next, we introduce an *auxiliary dissipation functional*. Since for fixed $(t, v) \in [0, T] \times \mathbb{R}^3$ the mapping $z \mapsto \Psi(t, z, v)$ is in general not continuous on $\mathcal{A}$, we will expand its definition for $z \in \partial\mathcal{A}$ to the whole set $\mathcal{A}$ and define the Lipschitz continuous functional

$$\tilde{\Psi} : [0, T] \times \mathcal{A} \times \mathbb{R}^3 \to [0, \infty) \quad \text{with} \quad \tilde{\Psi}(t, z, v) := \sigma(t, z)_+ \, \|\mathrm{M}(z)v\|. \tag{3.1}$$

Replacing the non-continuous functional Ψ by the Lipschitz continuous functional $\tilde{\Psi}$ will facilitate the construction of a Lipschitz continuous solution candidate z in Subsection 3.2. In fact, we will see in our construction that the obtained limit function z will satisfy $\Psi\big(t, z(t), v\big) = \tilde{\Psi}\big(t, z(t), v\big)$ for all $(t, v) \in [\tilde{T}, \tilde{T}+\Delta] \times \mathbb{R}^3$, which will allow us to rid ourselves of $\tilde{\Psi}$ again.

3.1 Time incremental minimization

To construct for a given time span $\Delta > 0$ a sequence of approximative solutions we solve a time discretized problem of the following type.

Definition 3.1 (Incremental Problem (IP)) *For a given partition Π of the time interval $[\tilde{T}, \tilde{T}+\Delta]$, i.e.*

$$\Pi : \ \tilde{T} = t_0 < t_1 < \cdots < t_{N_\Pi} = \tilde{T} + \Delta \qquad \text{with } N_\Pi \in \mathbb{N}$$

and a given initial value $\tilde{z} \in \mathcal{A}$ find a solution vector $(z_k)_{k=0,\ldots,N_\Pi}$ with $z_0 = \tilde{z}$ whose values z_k incrementally satisfy for $k = 1, \ldots, N_\Pi$

$$z_k \in \mathit{argmin}\Big\{\mathcal{E}(t_k, z) + \tilde{\Psi}(t_{k-1}, z_{k-1}, y - z_{k-1}) \ : \ y \in \mathcal{A}\Big\}. \tag{IP}$$

Here "argmin" denotes the set of all minimizers.

We are going to solve the incremental problem (IP) for a sequence of partitions $\big(\Pi^l\big)_{l\in\mathbb{N}}$ of the time interval $[\tilde{T}, \tilde{T}+\Delta]$:

$$\Pi^l : \ \tilde{T} = t_0^l < t_1^l < \cdots < t_{N_{\Pi^l}}^l = \tilde{T} + \Delta,$$

whose fineness f_{Π^l}, defined by $\mathrm{f}_{\Pi^l} := \max\big\{t_k^l - t_{k-1}^l \ : \ t_k^l, t_{k-1}^l \in \Pi^l \text{ for } 1 \le k \le N_{\Pi^l}\big\}$ tends to 0. Our aim is to show that the related sequence of solution vectors $(z_k^l)_{k=1,\ldots,N_{\Pi^l}}$ provides us with a good time discrete approximation of the solution $z \in \mathrm{W}^{1,\infty}([\tilde{T}, \tilde{T}+\Delta], \mathcal{A})$.

For simplicity of notation, we will assume in the following to be given an arbitrary partition Π of $[\tilde{T}, \tilde{T}+\Delta]$ and we will write z_k, t_k instead of z_k^l and t_k^l. A direct method in the calculus of variations provides us now immediately with the following result, since $\mathcal{E}(t_k, \cdot) + \tilde{\Psi}(t_{k-1}, z_{k-1}, \cdot - z_{k-1})$ is uniformly convex on the convex domain $\mathcal{A}$.

Lemma 3.2 (Existence and Uniqueness of the solution of (IP)) *Under the assumptions* (G1) *and* (G2) *there exists for any given partition* Π *and initial value* $\tilde{z} \in \mathcal{A}$ *a unique solution* $(z_k)_{k=0,\dots,N_\Pi}$ *of* (IP).

In the next lemma, we will show, that a discrete solution already has properties which are discrete versions of the properties the continuous solution will have. We recall, that due to assumption (G3) the initial condition $\tilde{z}$ is globally stable in the following sense:

$$\mathcal{E}(\tilde{T}, \tilde{z}) \leq \mathcal{E}(\tilde{T}, y) + \Psi(\tilde{T}, \tilde{z}, y - \tilde{z}) \quad \text{for all } y \in \mathcal{A}.$$

Since $\tilde{\Psi} \geq \Psi$, this implies global stability with respect to $\tilde{\Psi}$, too.

Lemma 3.3 (Properties of the solution of (IP)) *Let the assumptions* (G1) - (G5) *hold and assume* $\Pi : \tilde{T} = t_0 < \cdots < t_{N_\Pi} = \tilde{T} + \Delta$ *to be an arbitrary partition. Then the solution* $(z_k)_{k=0,\dots,N_\Pi}$ *of* (IP) *satisfies for each* $k = 0, \dots, N_\Pi$ *:*

1. *(stability)* $\mathcal{E}(t_k, z_k) \leq \mathcal{E}(t_k, y) + \tilde{\Psi}(t_{k-1}, z_{k-1}, y - z_k)$ *for all* $y \in \mathcal{A}$ *and*
2. *(unilateral contact condition)* $\langle z_k, \nu \rangle \sigma(t_k, z_k) = 0$.

Proof: ad 1. Since $(z_k)_{k=1,\dots,N_\Pi}$ is a solution of (IP) we have, for each $k = 1, \dots, N_\Pi$,

$$\mathcal{E}(t_k, z_k) + \tilde{\Psi}(t_{k-1}, z_{k-1}, z_k - z_{k-1}) \leq \mathcal{E}(t_k, y) + \tilde{\Psi}(t_{k-1}, z_{k-1}, y - z_{k-1}) \text{ for all } y \in \mathcal{A}.$$

Since $\tilde{\Psi}$ satisfies the triangle inequality

$$\forall (t, y) \in [0, T] \times \mathcal{A} \text{ and } \forall v_1, v_2 \in \mathbb{R}^3 : \quad \tilde{\Psi}(t, y, v_1 + v_2) \leq \tilde{\Psi}(t, y, v_1) + \tilde{\Psi}(t, y, v_2),$$

the stability follows easily.

ad 2. We only have to show that if $(z_k)_3 > 0$ then $\sigma(t_k, z_k) = 0$ holds. Assume $(z_k)_3 = c > 0$. Let us define $g(\lambda) := \mathcal{E}(t_k, z_k + \lambda\nu) + \tilde{\Psi}(t_{k-1}, z_{k-1}, \lambda\nu + z_k - z_{k-1})$. We obtain $g(\lambda) \geq g(0)$ for all $\lambda \leq c$, since z_k is a minimizer in $\mathcal{A}$. Because of $\mathrm{M}(z)\nu = 0$ we have $g(\lambda) = \mathcal{E}(t_k, z_k + \lambda\nu) + \tilde{\Psi}(t_{k-1}, z_{k-1}, z_k - z_{k-1})$ such that g is differentiable in 0 with

$$0 = \frac{\mathrm{d}}{\mathrm{d}\lambda} g(0) = \langle \mathrm{D}\mathcal{E}(t_k, z_k), \nu \rangle = \sigma(t_k, z_k),$$

which is the desired result for $k \in \{1, \dots, N_\Pi\}$. The argument for $k = 0$ is the same due to the stability of z_0. ■

3.2 Lipschitz continuity

The main step in the proof is to establish a uniform Lipschitz continuity of the discrete solutions independent of the partition. Since the proof is quite technical and perhaps difficult to read, we decided first to present a simplified version in Proposition 3.4, to make the reader familiar with the main ideas of the proof. The general case is presented in Proposition 3.6 with a complete proof. For the next proposition we define for a given time span $\Delta > 0$ the constants

$$\mathrm{c}_* := \mathrm{c}_1 + \mathrm{c}_2 \quad \text{with } \mathrm{c}_1 := \|\partial_t \mathrm{D}\mathcal{E}\|_{\mathrm{L}^\infty([\tilde{T}, \tilde{T}+\Delta] \times \mathcal{A})} \text{ and } \mathrm{c}_2 := \mathrm{c}_1 \|\mathrm{M}\|_{\mathrm{L}^\infty(\mathcal{A})}, \tag{3.2}$$

$$\mathrm{q}_* := \Big(\|\sigma\|_{\mathrm{L}^\infty([\tilde{T}, \tilde{T}+\Delta] \times \partial\mathcal{A})} \|\mathrm{DM}\|_{\mathrm{L}^\infty(\mathcal{A})} + \|\mathrm{M}\|_{\mathrm{L}^\infty(\mathcal{A})} \big\| \big(\mathrm{H}_{3,1}(t, z), \mathrm{H}_{3,2}(t, z)\big) \big\|_{\mathrm{L}^\infty([\tilde{T}, \tilde{T}+\Delta] \times \mathcal{A})} \Big). \tag{3.3}$$

Note that we have constants c_* and q_*, while elsewhere we consider functions depending on (t, z). The following global assumption (G5*) will guarantee for any given time span $\Delta > 0$ the existence of a solution on the whole interval $[\tilde{T}, \tilde{T} + \Delta]$.

Proposition 3.4 (Lipschitz continuity: global version) *Let us assume* (G1)–(G4) *and that*

$$\mathrm{c}_* < \infty, \quad \mathrm{q}_* < \alpha_* \tag{G5*}$$

hold with the constant α_* being defined as in assumption (G2)*, then for all partitions Π of $[\tilde{T}, \tilde{T}+\Delta]$ the unique solution $(z_k)_{k=0,\ldots,N_\Pi}$ of the corresponding incremental problem* (IP) *satisfies*

$$\|z_k - z_{k-1}\| \le \frac{\mathrm{c}_*}{\alpha_* - \mathrm{q}_*}\mathrm{f}_\Pi \quad \textit{for } k = 1,\ldots,N_\Pi. \tag{3.4}$$

Sketch of the Proof: We introduce the difference operator $\delta_k\zeta := \zeta_k - \zeta_{k-1}$ where ζ stands for t or z. Let $\Pi : \tilde{T} = t_0 < \cdots < t_{N_\Pi} = \tilde{T} + \Delta$ be a given partition. The existence of a solution $(z_k)_{k=0,\ldots,N_\Pi}$ of the corresponding incremental problem is clear due to Lemma 3.2. The key in proving (3.4) is to show for $k \in \{2,\ldots,N_\Pi\}$ the recursive estimate

$$\alpha_*\|\delta_k z\| \le \mathrm{c}_* \max\{\delta_k t, \delta_{k-1}t\} + \mathrm{q}_*\|\delta_{k-1}z\| \tag{3.5}$$

and for $k = 1$ the estimate

$$\alpha_*\|\delta_1 z\| \le \mathrm{c}_*\delta_1 t. \tag{3.6}$$

The rest will follow from an induction. We content ourselves with sketching the estimates for the prove of (3.5). The ideas for (3.6) are analogous. To keep notation simple we introduce for each $k \in \{1,\ldots,N_\Pi\}$ the functional $J_k(z) := \mathcal{E}(t_k, z) + \tilde{\Psi}(t_{k-1}, z_{k-1}, z - z_{k-1})$ which satisfies

$$J_k(y) - J_k(z_k) \ge \frac{\alpha_*}{2}\|y - z_k\|^2 \text{ for all } y \in \mathcal{A}, \tag{3.7}$$

with α_* being defined in assumption (G2).

By applying (3.7) twice, once for the choice k and $y = z_{k-1}$ and once for $k-1$ and $y = z_k$, we conclude that

$$\alpha_*\|\delta_k z\|^2 \le J_k(z_{k-1}) - J_k(z_k) + J_{k-1}(z_k) - J_{k-1}(z_{k-1}). \tag{3.8}$$

Note that $\tilde{\Psi}$ satisfies a triangle inequality with respect to its third argument, i.e.$\tilde{\Psi}(\tau, y, v) + \tilde{\Psi}(\tau, y, w) \ge \tilde{\Psi}(\tau, y, v + w)$ and $\tilde{\Psi}(\tau, y, 0) = 0$ holds. If we set $\tilde{\Psi}_k(v) := \tilde{\Psi}(t_k, z_k, v)$ the four terms involving $\tilde{\Psi}$ in (3.8) are $\tilde{\Psi}_{k-1}(0) - \tilde{\Psi}_{k-1}(z_k - z_{k-1}) + \tilde{\Psi}_{k-2}(z_k - z_{k-1}) - \tilde{\Psi}_{k-1}(z_{k-1} - z_{k-2})$. Hence, the right side of equation (3.8) is bounded by

$$\sum_{i,j=0}^{1}(-1)^{i+j+1}J_{k-i}(z_{k-j}) \le \sum_{i,j=0}^{1}(-1)^{i+j+1}\mathcal{E}(t_{k-i}, z_{k-j}) + \tilde{\Psi}_{k-1}(\delta_k z) - \tilde{\Psi}_{k-2}(\delta_k z). \tag{3.9}$$

Defining $t(r) := t_k - r \cdot \delta_k t$ and $z(s) := z_{k-1} + s \cdot \delta_k z$ we estimate the sum of the energies using the fundamental theorem of calculus by

$$\sum_{i,j=0}^{1}(-1)^{i+j+1}\mathcal{E}(t_{k-i}, z_{k-j}) = \int_0^1\int_0^1 \langle \partial_t \mathrm{D}\mathcal{E}\big(t(r), z(s)\big), \delta_k z\rangle(-\delta_k t)\ \mathrm{d}r\ \mathrm{d}s \le \mathrm{c}_1\delta_k t\|\delta_k z\|. \tag{3.10}$$

After some technical calculation we get for the difference of the dissipations

$$\tilde{\Psi}_{k-1}(\delta_k z) - \tilde{\Psi}_{k-2}(\delta_k z) \le \mathrm{q}_*\|\delta_k z\|\|\delta_{k-1}z\| + \mathrm{c}_2\delta_{k-1}t\|\delta_k z\|. \tag{3.11}$$

Now the estimates (3.8)–(3.11) yield $\alpha_*\|\delta_k z\|^2 \le (\mathrm{c}_* \max\{\delta_k t, \delta_{k-1}t\} + \mathrm{q}_*\|\delta_{k-1}z\|)\,\|\delta_k z\|$ but this proves exactly (3.5). Exploiting the stability of $(t_0, z_0) = (\tilde{T}, \tilde{z})$ one can prove in an analogous way (3.6) which is equivalent to $\|\delta_1 z\| \le \frac{\mathrm{c}_*}{\alpha_* - \mathrm{q}_*}\delta_1 t$. The proof of (3.4) is now done by induction. The estimate (3.6) represents the start of the induction. For the induction step we use the recursive estimate (3.5) and assume that (3.4) holds for $k-1$. We conclude

$$\alpha_*\|\delta_k z\| \leq \mathrm{c}_* \max\{\delta_k t, \delta_{k-1} t\} + \mathrm{q}_*\|\delta_{k-1} z\| \leq \mathrm{c}_* \mathrm{f}_\Pi + \mathrm{q}_* \frac{\mathrm{c}_*}{\alpha_* - \mathrm{q}_*} \mathrm{f}_\Pi \leq \alpha_* \frac{\mathrm{c}_*}{\alpha_* - \mathrm{q}_*} \mathrm{f}_\Pi .$$

This closes the induction and proves (3.4) for $k = 1, \ldots, N_\Pi$. ∎

Remark 3.5 *The above proof follows the ideas in [MR06], which treats a case where the dissipation potential Ψ is much better behaved. The main new point is the estimate* (3.11)*, which uses specific properties of the frictional contact problem. Here it is essential to use the fact that coming from non-contact into contact is quite different from losing contact. We refer to Step 2.3 in the proof of Proposition 3.6, in particular* (3.21)*.*

The observation that under the strong and global assumption (G5*) which includes the whole set $\mathcal{A}$ our discrete solutions are uniformly Lipschitz continuous reveals that for a short time span $\Delta > 0$ the solution values remain in a neighborhood of the initial value $\tilde{z}$. Hence the assumption (G5*), i.e. $\mathrm{q}_* < \alpha_*$, seems to be far too strong and we should be able to replace the assumption by a more *local* one. This motivates the definition of the functions q and c as in (2.4) and (2.5) respectively. The physical meaning of these functions will be illustrated in Section 4.

We now introduce definitions of local sets. For given $\gamma, \varepsilon > 0$ we denote by $B_\varepsilon(z)$ the closed ball $B_\varepsilon(z) := \left\{ w \in \mathbb{R}^3 \,:\, \|w - z\| \leq \varepsilon \right\}$ and by $C_{\gamma,\varepsilon}(t,z)$ the closed cylinder $C_{\gamma,\varepsilon}(t,z) := [t, t+\gamma] \times B_\varepsilon(z)$. Depending on γ, ε and corresponding to the function q and c we define, for fixed $(\tilde{T}, \tilde{z}) \in [0,T) \times \mathcal{A}$, the constants

$$\tilde{\mathrm{q}} := \left(\|\mathrm{DM}\|_{\mathrm{L}^\infty(B_\varepsilon(\tilde{z}))} |\sigma_+|_{\mathrm{L}^\infty(C_{\gamma,\varepsilon}(\tilde{T},\tilde{z}))} + \|\mathrm{M}\|_{\mathrm{L}^\infty(B_\varepsilon(\tilde{z}))} \|(\mathrm{H}_{31}, \mathrm{H}_{32})\|_{\mathrm{L}^\infty(C_{\gamma,\varepsilon}(\tilde{T},\tilde{z}))} \right), \tag{3.12}$$

$$\tilde{\mathrm{c}} := (\|\mathrm{M}\|_{\mathrm{L}^\infty(B_\varepsilon(\tilde{z}))} + 1) \|\partial_t \mathrm{D}\mathcal{E}\|_{\mathrm{L}^\infty(C_{\gamma,\varepsilon}(\tilde{T},\tilde{z}))} \text{ and} \tag{3.13}$$

$$\tilde{\alpha} := \inf \left\{ \alpha(\tau, y) \,:\, (\tau, y) \in C_{\gamma,\varepsilon}(\tilde{T}, \tilde{z}) \right\}. \tag{3.14}$$

This constants are local versions of the global constants q_* and c_* from above. The value $\tilde{\mathrm{q}}$ is situated between $\mathrm{q}(t,z)$ and q_*. Analog observations hold for $\tilde{\mathrm{c}}$ and $\tilde{\alpha}$.

Proposition 3.6 (Lipschitz continuity: local version) *Let* (G1)-(G5)*. Then there exists a time span $\Delta > 0$ and a constant $\tilde{C} > 0$ such that for any partition Π of the time interval $[\tilde{T}, \tilde{T} + \Delta]$ the solution $(z_k)_{k=0,\ldots,N_\Pi}$ of the corresponding incremental problem* (IP) *satisfies*

$$\|z_k - z_{k-1}\| \leq \tilde{C} \mathrm{f}_\Pi \tag{3.15}$$

for $k = 1, \ldots, N_\Pi$.

Further, for each $\rho > 0$ we can choose the time span $\Delta(\rho) > 0$ small enough to assure additionally $\tilde{C} \leq \frac{\mathrm{c}(\tilde{T},\tilde{z})}{\alpha(\tilde{T},\tilde{z}) - \mathrm{q}(\tilde{T},\tilde{z})} + \rho$.

Remark 3.7 *This implies a uniform Lipschitz continuity for a suitably large set of partitions including all equi-distant partitions. Choose $\delta \in (0,1)$, then for all partition $\Pi = \{t_k \,:\, k = 1, \ldots, N_\Pi\}$ satisfying*

$$\min \{ t_k - t_{k-1} \,:\, k = 1, \ldots, N_\Pi \} \geq \delta \mathrm{f}_\Pi$$

estimate (3.15) *implies $\|z_k - z_{k-1}\| \leq \dfrac{\tilde{C}}{\delta}(t_k - t_{k-1})$ for $k = 1, \ldots, N_\Pi$.*

Proof of Proposition 3.6:

Step 1. Localization. In the above proposition we replaced the global assumption (G5*) of Proposition 3.4 by the local assumption (G5). This forces us to restrict ourself to a small neighborhood of $(\tilde{T}, \tilde{z})$.

Step 1.1. Choosing the local set. Let us assume that $\mathrm{q}(\tilde{T}, \tilde{z}) < \alpha(\tilde{T}, \tilde{z})$ holds. Due to the continuity of $\mathcal{E}$ and M (see (G1) and (G4)) we now choose for a given $\rho > 0$ the values $\gamma, \varepsilon > 0$ such that the corresponding constants $\tilde{\mathrm{q}}, \tilde{\mathrm{c}}$ and $\tilde{\alpha}$, as they were defined in (3.12)–(3.14), satisfy $\tilde{\mathrm{q}} < \tilde{\alpha}$ and $\frac{\tilde{\mathrm{c}}}{\tilde{\alpha}-\tilde{\mathrm{q}}} \leq \frac{\mathrm{c}(\tilde{T},\tilde{z})}{\alpha(\tilde{T},\tilde{z})-\mathrm{q}(\tilde{T},\tilde{z})} + \rho$. In the following we will show that $\tilde{C} := \frac{\tilde{\mathrm{c}}}{\tilde{\alpha}-\tilde{\mathrm{q}}}$ is the desired Lipschitz constant. With the above constants $\tilde{\mathrm{c}}, \tilde{\mathrm{q}}$ and $\tilde{\alpha}$ we can do estimations on the cylinder $C_{\gamma,\varepsilon}(\tilde{T}, \tilde{z})$ and the ball $B_\varepsilon(\tilde{z})$ only. This motivates the introduction of the following local incremental problem (IP) $_{\mathrm{loc}}$.

Step 1.2. The localized incremental problem (IP)$_{\mathbf{loc}}$. This problem will depend on the two parameters $r > 0$ and $\Delta > 0$.

For any given partition $\Pi : \tilde{T} = t_0 < \cdots < t_{N_\Pi} = \tilde{T} + \Delta$, initial value $z_0 = \tilde{z}$ and radius $r > 0$ find, for $k = 1, \ldots, N_\Pi$,

$$z_k \in \operatorname{argmin}\Big\{\mathcal{E}(t_k, y) + \tilde{\Psi}(t_{k-1}, z_{k-1}, y - z_{k-1}) \,:\, y \in B_r(z_{k-1}) \cap \mathcal{A}\Big\}.$$

The existence and uniqueness of a solution $(z_k)_{k=0,\ldots,N_\Pi}$ is clear, see Lemma 3.2.

Step 1.3. Comparing (IP) and (IP)$_{\mathbf{loc}}$. Let us compare for fixed partition Π of $[\tilde{T}, \tilde{T}+\Delta]$, initial value z_0 and radius $r > 0$ the solution $(x_k)_{k=0,\ldots,N_\Pi}$ of the local incremental problem (IP)$_{\mathrm{loc}}$ with the solution $(y_k)_{k=0,\ldots,N_\Pi}$ of the original and global incremental problem (IP) (see 3.1).

In both problems we are looking, for each $k = 1, \ldots N_\Pi$, for minimizers of the functional $J_k(z) := \mathcal{E}(t_k, z) + \tilde{\Psi}(t_{k-1}, z_{k-1}, z - z_{k-1})$. Due to the uniform convexity of J_k, see (G2), both solutions are unique. Further, if for a given $k \in \{1, \ldots, N_\Pi\}$ the local solution satisfies $x_j \in \mathrm{int} B_r(x_{j-1})$ for $1 \leq j \leq k$ then if follows $x_k = y_k$.

Step 1.4. Choosing parameters in (IP)$_{\mathbf{loc}}$. Next we fix the parameters r and Δ in (IP)$_{\mathrm{loc}}$ such that we can expect the solutions to remain in the cylinder $C_{\gamma,\varepsilon}(\tilde{T}, \tilde{z})$. For this we choose the radius $r := \frac{\varepsilon}{2}$ and the time span $\Delta := \min\Big\{\gamma, \frac{\varepsilon}{2}\frac{\tilde{\alpha}-\tilde{\mathrm{q}}}{\tilde{\mathrm{c}}}\Big\}$. The latter choice is motivated by our conjecture that the solutions satisfy the Lipschitz constant $\tilde{C} = \frac{\tilde{\mathrm{c}}}{\tilde{\alpha}-\tilde{\mathrm{q}}}$.

Step 2: Recursive estimate. For the third and crucial part of the induction step we introduce the difference operator $\delta_k\zeta := \zeta_k - \zeta_{k-1}$, where ζ stands for t or z. Let us fix $k \in \{2, \ldots, N_\Pi\}$ and assume that z_{k-1} coincides with y_{k-1} of the solution of the global incremental problem (IP) defined in 3.1. Further we assume $z_{k-1} \in B_{\frac{\varepsilon}{2}}(\tilde{z})$. As a consequence we have $(t_j, z_j) \in C_{\gamma,\varepsilon}(\tilde{T}, \tilde{z})$ for $j \in \{k-2, k-1, k\}$. We next show the recursive estimate

$$\tilde{\alpha}\|\delta_k z\| \leq \tilde{\mathrm{c}} \max\{\delta_k t, \delta_{k-1} t\} + \tilde{\mathrm{q}}\|\delta_{k-1} z\|. \tag{3.16}$$

Step 2.1. Estimating by the functionals J_k. The first step in estimating $\|z_k - z_{k-1}\|$ is the inequality

$$J_k(y) - J_k(z_k) \geq \frac{\tilde{\alpha}}{2}\|y - z_k\|^2 \quad \text{for all } y \in B_{\frac{\varepsilon}{2}}(z_{k-1}) \cap \mathcal{A}. \tag{3.17}$$

In fact we express the difference $J_k(y) - J_k(z_k)$ by defining the function $z(\lambda) := z_k + \lambda(y - z_k)$ and using the fundamental theorem of calculus twice we get

$$J_k(y) - J_k(z_k) = \int_0^1 s \int_0^s \langle \mathrm{H}(t, z(rs))(y - z_k), (y - z_k)\rangle \mathrm{d}r\, \mathrm{d}s + g(y) \geq \tilde{\alpha}\frac{1}{2}\|y - z_k\|^2 + g(y).$$

with $g(y) := \langle \mathrm{D}\mathcal{E}(t, z_k), y - z_k\rangle + \tilde{\Psi}(t_{k-1}, z_{k-1}, y-z_{k-1}) - \tilde{\Psi}(t_{k-1}, z_{k-1}, z_k-z_{k-1})$.

Now, by definition the value z_k satisfies $z_k = \operatorname{argmin}\Big\{ J_k(y) \,:\, y \in B_{\frac{\varepsilon}{2}}(z_{k-1}) \cap \mathcal{A}\Big\}$, which is equivalent to $g(y) \geq 0$ for all $y \in B_{\frac{\varepsilon}{2}}(z_{k-1}) \cap \mathcal{A}$. This proves (3.17).

We apply (3.17) twice, once for the index k and $y = z_{k-1}$ and once for the index $k-1$ and $y = z_k$ and we conclude by adding the inequalities that

$$\tilde{\alpha}\|\delta_k z\|^2 \leq J_k(z_{k-1}) - J_k(z_k) + J_{k-1}(z_k) - J_{k-1}(z_{k-1}). \tag{3.18}$$

Our aim is now to estimate the right side of (3.18). Note that $\tilde{\Psi}$ satisfies a triangle inequality with respect to its third argument, i.e.$\tilde{\Psi}(\tau, y, v) + \tilde{\Psi}(\tau, y, w) \geq \tilde{\Psi}(\tau, y, v + w)$ and $\tilde{\Psi}(\tau, y, 0) = 0$ holds. Setting $\tilde{\Psi}_k(x) := \tilde{\Psi}(t_k, z_k, x)$ the right side of equation (3.18) is estimated by

$$\sum_{i,j=0}^{1} (-1)^{i+j+1} J_{k-i}(z_{k-j}) \leq \sum_{i,j=0}^{1} (-1)^{i+j+1}\mathcal{E}(t_{k-i}, z_{k-j}) + \tilde{\Psi}_{k-1}(\delta_k z) - \tilde{\Psi}_{k-2}(\delta_k z). \tag{3.19}$$

2.2. Estimating the energy terms. We define $t(r) := t_k - r \cdot \delta_k t$ and $z(s) := z_k + s \cdot \delta_k z$ and estimate the sum of the energies using the fundamental theorem of calculus by

$$\begin{aligned}\sum_{i,j=0}^{1} (-1)^{i+j+1}\mathcal{E}(t_{k-i}, z_{k-j}) &= \int_0^1 \int_0^1 \langle \partial_t \mathrm{D}\mathcal{E}\big(t(r), z(s)\big), \delta_k z\rangle(-\delta_k t) \; \mathrm{d}r \; \mathrm{d}s \\ &\leq \|\partial_t \mathrm{D}\mathcal{E}\|_{\mathrm{L}^\infty(C_{\gamma,\varepsilon}(\tilde{T},\tilde{z}))}\|\delta_k z\|\delta_k t.\end{aligned} \tag{3.20}$$

Step 2.3. Estimating the dissipation terms. The estimation of the difference of the dissipations in equation (3.19) is now quite technical and will be summarized in (3.22). We rewrite the difference by

$$\begin{aligned}\tilde{\Psi}_{k-1}(\delta_k z) - \tilde{\Psi}_{k-2}(\delta_k z) = \;& \sigma(t_{k-1}, z_{k-1})_+ \big(\|\mathrm{M}(z_{k-1})\delta_k z\| - \|\mathrm{M}(z_{k-2})\delta_k z\|\big) \\ &+ \big(\sigma(t_{k-1}, z_{k-1})_+ - \sigma(t_{k-2}, z_{k-2})_+\big)\|\mathrm{M}(z_{k-2})\delta_k z\|.\end{aligned}$$

We estimate the first term due the Lipschitz continuity of the matrix of friction M on $B_\varepsilon(z_0)$ (see (G4)) by $\|\sigma(t, z)_+\|_{\mathrm{L}^\infty(C_{\gamma,\varepsilon}(\tilde{T},\tilde{z}))}\|\mathrm{DM}\|_{\mathrm{L}^\infty(B_\varepsilon(z))}\|\delta_{k-1}z\|\|\delta_k z\|$ while we split the difference of the normal forces in the second term into

$$\big(\sigma(t_{k-1}, z_{k-1})_+ - \sigma(t_{k-1}, z_{k-2})_+\big) + \big(\sigma(t_{k-1}, z_{k-2})_+ - \sigma(t_{k-2}, z_{k-2})_+\big). \tag{3.21}$$

The second difference of the normal forces is dominated by $\|\partial_t \mathrm{D}\mathcal{E}\|_{\mathrm{L}^\infty(C_{\gamma,\varepsilon}(\tilde{T},\tilde{z}))}\delta_{k-1}t$. The first difference contains all the difficulties arising from switching between noncontact and contact. Note that we didn't use the modulus since we need to use sign conditions. We recall that the value z_{k-1} coincides with the solution of the global incremental problem (IP) and hence satisfies the unilateral contact condition due to the Lemma 3.3. Thus in the case $z_{k-1} \in \operatorname{int}\mathcal{A}$ we have $\sigma(t_{k-1}, z_{k-1})_+ = 0$ and the first difference of the normal forces is estimated by 0. Note that the above estimates would not work if the difference $\sigma^+_{k-1} - \sigma^+_{k-2}$ in (3.21) was splitted by inserting $\pm\sigma(t_{k-2}, z_{k-1})_+$.

In the case of $z_{k-1} \in \partial\mathcal{A}$ we have $\sigma(t_{k-1}, z_{k-1}) \geq 0$ and we control the first difference by

$$\begin{aligned}\langle \mathrm{D}\mathcal{E}(t_{k-1}, z_{k-1}), e_3\rangle - \langle \mathrm{D}\mathcal{E}(t_{k-1}, z_{k-2}), e_3\rangle &= \int_0^1 \langle \mathrm{H}(t_{k-1}, z(s))(\delta_{k-1}z), e_3\rangle \mathrm{d}s \\ &\leq \|(\mathrm{H}_{31}, \mathrm{H}_{32})\|_{\mathrm{L}^\infty(C_{\gamma,\varepsilon}(\tilde{T},\tilde{z}))}\|\delta_{k-1}z\|.\end{aligned}$$

Here we have used $\mathrm{H}_{33}(t_{k-1},z(s))(\delta_{k-1}z)_3 \le 0$ since $\mathrm{H}_{33}>0$ holds by the α-ellipticity (G2) and $(\delta_{k-1}z)_3 \le 0$ because of $(z_{k-1})_3=0$ while $(z_{k-2})_3 \ge 0$.

Summarizing the estimates for the difference of the dissipations we have

$$\tilde{\Psi}_{k-1}(\delta_k z)-\tilde{\Psi}_{k-2}(\delta_k z) \;\le\; \|\sigma_+\|_{\mathrm{L}^\infty(C_{\gamma,\varepsilon}(\tilde{T},\tilde{z}))}\|\mathrm{DM}\|_{\mathrm{L}^\infty(B_\varepsilon(\tilde{z}))}\|\delta_k z\|\|\delta_{k-1}z\| \\ +\Big(\|\partial_t \mathrm{D}\mathcal{E}\|_{\mathrm{L}^\infty(C_{\gamma,\varepsilon}(\tilde{T},\tilde{z}))}\delta_{k-1}t+\big\|\big(\mathrm{H}_{3,1},\mathrm{H}_{3,2}\big)\big\|_{\mathrm{L}^\infty(C_{\gamma,\varepsilon}(\tilde{T},\tilde{z}))}\|\delta_{k-1}z\|\Big)\|\mathrm{M}\|_{\mathrm{L}^\infty(B_\varepsilon(\tilde{z}))}\|\delta_k z\|. \tag{3.22}$$

Equations (3.18)–(3.22) together prove the desired recursive estimate (3.16).

Step 2.4. The case $k=1$. For $k=1$ we have to prove

$$\tilde{\alpha}\|\delta_1 z\| \le \tilde{\mathrm{c}}\delta_1 t. \tag{3.23}$$

Define $J_0(z):=\mathcal{E}(t_0,z)+\tilde{\Psi}(t_0,z_0,z-z_0)$, then the stability assumption (G3) on $(\tilde{T},\tilde{z})=(t_0,z_0)$ implies $z_0=\operatorname{argmin}\Big\{J_0(y)\ :\ y\in B_{\frac{\varepsilon}{2}}(\tilde{z})\cap\mathcal{A}\Big\}$. Analogously to (3.18) we get

$$\tilde{\alpha}\|\delta_1 z\|^2 \le J_1(z_0)-J_1(z_1)+J_0(z_1)-J_0(z_0). \tag{3.24}$$

Note, that this time the sum of the dissipation terms vanishes and we have

$$\sum_{i,j=0}^{1}(-1)^{i+j+1}J_{k-i}(z_{k-j})=\sum_{i,j=0}^{1}(-1)^{i+j+1}\mathcal{E}(t_{k-i},z_{k-j}) \le \|\partial_t \mathrm{D}\mathcal{E}\|_{\mathrm{L}^\infty(C_{\gamma,\varepsilon}(\tilde{T},\tilde{z}))}\|\delta_1 z\|\delta_1 t. \tag{3.25}$$

This proves (3.23).

Step 3: Induction. Let now Π be a given partition of $[\tilde{T},\tilde{T}+\Delta]$ and $(z_k)_{k=0,\dots,N_\Pi}$ a solution of the corresponding $(\mathrm{IP})_{\mathrm{loc}}$. Our aim is to prove by induction on $k=1,\dots,N_\Pi$ that

$$\|z_k - z_{k-1}\| \le \frac{\tilde{\mathrm{c}}}{\tilde{\alpha}-\tilde{\mathrm{q}}}\mathrm{f}_\Pi \tag{3.26}$$

holds with $z_k\in B_{\frac{\varepsilon}{2}}(\tilde{z})$ and z_k coincides with the value y_k of the corresponding global solution.

Step 3.1. Start of induction. The start of the induction follows from (3.23). Due to our choice of $\Delta>0$ we have

$$\|z_1-\tilde{z}\| \le \frac{\tilde{\mathrm{c}}}{\tilde{\alpha}}\mathrm{f}_\Pi \le \frac{\tilde{\mathrm{c}}}{\tilde{\alpha}}\Delta < \frac{\varepsilon}{2}.$$

As seen in Step 1.3 this also proves $z_1=y_1$ with y_1 being the global solution.

Step 3.2. Induction step. For the induction step we assume for $k\in\{2,\dots,N_\Pi\}$ that $\|z_{k-1}-z_{k-2}\| \le \frac{\tilde{\mathrm{c}}}{\tilde{\alpha}-\tilde{\mathrm{q}}}\mathrm{f}_\Pi$ holds with $z_{k-1}\in B_{\frac{\varepsilon}{2}}(\tilde{z})$ and z_k coincides with the global solution value y_k. Using the recursive estimate (3.16) we get

$$\tilde{\alpha}\|z_k-z_{k-1}\| \le \tilde{\mathrm{c}}\mathrm{f}_\Pi+\tilde{\mathrm{q}}\frac{\tilde{\mathrm{c}}}{\tilde{\alpha}-\tilde{\mathrm{q}}}\mathrm{f}_\Pi=\tilde{\alpha}\frac{\tilde{\mathrm{c}}}{\tilde{\alpha}-\tilde{\mathrm{q}}}\mathrm{f}_\Pi.$$

This proves (3.26). Again our choice of $\Delta>0$ together with the argument of Step 1.3 proves $z_k\in\operatorname{int}B_{\frac{\varepsilon}{2}}(z_{k-1})$ and $z_k=y_k$ with y_k being the global solution. To show $z_k\in B_{\frac{\varepsilon}{2}}(\tilde{z})$ we estimate

$$\|z_k-\tilde{z}\| \le \sum_{j=1}^{k}\|z_j-z_{j-1}\| \le \frac{\tilde{\mathrm{c}}}{\tilde{\alpha}-\tilde{\mathrm{q}}}k\mathrm{f}_\Pi \le \frac{\tilde{\mathrm{c}}}{\tilde{\alpha}-\tilde{\mathrm{q}}}\Delta \le \frac{\varepsilon}{2}.$$

Step 3.3. Conclusion. We have shown that the local solution $(z_k)_{k=1,\dots,N_\Pi}$ of $(\mathrm{IP})_{\mathrm{loc}}$ coincides with the solution of the corresponding global incremental problem (IP). Further the solution satisfies $\|z_k - z_{k-1}\| \leq \frac{\tilde{\mathrm{c}}}{\tilde{\alpha}-\tilde{\mathrm{q}}}$ for all $k = 1, \dots, N_\Pi$. Since we have assured in Step 1.1 that $\frac{\tilde{\mathrm{c}}}{\tilde{\alpha}-\tilde{\mathrm{q}}} \leq \frac{\mathrm{c}(\tilde{T},\tilde{z})}{\alpha(\tilde{T},\tilde{z})-\mathrm{q}(\tilde{T},\tilde{z})} + \rho$ holds, this completes our proof. ■

With the help of our discrete solutions we now construct piecewise linear approximands.

Definition 3.8 (Approximative solution) *For a given partition Π of $[\tilde{T}, \tilde{T}+\Delta]$ and initial value $\tilde{z}$ let $(z_k)_{k=0,\dots,N_\Pi}$ be the unique solution of the incremental problem* (IP) *(see 3.1). Then we call the piecewise linear function $\hat{z} \in \mathrm{W}^{1,\infty}\big([\tilde{T}, \tilde{T}+\Delta], \mathcal{A}\big)$ defined by $\hat{z}(t) := z_{k-1} + (z_k - z_{k-1})\frac{t-t_{k-1}}{t_k - t_{k-1}}$ for $t \in [t_{k-1}, t_k]$ and $k = 1, \dots, N_\Pi$ the* approximative solution*, that corresponds to the incremental problem* (IP).

Proposition 3.9 (Lipschitz continuous limit function) *If the assumptions* (G1)–(G5) *hold, then there exists a positive time span $0 < \Delta \leq T - \tilde{T}$ such that the following situation holds.*

For each equi-distant sequence of partitions $\big(\Pi^l\big)_{l\in\mathbb{N}}$ of the time interval $[\tilde{T}, \tilde{T}+\Delta]$ with $\mathrm{f}_{\Pi^l} \to 0$ for $l \to \infty$ the corresponding sequence of approximative solutions $\big(\hat{z}^l\big)_{l\in\mathbb{N}}$ has a convergent subsequence $(\hat{z}^{l_k})_{k\in\mathbb{N}}$, i.e. there exists a limit function *$z \in \mathrm{W}^{1,\infty}\big([\tilde{T}, \tilde{T}+\Delta], \mathcal{A}\big)$ such that*

$$\|\hat{z}^{l_k} - z\|_{\mathrm{L}^\infty([\tilde{T},\tilde{T}+\Delta],\mathcal{A})} \to 0 \quad \text{for} \quad k \to \infty$$

holds.

Further we can choose for each $\rho > 0$ a time span $\Delta(\rho) > 0$ that is small enough to assure that the limit function satisfies, on $[\tilde{T}, \tilde{T}+\Delta(\rho)]$, a Lipschitz constant $\tilde{C}$ with

$$\tilde{C} \leq \frac{\mathrm{c}\big(\tilde{T},\tilde{z}\big)}{\alpha\big(\tilde{T},\tilde{z}\big) - \mathrm{q}\big(\tilde{T},\tilde{z}\big)} + \rho.$$

Proof: The result follows directly from Remark 3.7 and the Arzela-Ascoli theorem. ■

3.3 Existence of solutions

Our remaining task is to show, that the limit function $z \in \mathrm{W}^{1,\infty}\big([\tilde{T}, \tilde{T}+\Delta], \mathcal{A}\big)$ of Proposition 3.9 provides us with a solution. For this we assume for the whole subsection that the assumptions (G1)–(G5) hold and we further denote our limit function by z, the corresponding (sub)sequence of approximative solutions by $\big(\hat{z}^l\big)_{l\in\mathbb{N}}$ and by $\Pi^l: \ \tilde{T} = t_0^l < \dots < t^l_{N_{\Pi^l}} = \tilde{T}+\Delta$ the corresponding (sub)sequence of uniform partitions that satisfies $\mathrm{f}_{\Pi^l} \to 0$ for $l \to \infty$.

Proposition 3.10 (Unilateral contact condition) *If our limit function z satisfies $z(t) \in \mathrm{int}\mathcal{A}$, for some $t \in [\tilde{T}, \tilde{T}+\Delta]$, then $\big\langle \sigma(t, z(t)), \nu \big\rangle = 0$ holds.*

Proof: This is a direct consequence of the unilateral contact condition of the discrete solutions (see Lemma 3.3) and Proposition 3.9. ■

Corollary 3.11 *As a consequence of Proposition 3.10 we have, along any limit function $z : [\tilde{T}, \tilde{T}+\Delta] \to \mathcal{A}$, the equality $\tilde{\Psi}\big(t, z(t), v\big) = \Psi\big(t, z(t), v\big)$ for all $t \in [\tilde{T}, \tilde{T}+\Delta]$ and $v \in \mathbb{R}^3$.*

Lemma 3.12 (Stability of z) *For all $t \in [\tilde{T}, \tilde{T}+\Delta]$ the limit function z is stable, i.e.*

$$\mathcal{E}(t, z(t)) \leq \mathcal{E}(t, y) + \Psi\left(t, z(t), y - z(t)\right) \quad \textit{for all } y \in \mathcal{A}. \tag{3.27}$$

Proof: The discrete stability of solutions of (IP) in Lemma 3.3 together with the definition of the approximative solutions 3.8 provide us with

$$\mathcal{E}\left(t_k^l, \hat{z}^l(t_k^l)\right) \leq \mathcal{E}\left(t_k^l, y\right) + \tilde{\Psi}\left(t_{k-1}^l, \hat{z}^l(t_{k-1}^l), y - \hat{z}^l(t_{k-1}^l)\right) \tag{3.28}$$
$$\text{for all } l \in \mathbb{N}, k = 1, \dots, N_{\Pi^l} \text{ and all } y \in \mathcal{A}.$$

Let us now fix $t \in [\tilde{T}, \tilde{T}+\Delta]$. For each $l \in \mathbb{N}$ we choose $k(l) \in (0, \dots, N_{\Pi^l})$ such that $|t_{k(l)}^l - t| \leq \mathrm{f}_{\Pi^l}$ holds. Hence we have $|t_{k(l)}^l - t| \to 0$.
As a consequence we obtain

$$\|\hat{z}^l(t_{k(l)}^l) - z(t)\| \leq \|\hat{z}^l(t_{k(l)}^l) - \hat{z}^l(t)\| + \|\hat{z}^l(t) - z(t)\| \to 0 \quad \text{for } l \to \infty,$$

due to the uniform Lipschitz continuity of the approximative solutions (see Remark 3.7). Our regularity assumptions on $\mathcal{E}$ and M allow us to pass to the limit in (3.28) and we get (3.27). Finally we replace $\tilde{\Psi}$ by Ψ, using Corollary 3.11. ■

A direct consequence of Lemma 3.12 is

Lemma 3.13 (Lower energy estimate) *For all $t \in [\tilde{T}, \tilde{T}+\Delta]$ we have the lower energy estimate*

$$\mathcal{E}(\tilde{T}, \tilde{z}) + \int_{\tilde{T}}^{t} \partial_t \mathcal{E}(\tau, z(\tau))\ d\tau \geq \mathcal{E}(t, z(t)) + \int_{\tilde{T}}^{t} \Psi\big(\tau, z(\tau), \dot{z}(\tau)\big)\ d\tau.$$

Proof: Since $z \in \mathrm{W}^{1,\infty}([\tilde{T}, \tilde{T}+\Delta], \mathcal{A})$, it is differentiable almost everywhere. Take a $t \in [\tilde{T}, \tilde{T}+\Delta]$ where z is differentiable and insert $y = z(t+h)$ into (3.27) , move $\mathcal{E}(t, z(t))$ to the right-hand side and divide by h. Since the mapping $\Psi(t, z, \cdot) : \mathbb{R}^3 \to [0, \infty)$ is homogeneous of degree 1 we conclude

$$0 \leq \langle \mathrm{D}\mathcal{E}(t, z(t)), \dot{z}(t)\rangle + \Psi(t, z(t), \dot{z}(t)) \quad \text{ for almost all } t \in [\tilde{T}, \tilde{T}+\Delta].$$

Integration from $\tilde{T}$ to t completes the proof. ■

In the following lemma, we will complete the proof of the Energy Equality (E) by deriving the opposite estimate. But let us say something about the convergence of $(\dot{\hat{z}}^l)_{l\in\mathbb{N}}$ first. By our construction we know that the sequence $(\hat{z}^l)_{l\in\mathbb{N}}$ and its limit z belong to $\mathrm{W}^{1,\infty}([\tilde{T}, \tilde{T}+\Delta], \mathcal{A})$ and are uniformly bounded. Since for $1 < p < \infty$ the space $\mathrm{W}^{1,p}([\tilde{T}, \tilde{T}+\Delta], \mathcal{A})$ is reflexive and the following embeddings $\mathrm{W}^{1,\infty} \hookrightarrow \mathrm{W}^{1,p} \hookrightarrow \mathrm{W}^{1,1}$ are all continuous, we choose a subsequence of $(\hat{z}^l)_{l\in\mathbb{N}}$, which we will still denote by $(\hat{z}^l)_{l\in\mathbb{N}}$, such that $\hat{z}^l \rightharpoonup z$ in $\mathrm{W}^{1,1}([\tilde{T}, \tilde{T}+\Delta], \mathcal{A})$.

Lemma 3.14 (Upper energy estimate) *For all $t \in [\tilde{T}, \tilde{T}+\Delta]$ the limit function z satisfies*

$$\mathcal{E}(\tilde{T}, \tilde{z}) + \int_{\tilde{T}}^{t} \partial_t \mathcal{E}(\tau, z(\tau)) d\tau \leq \mathcal{E}(t, z(t)) + \int_{\tilde{T}}^{t} \Psi\big(\tau, z(\tau), \dot{z}(\tau)\big) d\tau. \tag{3.29}$$

Proof: We start by deriving a discrete version of (3.29). Let $(\Pi^l)_{l\in\mathbb{N}}$ be a sequence of partitions with $\mathrm{f}_{\Pi^l} \to 0$. Let us fix $l \in \mathbb{N}$ and choose $y = z_{k-1}^l$ in (IP) on page 7. Then for any

$k = 1, \ldots, N_{\Pi^l}$, we have $\mathcal{E}(t_k^l, z_{k-1}^l) \le \mathcal{E}(t_k^l, z_k^l) + \tilde{\Psi}(t_{k-1}^l, z_{k-1}^l, z_k^l - z_{k-1}^l)$. Since $\tilde{\Psi}(t, z, \cdot) : \mathbb{R}^3 \to [0, \infty)$ is homogeneous of degree 1 this inequality is equivalent to

$$\mathcal{E}(t_{k-1}^l, \hat{z}^l(t_{k-1}^l)) + \int_{t_{k-1}^l}^{t_k^l} \partial_\tau \mathcal{E}(\tau, z_{k-1}^l)\ \mathrm{d}\tau \le \mathcal{E}(t_k^l, \hat{z}^l(t_k^l)) + \int_{t_{k-1}^l}^{t_k^l} \tilde{\Psi}\big(t_{k-1}^l, z_{k-1}^l, \dot{\hat{z}}^l(\tau)\big)\ \mathrm{d}\tau. \quad (3.30)$$

Next, we want to sum equation (3.30) over k. For this reason we make both integrands independent of the index k and we define the piecewise constant functions $\bar{z}^l$ and $\bar{\Psi}^l$ by $\bar{z}^l(t) := z_k^l$ and $\bar{\Psi}^l(t, z, w) := \tilde{\Psi}(t_k^l, z, w)$ for all $t \in [t_{k-1}^l, t_k^l)$ and all $k = 1, \ldots, N_{\Pi^l}$. Hence by adding up (3.30), we obtain for arbitrary $t^l \in \Pi^l$

$$\mathcal{E}(\tilde{T}, \tilde{z}) + \int_{\tilde{T}}^{t^l} \partial_\tau \mathcal{E}\big(\tau, \bar{z}^l(\tau)\big)\mathrm{d}\tau \le \mathcal{E}(t^l, \hat{z}^l(t^l)) + \int_{\tilde{T}}^{t^l} \bar{\Psi}^l\big(\tau, \bar{z}^l(\tau), \dot{\hat{z}}^l(\tau)\big)\mathrm{d}\tau. \quad (3.31)$$

It remains for us to choose a subsequence, such that (3.31) converges to (3.29). Let us fix $t \in [\tilde{T}, \tilde{T}+\Delta]$ and for each $l \in \mathbb{N}$ we choose $t^l \in \Pi^l$ such that $|t^l - t| \le \mathrm{f}_{\Pi^l}$. Due to the Lipschitz continuity of the solutions with a Lipschitz constant independent of the partition (see Remark 3.7) and our smoothness assumption (G1) on $\mathcal{E}$ we immediately obtain the convergence of the energy terms on the right side of equation (3.31),i.e.

$$\mathcal{E}\left(t^l, \hat{z}^l(t^l)\right) \to \mathcal{E}\left(t, z(t)\right) \quad \text{for } l \to \infty.$$

The integrands of the integrals on both sides are uniformly bounded due to the uniform boundedness of our approximative solutions and the continuity of the functions Ψ and $\partial_t \mathcal{E}(t, \cdot)$. Hence, for l tending towards infinity, we replace the integrals $\int_{\tilde{T}}^{t^l} \mathrm{d}\tau$ by $\int_{\tilde{T}}^{t} \mathrm{d}\tau$. Note that by our construction of the piecewise constant functions $\bar{z}^l$ we get the pointwise convergence $\bar{z}^l(t) \to z(t)$ for all $t \in [\tilde{T}, \tilde{T} + \Delta]$. This implies, on the one hand, the uniform convergence of the uniformly bounded integrands due to the continuity of $\partial_t \mathcal{E}(t, \cdot)$ and hence we obtain

$$\int_{\tilde{T}}^{t} \partial_\tau \mathcal{E}(\tau, \bar{z}^l(\tau))\mathrm{d}\tau \to \int_{\tilde{T}}^{t} \partial_\tau \mathcal{E}(\tau, z(\tau))\mathrm{d}\tau \quad \text{for } l \to \infty.$$

On the other side it implies the uniform convergence of the difference $|\bar{\Psi}^l\big(t, \bar{z}^l(t), \dot{\hat{z}}^l(t)\big) - \tilde{\Psi}\big(t, z(t), \dot{\hat{z}}^l(t)\big)| \to 0$. Here we had to exploit the regularity of $\mathcal{E}$ and M and the uniform boundedness of the derivatives $\dot{\hat{z}}^l$. Further we replace, using again Corollary 3.11, the functional $\tilde{\Psi}$ by Ψ in the above difference.

Hence, due to the convergence theorem of Lebesgue, we obtain

$$\int_{\tilde{T}}^{t} \bar{\Psi}\big(\tau, \bar{z}^l(\tau), \dot{\hat{z}}^l(\tau)\big) - \Psi\big(\tau, z(\tau), \dot{\hat{z}}^l(\tau)\big)\mathrm{d}\tau \to 0 \quad \text{for } l \to \infty.$$

Summarizing the last convergence results and taking the $\liminf_{l\to\infty}$ on both sides of equation (3.31) we get

$$\mathcal{E}(\tilde{T}, \tilde{z}) + \int_{\tilde{T}}^{t} \partial_\tau \mathcal{E}(\tau, z(\tau))\mathrm{d}\tau \le \mathcal{E}(t, z(t)) + \liminf_{l\to\infty} \underbrace{\int_{\tilde{T}}^{t} \Psi\big(\tau, z(\tau), \dot{\hat{z}}^l(\tau)\big)\mathrm{d}\tau}_{=:F(\hat{z}^l)}. \quad (3.32)$$

Remember that we have $\hat{z}^l \rightharpoonup z$ in $\mathrm{W}^{1,\infty}([\tilde{T}, \tilde{T} + \Delta], \mathcal{A})$. Since the mapping $F : \mathrm{W}^{1,\infty}([\tilde{T}, \tilde{T} + \Delta], \mathcal{A}) \to \mathbb{R}$ is continuous and convex, it is also weakly lower sequentially continuous and with (3.32) we directly obtain the desired (3.29). ■

Proof: (Theorem 2.2) In Proposition 3.9 we have shown, under the assumptions (G1)–(G5), that there exists a time span $\Delta > 0$ and a sequence of approximative solutions that uniformly converges to a limit function $z \in \mathrm{W}^{1,\infty}\big([\tilde{T}, \tilde{T}+\Delta], \mathcal{A}\big)$. Lemmas 3.12–3.14 show that the limit function satisfies the conditions (S) and (E) and hence is a solution of Problem 2.1. The estimate for the Lipschitz constant follows again from Proposition 3.9. ■

4 Examples of non-existence

We present two examples of non-existence of a Lipschitz continuous solution by violating the assumption $\mathrm{q}(t,z) < \alpha(t,z)$ in (G5). From a physical point of view this assumption assures that no sliding direction exists for which the frictional force declines faster than the elastic force. Otherwise the sliding velocity becomes unbounded in such a direction.

In the examples we restrict ourselves to a two-dimensional setting $\mathcal{A} = \big\{z \in \mathbb{R}^2 \,:\, z_2 \geq 0\big\}$, a purely quadratic energy $\mathcal{E}(t,z) := \langle \mathrm{H}z, z\rangle - \langle \mathrm{f}(t), z\rangle$ with constant Hessian matrix $\mathrm{H} \in \mathbb{R}^{2\times 2}$ and given external forces $\mathrm{f} \in \mathrm{C}^2([0,T], \mathbb{R}^2)$. We assume isotropic friction and hence $\mathrm{M}(z) = \begin{pmatrix} \mu(z_1) & 0 \\ 0 & 0 \end{pmatrix}$ with μ being the classical coefficient of friction.

Consequently the normal force is $\sigma(t,z) = \langle \mathrm{H}z - \mathrm{f}(t), \mathrm{e}_2\rangle$ and the dissipation potential turns out to be $\Psi(t,z,v) = \sigma(t,z)\mu(z_1)|v_1|$. For the function q we obtain $\mathrm{q}(t,z) = |\mathrm{D}\mu(z_1)|\,\sigma(t,z) + \mu(z_1)\mathrm{H}_{21}$.

Using the equivalent subdifferential formulation (see Section 2.1) our problem to solve is

$$-\mathrm{H}z(t) + \mathrm{f}(t) \in \left\{ \begin{array}{c} \sigma\left(t, z(t)\right) \mu\left(z_1(t)\right) \partial |\cdot| \left(\dot{z}_1(t)\right) \\ 0 \end{array} \right\} + \left\{ \begin{array}{c} 0 \\ \partial \mathcal{X}_{[0,\infty)}\big(z_2(t)\big) \end{array} \right\} \subset \mathbb{R}^2. \tag{4.1}$$

4.1 First example: varying coefficient of friction

We consider a situation with one degree of freedom only and choose $\tilde{T} = 0, \tilde{z} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathrm{H} = \begin{pmatrix} \mathrm{H}_{11} & 0 \\ 0 & \mathrm{H}_{22} \end{pmatrix}$. We make an ansatz of persistent contact $z_2(t) \equiv 0$ or $z(t) = \begin{pmatrix} z_1(t) \\ 0 \end{pmatrix}$. In fact the second line in (4.1) reads

$$-\sigma(t,z) = f_2(t) \in (-\infty, 0]$$

and if we choose $f_2(t) = -\sigma_*$ for some constant normal force $\sigma_* > 0$, the above ansatz is justified. It remains to solve the first line in (4.1) that simplifies to

$$-\mathrm{H}_{11} z_1(t) + \mathrm{f}_1(t) \in \sigma_* \mu(z_1(t)) \partial\,|\cdot|\,(\dot{z}_1(t)).$$

Note that our functions α and q are here

$$\begin{aligned} \alpha(t,z) &= \alpha_{\mathrm{const}} = \min\{\mathrm{H}_{11}, \mathrm{H}_{22}\} \\ \mathrm{q}(t,z) &= |\mathrm{D}\mu(z_1)|\,\sigma_*. \end{aligned}$$

To violate $\alpha(t,z) > \mathrm{q}(t,z)$ we choose a coefficient of friction that depends on z_1.

$$\mu(z_1) = \left\{ \begin{array}{ccc} \mu_0 & \text{for} & z_1 < 2 \\ \mu_0 + (\mu_1 - \mu_0)(z_1 - 1) & \text{for} & z_1 \in [2,3]. \\ \mu_1 & \text{for} & z_1 > 3 \end{array} \right.$$

Note that our assumption (G5) $\min\{\mathrm{H}_{11}, \mathrm{H}_{22}\} = \alpha(t,z) > \mathrm{q}(t,z) = |\mu_0 - \mu_1|\sigma_*$ immediately implies $(\mu_0 - \mu_1)\sigma_* < \mathrm{H}_{11}$. In fact, for the simple force $\mathrm{f}_1(t) = \mathrm{H}_{11} + \mathrm{a}t$ with fixed $\mathrm{a} > 0$ and $\mathrm{H}_{11} > (\mu_0 - \mu_1)\sigma_*$, we obtain the solution

$$z_1(t) = \begin{cases} 1 & \text{for} \quad t \leq t_1 \\ \frac{\mathrm{a}}{\mathrm{H}_{11}} t + \mathrm{c}_1 & \text{for} \quad t_1 \leq t \leq t_2 \\ \frac{\mathrm{a}}{\mathrm{H}_{11} - (\mu_0 - \mu_1)\sigma_*} t + \mathrm{c}_2 & \text{for} \quad t_2 \leq t \leq t_3 \\ \frac{\mathrm{a}}{\mathrm{H}_{11}} t + \mathrm{c}_3 & \text{for} \quad t_3 \leq t \end{cases}$$

for appropriate values $\mathrm{c}_1, \mathrm{c}_2, \mathrm{c}_3$ and times t_1, t_2 and t_3, see Figure 4.1. Hence, for $\mathrm{H}_{11} = (\mu_0 - \mu_1)\sigma_*$ the assumption is violated and regarding our solution we expect a jump to occur at the time $t_2 = t_3$.

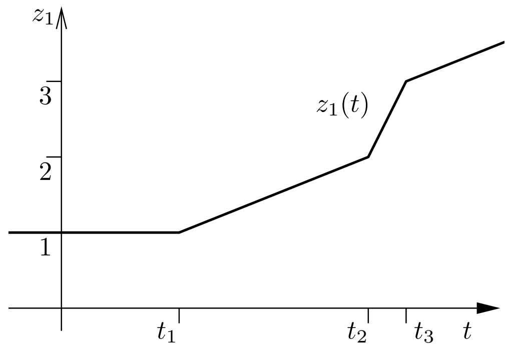

Figure 1: First example - solution $z(t) = (z_1(t), 0)^\top$

This example indicates that one should be able to replace the function $\alpha(t,z)$ by $\bar{\alpha}(t,z) := \min\left\{\langle \mathrm{H}(t,z)v, v\rangle \; : \; v \in \mathbb{R}^3, v_3 = 0, \|v\| = 1\right\}$ in assumption (G5) as, for example, Ballard did in [Bal99] or will be done in [Sch07].

4.2 Second example: varying normal force

The second example of non-existence was introduced by Klarbring [Kla90]. As in the first example a jump will occur. Since the example is two-dimensional $\alpha(t,z)$ is defined by $\alpha(t,z) = \min\left\{\langle \mathrm{H}v, v\rangle \; : \; v \in \mathbb{R}^2, \|v\| = 1\right\} < \mathrm{H}_{11}$. This time we assume μ to be constant with $\mu(z_1) = \mu_* > 0$ and we obtain $\mathrm{q}(t,z) = \mu_* \mathrm{H}_{21}$. Note that we can violate the condition $\alpha > \mathrm{q}$ by choosing $\mathrm{H}_{11} < \mu_* \mathrm{H}_{12}$. For the initial value we choose again $\tilde{T} = 0$ and $\tilde{z} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Hence the elastic system is prestressed in normal and tangential direction at time $\tilde{T} = 0$. We assume for the external tangential force $\mathrm{f}_1(t) \equiv 0$ and choose the external normal force affine in time with $\mathrm{f}_2(t) = \mathrm{f}_* + \mathrm{a}t$ and $\mathrm{a} > 0$. We have to choose $\mathrm{f}_* < 0$ such that the initial state is stable, i.e. the frictional forces are greater then the elastic forces $\mu_* \sigma(0, \tilde{z}) = \mu_* (\mathrm{H}_{21} - \mathrm{f}_*) \geq \mathrm{H}_{11}$. Note that due to $a > 0$ the frictional forces diminish in time and if $\mathrm{H}_{11} > \mu_* \mathrm{H}_{21}$ holds we have the solution

$$z(t) = \begin{cases} \tilde{z} & \text{for} \quad t \leq t_1 \\ \begin{pmatrix} \frac{-\mathrm{a}}{\mathrm{H}_{11} - \mu_* \mathrm{H}_{12}} \mathrm{f}_2(t) \\ 0 \end{pmatrix} & \text{for} \quad t_1 \leq t \leq t_2 \\ \mathrm{H}^{-1}\mathrm{f}(t) & \text{for} \quad t_2 \leq t \end{cases}$$

for appropriate times $0 \leq t_1 < t_2$. Hence, for $t \in [t_1.t_2]$ the body slides from $\tilde{z}$ to the origin 0, while for $t > t_2$ we have loss of contact and the position $z(t)$ coincides with the minimizer of $\mathcal{E}$. However, for $\mathrm{H}_{11} \leq \mu_* |\mathrm{H}_{21}|$ a jump occurs from $\tilde{z}$ to $\mathrm{H}^{-1}\mathrm{f}(t_1)$ at time $t_1 = t_2$.

As in the first example we see that we should replace the function α by the function $\bar{\alpha}(t,z) := \min\{\langle \mathrm{H}(t,z)v, v\rangle \;:\; v \in \mathbb{R}^3, v_3 = 0, \|v\| = 1\}$ in assumption (G5).

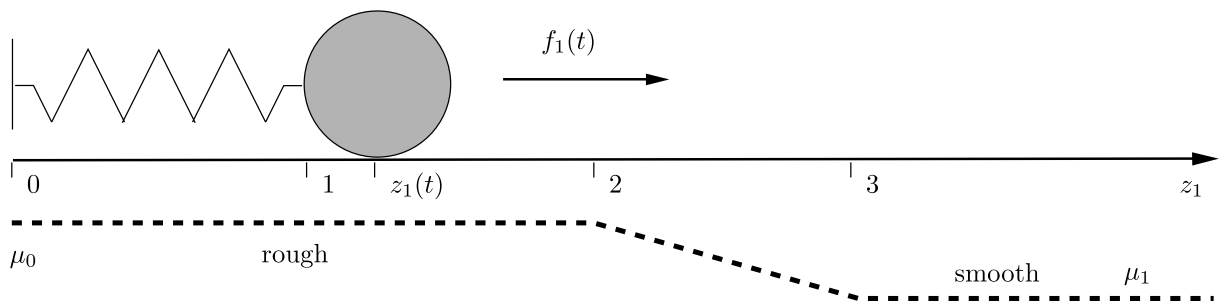

Figure 2: First example - varying coefficient of friction. The elastic system consists of a spring with origin in zero whose shape is determined by the position $z_1(t)$.

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References

[Ala93] P. Alart. Critères d'injectivité et de surjectivité pour certaines applications de $\mathbb{R}^n$ dans luimême; application a la mécanique du contact. *Math. Modelling Numer. Anal.*, 27:203–202, 1993.

[And99] L.-E. Andersson. *Quasistatic fricional contact problem with finitely many degrees of freedom*, LiTH-MAT-R-1999-22, Department of Mathematics. Linköping University, Sweden, 1999.

[And00] L.-E Andersson. Existence results for quasistatic contact problems with coulomb friction. *Appl. Math. Optim.*, 42(2):169–202, 2000.

[Bal99] P. Ballard. A counter-example to uniqueness in quasi-static elastic contact problems with small friction. *Internat.J.Engrg.Sci.*, 37(2):163–178, 1999.

[BB05] P. Ballard and S. Basseville. Existence and uniqueness for dynamical unilateral contact with coulomb friction: a model problem. *M2AN Math. Model. Numer. Anal.*, 39(1):59–77, 2005.

[DL72] G. Duvaut and J.-L. Lions. *Les inéquations en mécanique et en physique*. Dunod, Paris, 1972. French, Travaux et Recherches Mathématiques, 21.

[DL76] G. Duvaut and J.-L. Lions. *Inequalities in mechanics and physics*. Springer-Verlag, Berlin, 1976. Translated from the French by C. W. John, Grundlehren der Mathematischen Wissenschaften, 219.

[EJ00] C. Eck and J. Jarušek. Existence results for the semicoercive static contact problem with Coulomb friction. *Nonlinear Anal.*, 42(6, Ser. A: Theory Methods):961–976, 2000.

[Fic72] G. Fichera. *Boundary value problems in elastictiy with unilateral constraints*. Springer-Verlag, Berlin, 1972. In Handbuch der Physik, VI a/2.

[GMM98] F. Gastaldi, M. D. P. M. Marques, and J. A. C. Martins. Mathematical analysis of a two degree-of-freedom frictional contact problem with discontinuous solutions. *Math. Comput. Modelling*, 28:247–261, 1998.

[Jan81] V. Jankovský. Catastrophic features of coulomb friction model. In J.R. Whitemann, editor, *The Mathematics of Finite Elements and Applications*, pages 259–264. Academic Press, London, 1981.

[Kla90] A. Klarbring. Example of nonuniqueness and nonexistence of solutions to quasi-static contact problems with friction. *Ingenieur Archiv*, 60(8):529–541, 1990.

[KMS88] A. Klarbring, A. Mikelić, and M. Shillor. Frictional contact problems with normal compliance. *Internat. J. Engrg. Sci.*, 26(8):811–832, 1988.

[KMS89] A. Klarbring, A Mikelić, and M. Shillor. On friction problems with normal compliance. *Nonlinear Anal.*, 13(8):935–955, 1989.

[KMS91] A. Klarbring, A. Mikelić, and M. Shillor. A global existence result for the quasistatic frictional contact problem with normal compliance. In *Unilateral problems in structural analysis, VI (Capri, 1989)*, pages 85–111. Internat. Ser. Numer. Math. 101. Birkhäuser, Basel, 1991.

[Mie05] A. Mielke. Evolution in rate-independent systems (ch.6). In C.Dafermos and E.Feireisl, editors, *Handbook of Differential Equations, Evolutionary Equations*, volume 2, pages 461–559. Elsevier B.V., 2005.

[MM93] Manuel D. P. Monteiro Marques. *Differential inclusions in nonsmooth mechanical problems :Shocks and dry friction*. Birkhäuser Verlag, Basel, 1993.

[MMMP05] J.A.C. Martins, M.D.P. Monteiro Marques, and A. Petrov. Dynamics with friction and persistent contact. *ZAMM Z. Angew. Math. Mech.*, 85(8):531–538, 2005.

[MR06] A. Mielke and R. Rossi. Existence and uniqueness results for a class of rate-independent hystersis problems. *M^3AS Math. Models Methods Appl. Sci.*, 2006. To appear (WIAS Preprint 1078).

[MT99] A. Mielke and F. Theil. A mathematical model for rate–independent phase transformations with hysteresis. In H.-D. Alber, R.M. Balean, and R. Farwig, editors, *Proceedings of the Workshop on "Models of Continuum Mechanics in Analysis and Engineering"*, pages 117–129. Shaker–Verlag, 1999.

[MT04] Alexander Mielke and Florian Theil. On rate-independent hysteresis models. *NoDEA Nonlinear Differential Equations Appl.*, 11(2):151–189, 2004.

[MTL02] A. Mielke, F. Theil, and V.I. Levitas. A variational formulation of rate–independent phase transformations using an extremum principle. *Arch. Rational Mech. Anal.*, 162:137–177, 2002.

[NJH80] J. Nečas, J. Jarušek, and J. Haslinger. On the solution of the variational inequality to the signorini problem with small friction. *Boll.Un.Mat.Ital.B(5)*, 17(2):796–811, 1980.

[PdCM03] A. Pinto da Costa and J. A. C. Martins. The evolution and rate problems and the computation of all possible evolutions in quasi-static frictional contact. *Comput. Methods Appl. Mech. Engrg.*, 192:2791–2821, 2003.

[Roc99] R. Rocca. Existence of a solution for a quasistatic problem of unilateral contact with local friction. *C.R.Acad.Sci.Paris Sér. I Math.*, 328(12):1253–1258, 1999.

[Roc01] R. Rocca. Existence of approximation of a solution to quasi-static problem signorini problem with local friction. *Internat. J. Engrg. Sci.*, 39:1233–1255, 2001.

[Sch07] F. Schmid. 2007. In preparation.

[Sig59] A. Signorini. Questioni di elasticiá non linearizzata e semilinearizzata. *Pend.Mat. e Appl.*, 18(5):95–139, 1959.

[SM05] F. Schmid and A. Mielke. Vortex pinning in super-conductivity as a rate-independent process. *European J. Appl. Math.*, 16(6):799–808, 2005.

[TM05] A. Touzlaine and A. Mignot. Existence of solutions for quasistatic problems of unilateral contact with nonlocal friction for nonlinear elastic materials. *Electron. J. Differential Equations*, 99:13 pp., 2005.