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Analytical and Numerical Methods for Finite-Strain Elastoplasticity

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Abstract

An important class of finite-strain elastoplasticity is based on the multiplicative decomposition of the strain tensor $\mathbf{F} = \mathbf{F}_{\text{el}}\mathbf{F}_{\text{pl}}$ and hence leads to complex geometric nonlinearities. This survey describes recent advances on the analytical treatment of time-incremental minimization problems with or without regularizing terms involving strain gradients. For a regularization controlling all of $\nabla\mathbf{F}_{\text{pl}}$ we provide an existence theory for the time-continuous rate-independent evolution problem, which is based on a recently developed energetic formulation for rate-independent systems in abstract topological spaces.

In systems without gradient regularization one encounters the formation of microstructures, which can be described by sequential laminates or more general gradient Young measures. We provide a mathematical framework for the evolution of such microstructure and discuss algorithms for solving the associated space-time discretizations. We outline in a finite-step-sized incremental setting of standard dissipative materials details of relaxation-induced microstructure development for strain softening von Mises plasticity and single-slip crystal plasticity. The numerical implementations are based on simplified assumptions concerning the complexity of the microstructures.

1 Introduction

We study the theory of elastoplasticity in the case of finite strains in applications such as visualized in Fig. 1, where the deformation gradient $\mathbf{F} = \nabla\boldsymbol{\varphi}$ is considered as a matrix with positive determinant. Moreover, we work under the basic assumption that the multiplicative decomposition

$$\mathbf{F} = \nabla\boldsymbol{\varphi} = \mathbf{F}_{\text{el}}\,\mathbf{P} \quad \text{with } \mathbf{P} = \mathbf{F}_{\text{pl}}$$

can be used to describe the elastic properties via the elastic part $\mathbf{F}_{\text{el}}$ of the deformation tensor and the plastic evolution via the plastic tensor $\mathbf{P}$. In contrast to this, the additive decomposition $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_{\text{el}} + \boldsymbol{\varepsilon}_{\text{pl}}$ is well-established in small-strain elastoplasticity and has nice mathematical features since it can be easily combined with convexity tools. The assumption of finite strains and the multiplicative split destroy classical convexity properties and the more general notions of poly- and quasi-convexity need to be employed for the energy-storage potential

$$\psi(\mathbf{F}, \mathbf{P}, p) = \widetilde{\psi}(\mathbf{F}\,\mathbf{P}^{-1}, p)\,,$$

where p are additional hardening variables.

The subsequent mathematical analysis as well as the numerical implementations are based on the time-incremental minimization problems introduced in Sect. 2 which are phrased in terms of the full stored energy

$$\mathcal{E}(t, \boldsymbol{\varphi}, \mathbf{P}, p) = \int_{\mathcal{B}} \psi(\nabla\boldsymbol{\varphi}, \mathbf{P}, p) + U(\mathbf{P}, p, \nabla\mathbf{P}, \nabla p)\,\mathrm{d}x - \langle \Pi_{\text{ext}}(t), \boldsymbol{\varphi}\rangle$$

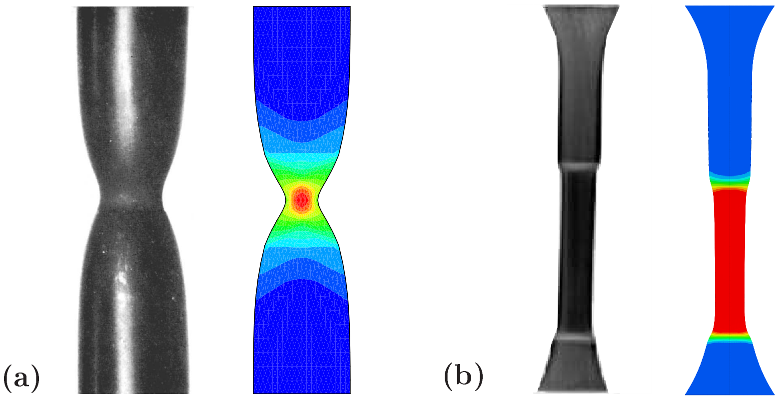


Figure 1: Experiments and numerical simulations of finite plastic deformations. **(a)** Necking of a polycrystalline material. **(b)** Neck propagation in a tensile test of an amorphous glassy polymer

and a dissipation distance

$$\mathcal{D}((\mathbf{P}_0, p_0), (\mathbf{P}_1, p_1)) = \int_{\mathcal{B}} D(\mathbf{P}_0, p_0), (\mathbf{P}_1, p_1))\, \mathrm{d}x\,.$$

For a given partition $0 = t_0 < t_1 < \cdots < t_N = T$ of the time interval $[0, T]$ the *time-incremental minimization problem* has the form

$$\textbf{(IP)} \qquad (\boldsymbol{\varphi}_j, \mathbf{P}_j, p_j) \in \operatorname*{Arg\,Min}_{\widetilde{\boldsymbol{\varphi}}, \widetilde{\mathbf{P}}, \widetilde{p}} \Big(\mathcal{E}(t_j, \widetilde{\boldsymbol{\varphi}}, \widetilde{\mathbf{P}}, \widetilde{p}) + \mathcal{D}((\mathbf{P}_{j-1}, p_{j-1}), (\widetilde{\mathbf{P}}, \widetilde{p})) \Big).$$

In Section 2.2 (cf. [Mie03a]) it is shown that this incremental problem occurs naturally as the fully implicit (backward Euler) scheme for the *energetic formulation* (S) & (E), which is a weak formulation of the time-continuous problem consisting of the elastic equilibrium together with the plastic flow law, see (2.3). In Sect. 2.3 we discuss the arising nonlinearities, which are best understood when considering the matrix groups $\mathrm{GL}_+(d) = \{\mathbf{F} \in \mathbb{R}^{d\times d} \mid \det \mathbf{F} > 0\}$ and $\mathrm{SL}(d) = \{\mathbf{P} \in \mathbb{R}^{d\times d} \mid \det \mathbf{P} = 1\}$ as Lie groups.

In Section 3 several existence results are surveyed. In the situation without any length scale (i.e., the term U involving $\nabla\mathbf{P}$ in $\mathcal{E}$ is not present) the variables $\mathbf{P}$ and p can be minimized pointwise for each $x \in \mathcal{B}$ in the incremental problem (IP). This leads to the condensed potential

$$W^{\mathrm{cond}}((\mathbf{P}_0, p_0); \mathbf{F}) = \min\{\, \psi(\mathbf{F}, \mathbf{P}_1, p_1) + D((\mathbf{P}_0, p_0), (\mathbf{P}_1, p_1)) \mid (\mathbf{P}_1, p_1)\,\}\,,$$

which plays a fundamental rôle in the existence theory in Sect. 3.1. Under the assumptions that $W^{\mathrm{cond}}((\mathbf{I}, p_0); \cdot)$ is polyconvex and that it satisfies the usual coercivity assumptions, an existence theory for (IP) was derived in [Mie04b]. If polyconvexity of W^{cond} fails, then existence of solutions is not to be expected because of the formation of microstructure. In this situation the relaxation techniques of Sect. 4 have to be used to derive effective properties.

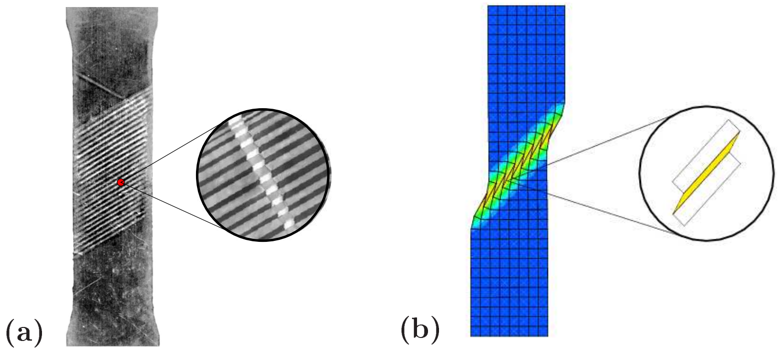


Figure 2: Experiment and numerical simulation of microstructures in finite plastic deformations. **(a)** Experimentally observed microstructure. **(b)** Numerical simulation based on rank–one laminate microstructure

In Sect. 3.2 a regularization of (IP) is considered which involves the geometric dislocation tensor $\mathbf{G_P} = (\operatorname{curl}\mathbf{P})\mathbf{P}^\top$ via the potential U in $\mathcal{E}$, namely $U(\mathbf{P},\nabla\mathbf{P}) = V(\mathbf{G_P})$. In [MM06b] it was observed that the multiplicative decomposition $\mathbf{F}_{\rm el} = \nabla\boldsymbol{\varphi}\,\mathbf{P}^{-1}$ is perfectly suited to be controlled in the sense of polyconvexity, if $\operatorname{curl}\mathbf{P}$ can be bounded by the energy. Hence, the solvability of (IP) can be proved under suitable assumptions on the dissipation distance.

Finally, in Sect. 3.3 we discuss work in progress (cf. [MM06a]) which uses a full regularization of the internal variables $(\mathbf{P}, p)$ in the energy-storage functional $\mathcal{E}$, i.e., $U(\mathbf{P}, p, \nabla\mathbf{P}, \nabla p) \geq c|(\nabla\mathbf{P}, \nabla p)|^r$. Using the abstract theory for rate-independent systems developed in [MM05, Mie05b, FM06], it is possible to show first existence of solutions for (IP) and then to pass to the limit for time step going to 0. The limit function obtained along a carefully chosen subsequence can finally be identified as a solution of the original energetic formulation (S) & (E).

In Section 4 we define material instabilities in rate independent standard dissipative solids based on *finite-step-sized incremental energy minimization principles* and apply the results in Sections 5 and 6 to the prediction of deformation microstructures in strain-softening von Mises and single-slip crystal plasticity. The formulation offers two important perspectives. First, the *definition of material stability* of standard dissipative materials is based on weak convexity conditions of incremental stress potentials in analogy to finite hyper-elasticity. Second, *microstructure developments* in unstable inelastic solids such as visualized in Fig. 2 are associated with non-convex incremental stress potentials similar to elastic phase transformation problems. These deformation microstructures can be resolved by a *relaxation of incremental energy functionals* based on a convexification of the non-convex stress potential. The subsequent developments are structured into three parts as overviewed in Table 1.

An incremental variational formulation for standard dissipative materials is outlined in the works [Mie02a, MSL02, Mie03a], which generalized treatments on the deformation theory of plasticity [Mar75], its application to a finite-step-sized incremental setting [OR99, OS99] and the formulation [CHM02] restricted to finite plasticity. It describes the response of an inelastic material by only two scalar functions: the energy storage function and the dissipation function. The general set up of this generic type of material model can be traced back to the works [Bio65, Zie63, Ger73]. It covers a broad spectrum of models in viscoelasticity, plasticity and damage mechanics. For this class of materials we define a variational formulation, where a quasi-hyperelastic stress potential at discrete time is obtained from a local *minimization problem of the constitutive response*, in Table 1 denoted by problem (C). Algorithms for a discrete setting of this constitutive minimization problem are outlined in the works [OS99, MA04, MAL02, MS04a, MS04b] for different approaches to finite plasticity.

A key advantage of the outlined variational formulation is the opportunity to define the stability of the incremental inelastic response in terms of terminologies used in elasticity theory, see for example [Dac89, Kra86, Cia88, MH94, Šil97]. Here, a necessary condition for the existence of minimizers forces the energy functional to be sequentially weakly lower semicontinuous (s.w.l.s.). An important implication of this desired property is the quasiconvexity of the stored elastic energy function, a terminology introduced in [Mor52]. The above outlined constitutive variational formulation enables us to extend these results to the finite-step-sized incremental response of inelasticity. To this end, we introduce an *incremental energy minimization principle for standard dissipative solids* that contains the incremental stress potential. The inelastic solid is then considered to be stable if this potential is quasiconvex, see condition (S) in Table 1. However, quasiconvexity is a global integral condition which is hard to verify in practice. More manageable is the slightly weaker rank-one convexity that is considered to be a close approximation of quasiconvexity. As presented in [ML03b, ML03a, MLG04], classical conditions of material stability of elastic-plastic solids outlined in [Tho61, Hil62, Ric76] are consistent with the infinitesimal form of the rank-one convexity, i.e. the strong ellipticity or Legendre-Hadamard condition.

As pointed out in the recent papers [LMD03, ML03b, ML03a, MLG04, GM06] the incremental variational formulation for the inelastic response opens up the opportunity to resolve a developing microstructure in unstable standard dissipative solids by a relaxation of the associated non-convex incremental variational problem, in Table 1 denoted by problem (R). If the above outlined material stability analysis detects a non-convex incremental stress potential, an energy-minimizing deformation microstructure is assumed to develop such as indicated in Fig. 2. A relaxation is associated with a convexification of the non-convex stress potential by constructing its quasi-or rank-one convex envelope. We refer to [KS86, Dac89, Šil97, M ̈99, Dol03] for a sound mathematical basis. The concept of relaxation has been applied to elastic phase decomposition problems in [Koh91, Lus96, CP97, DD00, GMH02, AFO03, KMR05], single crystal plasticity in [OR99, ORS00, MLG04], strain-softening von Mises plasticity in [LMD03, ML03b, ML03a] and damage mechanics in [GM06]. We comment on these results in Sections 5 and 6.

2 Modeling of Rate-Independent Elastoplasticity

2.1 Standard Generalized Materials

We consider a body with reference configuration $\mathcal{B} \subset \mathbb{R}^d$. The deformation is denoted by $\boldsymbol{\varphi} : \mathcal{B} \to \mathbb{R}^d$, and the deformation gradient $\mathbf{F} = \nabla\boldsymbol{\varphi}$ is called strain tensor. Additionally, in the sense of standard generalized materials (cf. [ZW87, Hac97]) we consider a set of internal variables $\boldsymbol{\mathcal{I}} : \mathcal{B} \to Z$, where Z is a suitable finite-dimensional manifold. The theory is based on the elastic potential ψ and the dissipation potential ϕ as the underlying constitutive functions

$$\psi = \psi(\mathbf{F}, \boldsymbol{\mathcal{I}}) \text{ and } \phi = \phi(\boldsymbol{\mathcal{I}}, \dot{\boldsymbol{\mathcal{I}}}) \geq 0 \,,$$

Finite-strain elastoplasticity is based on the multiplicative decomposition $\mathbf{F} = \nabla\boldsymbol{\varphi} = \mathbf{F}_{\text{el}}\,\mathbf{F}_{\text{pl}}$ of the deformation gradient, where $\mathbf{F}_{\text{el}}$ is the elastic part of the strain tensor and $\mathbf{P} := \mathbf{F}_{\text{pl}}$ the plastic part, shortly the *plastic tensor*. The internal variable takes the form $\boldsymbol{\mathcal{I}} = (\mathbf{P}, p) \in Z$, where $p \in \mathbb{R}^m$ denotes some hardening variable. For simplicity we neglect any dependence on the material point $x \in \mathcal{B}$

The deformation gradient $\mathbf{F}$ is best considered as an element of the Lie group $\mathrm{GL}_+(d) = \{\, \mathbf{F} \in \mathbb{R}^{d\times d} \mid \det \mathbf{F} > 0 \,\}$ and the plastic tensor $\mathbf{P}$ is usually assumed to have determinant 1, i.e. $\mathbf{P}$ is element of the special linear group $\mathrm{SL}(d) = \{\, \mathbf{P} \in \mathbb{R}^{d\times d} \mid \det \mathbf{P} = 1 \,\}$. We will investigate the arising geometric nonlinearities in Sect. 2.3. Consequently, ϕ is defined on the tangent bundle $\mathrm{T}Z$ of the manifold Z of internal variables. The multiplicative decomposition or equivalently the axiom of plastic indifference (cf. (Sy2) on p. 359 in [Mie03a]) means

$$\psi = \psi(\mathbf{F}, \boldsymbol{\mathcal{I}}) = \psi(\mathbf{F}, \mathbf{P}, p) = \widetilde{\psi}(\mathbf{F}_{\text{el}}, p) = \widetilde{\psi}(\mathbf{F}\,\mathbf{P}^{-1}, p), \tag{2.1}$$

$$\phi = \phi(\boldsymbol{\mathcal{I}}, \dot{\boldsymbol{\mathcal{I}}}) = \phi(\mathbf{P}, p, \dot{\mathbf{P}}, p) = \widetilde{\phi}(p, \dot{\mathbf{P}}\,\mathbf{P}^{-1}, \dot{p}). \tag{2.2}$$

Rate-independency is expressed in the fact, that ϕ is homogeneous of degree 1 in the rate $\dot{\boldsymbol{\mathcal{I}}} = (\dot{\mathbf{P}}, \dot{p})$, i.e., $\phi(\boldsymbol{\mathcal{I}}, \delta\dot{\boldsymbol{\mathcal{I}}}) = \delta\phi(\boldsymbol{\mathcal{I}}, \dot{\boldsymbol{\mathcal{I}}})$ for all $\delta \geq 0$.

The local balance laws involve the conjugated forces

$$\mathbb{P} = \partial_{\mathbf{F}}\psi(\mathbf{F}, \boldsymbol{\mathcal{I}}) = \partial_{\mathbf{F}_{\text{el}}}\widetilde{\psi}(\mathbf{F}_{\text{el}}, \boldsymbol{\mathcal{I}})\mathbf{P}^{-\top} \text{ and } \mathcal{F} = -\partial_{\boldsymbol{\mathcal{I}}}\psi(\mathbf{F}, \boldsymbol{\mathcal{I}}) \in \mathrm{T}^*_{\boldsymbol{\mathcal{I}}}Z$$

and take the following form

$$-\mathrm{div}\mathbb{P} = f_{\text{ext}} \quad \text{and} \quad 0 \in \partial_{\dot{\boldsymbol{\mathcal{I}}}}\phi(\boldsymbol{\mathcal{I}}, \dot{\boldsymbol{\mathcal{I}}}) - \mathcal{F} \quad \text{ in } \mathcal{B}. \tag{2.3}$$

The first equation, together with suitable boundary conditions, is the elastic equilibrium equation and the second is the plastic flow law which is defined on $\mathrm{T}^*_{\boldsymbol{\mathcal{I}}}Z$.

2.2 Energetic Formulation Using Dissipation Distances

We now use the abstract theory developed in [MTL02, MT04] in the Banach space setting and in [MM05, Mie05b, FM06] in the fully nonlinear setting to formulate time-continuous problem, which contains the full initial-boundary value problem of elastoplasticity. Because of the rate-independence and the strong nonconvexities we cannot expect that the rates $\dot{\boldsymbol{\mathcal{I}}}$ exist and hence we need a derivative-free formulation.

The function ϕ can be understood as an infinitesimal metric on $\boldsymbol{\mathcal{I}}$ which defines a (global) distance D, called *dissipation distance* in the sequel:

$$D(\boldsymbol{\mathcal{I}}_0,\boldsymbol{\mathcal{I}}_1) = \inf\{ \int_0^1 \phi(\boldsymbol{\mathcal{I}}(s),\dot{\boldsymbol{\mathcal{I}}}(s))\,\mathrm{d}s \mid \boldsymbol{\mathcal{I}}\in\mathrm{C}^1([0,1],Z),\ \boldsymbol{\mathcal{I}}(0)=\boldsymbol{\mathcal{I}}_0,\ \boldsymbol{\mathcal{I}}(1)=\boldsymbol{\mathcal{I}}_1 \} .$$

The definition provides immediately the triangle inequality

$$D(\boldsymbol{\mathcal{I}}_1,\boldsymbol{\mathcal{I}}_3) \leq D(\boldsymbol{\mathcal{I}}_1,\boldsymbol{\mathcal{I}}_2) + D(\boldsymbol{\mathcal{I}}_2,\boldsymbol{\mathcal{I}}_3) \text{ for all } \boldsymbol{\mathcal{I}}_1,\boldsymbol{\mathcal{I}}_2,\boldsymbol{\mathcal{I}}_3 \in Z . \tag{2.4}$$

The plastic indifference (2.2) provides the invariance

$$D((\mathbf{P}_0\mathbf{P}_*,p_0),(\mathbf{P}_1\mathbf{P}_*,p_1)) = D((\mathbf{P}_0,p_0),(\mathbf{P}_1,p_1)) \tag{2.5}$$

for all $\mathbf{P}_0,\mathbf{P}_1,\mathbf{P}_*$ and p_0,p_1.

For deformations $\boldsymbol{\varphi}:\mathcal{B}\mapsto\mathbb{R}^d$ and internal states $\boldsymbol{\mathcal{I}}:\mathcal{B}\mapsto Z$ we define global energy functionals by integration over the whole body $\mathcal{B}$ as follows

$$\begin{aligned}\mathcal{E}(t,\boldsymbol{\varphi},\boldsymbol{\mathcal{I}}) &= \int_{\mathcal{B}} \psi(\nabla\boldsymbol{\varphi}(x),\boldsymbol{\mathcal{I}}(x))\,\mathrm{d}x - \langle \Pi_{\mathrm{ext}}(t),\boldsymbol{\varphi}\rangle,\\ \mathcal{D}(\boldsymbol{\mathcal{I}}_0,\boldsymbol{\mathcal{I}}_1) &= \int_{\mathcal{B}} D(\boldsymbol{\mathcal{I}}_0(x),\boldsymbol{\mathcal{I}}_1(x))\,\mathrm{d}x,\end{aligned} \tag{2.6}$$

where $\Pi_{\mathrm{ext}}(t)$ denotes the external loading depending on the process time $t\in[0,T]$. Here, $\mathcal{E}(t,\boldsymbol{\varphi},\mathbf{P},p)$ is the Gibbs energy at time t associated with the state $(\boldsymbol{\varphi},\boldsymbol{\mathcal{I}}):\mathcal{B}\to\mathbb{R}^d\times Z$, and $\mathcal{D}(\boldsymbol{\mathcal{I}}_0,\boldsymbol{\mathcal{I}}_1)$ is the minimal amount of dissipation occurring when the internal state $\boldsymbol{\mathcal{I}}_0$ is changed continuously into $\boldsymbol{\mathcal{I}}_1$.

A pair $(\boldsymbol{\varphi},\boldsymbol{\mathcal{I}}):[0,T]\times\mathcal{B}\mapsto\mathbb{R}^d\times Z$ is called an *energetic solution* for the functionals $(\mathcal{E},\mathcal{D})$, if it satisfies for all $t\in[0,T]$ the following *stability condition* (S) and the *energy balance* (E):

(S) Stability: For all comparison states $(\widetilde{\boldsymbol{\varphi}},\widetilde{\boldsymbol{\mathcal{I}}})$ we have

$$\mathcal{E}(t,\boldsymbol{\varphi}(t),\boldsymbol{\mathcal{I}}(t)) \leq \mathcal{E}(t,\widetilde{\boldsymbol{\varphi}},\widetilde{\boldsymbol{\mathcal{I}}}) + \mathcal{D}(\boldsymbol{\mathcal{I}}(t),\widetilde{\boldsymbol{\mathcal{I}}}) .$$

(E) Energy balance:

$$\mathcal{E}(t,\boldsymbol{\varphi}(t),\boldsymbol{\mathcal{I}}(t)) + \mathrm{Diss}_{\mathcal{D}}(\boldsymbol{\mathcal{I}},[0,t]) \leq \mathcal{E}(0,\boldsymbol{\varphi}(0),\boldsymbol{\mathcal{I}}(t)) - \int_0^t \langle \dot{\Pi}_{\mathrm{ext}}(s),\boldsymbol{\varphi}(s)\rangle\,\mathrm{d}s .$$

The dissipated energy $\mathrm{Diss}_{\mathcal{D}}(\boldsymbol{\mathcal{I}},[r,s])$ along a process $\boldsymbol{\mathcal{I}}:[0,T]\times\mathcal{B}\mapsto Z$ is

$$\mathrm{Diss}_{\mathcal{D}}(\boldsymbol{\mathcal{I}},[r,s]) = \sup\{ \textstyle\sum_1^N \mathcal{D}(\boldsymbol{\mathcal{I}}(t_{j-1}),\boldsymbol{\mathcal{I}}(t_j)) \mid r\leq t_0<t_1<\cdots<t_N\leq s \}$$

and coincides with $\int_r^s \int_{\mathcal{B}} \phi(\boldsymbol{\mathcal{I}}(x,t),\dot{\boldsymbol{\mathcal{I}}}(x,t))\,\mathrm{d}x\,\mathrm{d}s$ for smooth processes.

The *energetic formulation* (S) & (E) characterizes the process completely and it does neither involve derivatives of $\mathbf{F}=\nabla\boldsymbol{\varphi}$ and $\boldsymbol{\mathcal{I}}$ with respect to t or x nor derivatives of the constitutive functions ψ and ϕ. It is shown in [Mie03a, Mie05b] that the energetic formulation is consistent with the classical local balance laws (2.3), i.e., they are satisfied for any sufficiently smooth energetic solution. Moreover, in smooth and convex cases we have uniqueness of energetic solutions if a suitable initial state $(\boldsymbol{\varphi}_0, \boldsymbol{\mathcal{I}}_0)$ is specified.

The energetic formulation is intrinsically linked to the time-incremental problem, which has the major advantage that it is a minimization problem. For a given partition $0 = t_0 < t_1 \cdots < t_N = T$ of the time interval $[0, T]$ and a given initial value $(\boldsymbol{\varphi}_0, \boldsymbol{\mathcal{I}}_0)$ we define the *incremental problem*

(IP) Incremental minimization problem:
Find iteratively $(\boldsymbol{\varphi}_j, \boldsymbol{\mathcal{I}}_j)$ for $j = 1, ..., N$ such that

$$(\boldsymbol{\varphi}_j, \boldsymbol{\mathcal{I}}_j) \in \operatorname{Arg\,min}\{\, \mathcal{E}(t_j, \widetilde{\boldsymbol{\varphi}}, \widetilde{\boldsymbol{\mathcal{I}}}) + \mathcal{D}(\boldsymbol{\mathcal{I}}_{j-1}, \widetilde{\boldsymbol{\mathcal{I}}}) \mid \text{ all } (\widetilde{\boldsymbol{\varphi}}, \widetilde{\boldsymbol{\mathcal{I}}}) \,\} \, .$$

This is a fully backward, hence fully implicit scheme which is difficult to solve numerically. Moreover, the dissipation distance $D : Z \times Z \to [0, \infty]$, which defines $\mathcal{D}$, is usually not known explicitly. Hence, the algorithms discussed in [Mie02a, MAL02, MSL02] are suitable variants of (IP).

The big advantage of (IP) is its mathematical consistency which arises from $\mathcal{D}$ satisfying the triangle inequality (2.4). Just using this and the minimization property, we obtain that every solution $(\boldsymbol{\varphi}_j, \boldsymbol{\mathcal{I}}_j)_{j=1,...,N}$ of (IP) satisfies for $j = 1, ..., N$ the following discretize versions of (S) and (E):

$$\mathcal{E}(t_j, \boldsymbol{\varphi}_j, \boldsymbol{\mathcal{I}}_j) \leq \mathcal{E}(t_j, \widetilde{\boldsymbol{\varphi}}, \widetilde{\boldsymbol{\mathcal{I}}}) + \mathcal{D}(\boldsymbol{\mathcal{I}}_j, \widetilde{\boldsymbol{\mathcal{I}}}) \text{ for all } (\widetilde{\boldsymbol{\varphi}}, \widetilde{\boldsymbol{\mathcal{I}}}) \, , \tag{2.7}$$

$$\begin{aligned} &\mathcal{E}(t_j, \boldsymbol{\varphi}_j, \boldsymbol{\mathcal{I}}_j) + \mathcal{D}(\boldsymbol{\mathcal{I}}_{j-1}, \boldsymbol{\mathcal{I}}_j) \\ &\leq \mathcal{E}(t_{j-1}, \boldsymbol{\varphi}_{j-1}, \boldsymbol{\mathcal{I}}_{j-1}) + \int_{t_{j-1}}^{t_j} \partial_s \mathcal{E}(s, (\boldsymbol{\varphi}_{j-1}, \boldsymbol{\mathcal{I}}_{j-1}))\, \mathrm{d}s \, . \end{aligned} \tag{2.8}$$

These estimates will be crucial for the subsequent analysis.

2.3 Lie Groups and Geometric Nonlinearities

Before dealing with an existence theory for the energetic formulation we work out a little more the geometry which arises from the fact that we are dealing with finite strains and that we are using the multiplicative decomposition. In finite-strain elasticity the stored-energy density ψ should be considered as a mapping from the Lie group

$$\mathfrak{G} := \mathrm{GL}_+(d) = \{\, \mathbf{F} \in \mathbb{R}^{d \times d} \mid \det \mathbf{F} > 0 \,\} \, .$$

The plastic tensor $\mathbf{P}$ is assumed to lie in the Lie subgroup

$$\mathfrak{P} := \mathrm{SL}(d) = \{\, \mathbf{P} \in \mathbb{R}^{d \times d} \mid \det \mathbf{P} = 1 \,\},$$

or even a smaller subgroup. Note that $\mathfrak{P}$ can be seen as the matrix group that maps the crystal lattice onto itself. We write $Z = \mathfrak{P} \times H$ for the manifold of internal states, where $\boldsymbol{\mathcal{I}} = (\mathbf{P}, p)$ with $p \in H$.

The conjugated forces have the following properties

$$\begin{aligned}
&\mathbb{P} = \partial_{\mathbf{F}}\psi(\mathbf{F},\mathbf{P},p) \in \mathrm{T}^*_{\mathbf{F}}\mathfrak{G} \ ,\\
&\mathbf{Q} = -\partial_{\mathbf{P}}\psi(\mathbf{F},\mathbf{P},p) \in \mathrm{T}^*_{\mathbf{P}}\mathfrak{P} \ ,\\
&q = -\partial_{p}\psi(\mathbf{F},\mathbf{P},p) \in \mathrm{T}^*_{p}H \ .
\end{aligned}$$

However, using the multiplicative structure of the Lie groups it is more advantageous to use the multiplicative derivatives defined via

$$\begin{aligned}
&\mathbf{K}{:}\mathbf{A} = \tfrac{\mathrm{d}}{\mathrm{d}s}\psi(\mathrm{e}^{s\mathbf{A}}\mathbf{F},\mathbf{P},p)|_{s=0} = \partial_{\mathbf{F}_{\mathrm{el}}}\widetilde{\psi}(\mathbf{F}_{\mathrm{el}},p)\mathbf{F}_{\mathrm{el}}^\top{:}\mathbf{A} \text{ for } \mathbf{A}\in \mathrm{T}_{\mathbf{I}}\mathfrak{G} \ ,\\
&\mathbf{M}{:}\mathbf{B} = -\tfrac{\mathrm{d}}{\mathrm{d}s}\psi(\mathbf{F},\mathrm{e}^{s\mathbf{B}}\mathbf{P},p)|_{s=0} = \mathbf{F}_{\mathrm{el}}^\top\partial_{\mathbf{F}_{\mathrm{el}}}\widetilde{\psi}(\mathbf{F}_{\mathrm{el}},p)\mathbf{P}^{-\top}{:}\mathbf{B} \text{ for } \mathbf{B}\in \mathrm{T}_{\mathbf{I}}\mathfrak{P} \ .
\end{aligned}$$

Hence, we find stress tensor in the dual Lie algebras $\mathfrak{g}$ and $\mathfrak{p}$:

$$\mathbf{K} = \partial_{\mathbf{F}_{\mathrm{el}}}\widetilde{\psi}(\mathbf{F}_{\mathrm{el}},p)\mathbf{F}_{\mathrm{el}}^\top \in \mathfrak{g} := \mathrm{T}^*_{\mathbf{I}}\mathfrak{G} \quad\text{and}\quad \mathbf{M} = \mathbf{F}_{\mathrm{el}}^\top\partial_{\mathbf{F}_{\mathrm{el}}}\widetilde{\psi}(\mathbf{F}_{\mathrm{el}},p) \in \mathfrak{p} := \mathrm{T}^*_{\mathbf{I}}\mathfrak{P} \ .$$

The tensors are know as the Kirchhoff stress tensor $\mathbf{K} = \mathbb{P}\mathbf{F}^T$ and the Mandel stress tensor $\mathbf{M} = \mathbf{Q}\mathbf{P}^\top$.

The first fact about these tensors is that we obtain another insight into the flow law $0 \in \partial_{\dot{\mathcal{I}}}\phi(\mathcal{I};\dot{\mathcal{I}}) - \mathcal{F}$ which is a differential inclusion on $\mathrm{T}^*_{\mathcal{I}}Z$. Using the plastic invariance of ϕ we define the elastic domain at $\mathbf{P} = \mathbf{I}$ via

$$\begin{aligned}
\mathbb{Q}(p) &= \partial_{(\mathbf{P},p)}\phi((\mathbf{I},p),(0,0))\\
&= \{\ (\mathbf{M},q) \mid \forall \dot{\mathcal{I}} \in Z{:}\ \phi((\mathbf{I},p),\widetilde{\mathcal{I}}) \geq (\mathbf{M},q){:}\widetilde{\mathcal{I}}\ \} \subset \mathfrak{p}\times \mathrm{T}^*_p H
\end{aligned}$$

and obtain, with $\mathbf{M} = \mathbf{Q}\mathbf{P}^\top$, the flow law in invariant form

$$(\dot{\mathbf{P}}\mathbf{P}^{-1},\dot{p}) \in \mathrm{N}_{(\mathbf{M},q)}\mathbb{Q}(p) = \partial\mathcal{X}_{\mathbb{Q}(p)}(\mathbf{M},q) \subset \mathfrak{p}\times \mathrm{T}_p H \ .$$

The second fact about these tensors is that they satisfy much better estimates in terms of the energy potential ψ. In fact, following [Bal02] it is reasonable to work with the following multiplicative stress control estimates:

$$\exists\, C_{\mathbf{K}} > 0\ \forall\, \mathbf{F}_{\mathrm{el}} \in \mathfrak{G}:\ |\mathbf{K}(\mathbf{F}_{\mathrm{el}},p)| \leq C_{\mathbf{K}}\big(\psi(\mathbf{F}_{\mathrm{el}},p){+}1\big) \ , \tag{2.9}$$

$$\exists\, C_{\mathbf{M}} > 0\ \forall\, \mathbf{F}_{\mathrm{el}} \in \mathfrak{G}:\ |\mathbf{M}(\mathbf{F}_{\mathrm{el}},p)| \leq C_{\mathbf{M}}\big(\psi(\mathbf{F}_{\mathrm{el}},p){+}1\big) \ , \tag{2.10}$$

In fact, (2.10) implies (2.9) but not vice versa. These conditions are satisfied by polyconvex potentials ψ that go to ∞ for $\det \mathbf{F}_{\mathrm{el}} \to 0$. In fact, most Ogden materials satisfy both conditions. For instance consider

$$\psi(\mathbf{F}_{\mathrm{el}},p) = c_1|\mathbf{F}_{\mathrm{el}}|^{r_1} + c_2(\det \mathbf{F}_{\mathrm{el}})^{-r_2} + c_3(\det \mathbf{F}_{\mathrm{el}})^{r_3} + \gamma(p)$$

with $c_j, r_j > 0$ and $\gamma(p) \geq 0$. Using $\partial_{\mathbf{F}}\det \mathbf{F} = \operatorname{cof}\mathbf{F}$ and $(\operatorname{cof}\mathbf{F})\mathbf{F}^\top = \mathbf{F}^\top\operatorname{cof}\mathbf{F} = (\det \mathbf{F})\mathbf{I}$ it is easy to see that (2.9) and (2.10) hold. A similar estimate does not hold for $\mathbb{P}(\mathbf{F}_{\mathrm{el}},p)$, since $(\det \mathbf{F}_{\mathrm{el}})^{-1-r_2}\operatorname{cof}\mathbf{F}$ cannot be estimated by $(\det \mathbf{F}_{\mathrm{el}})^{-r_2}$. It was observed in [FM06, KM06] that these estimates can be used effectively in rate-independent system to control the power of the external forces.

On the Lie groups $\mathfrak{G}$ it is possible to introduce right-invariant distance

$$d_{\mathfrak{G}}(\mathbf{F}_0,\mathbf{F}_1) = \inf\{ \int_0^1 |\dot{\mathbf{F}}(s)\mathbf{F}(s)^{-1}|\,\mathrm{d}s \mid \mathbf{F}{\in}\mathrm{C}^1([0,1],\mathfrak{G}), \mathbf{F}(0){=}\mathbf{F}_0, \mathbf{F}(1){=}\mathbf{F}_1 \},$$

which satisfies $d_{\mathfrak{G}}(\mathbf{F}_0,\mathbf{F}_1) = d_{\mathfrak{G}}(\mathbf{F}_0\mathbf{F}_1^{-1},\mathbf{I})$. Only in very few cases $d_{\mathfrak{G}}$ can be calculated explicitly, see [Mie02b, HMM03]. The condition (2.9) or (2.10) now implies that $\log(\psi{+}1)$ is globally Lipschitz continuous

$$\Big|\log\big(\psi(\mathbf{F},p){+}1\big) - \log\big(\psi(\widetilde{\mathbf{F}},p){+}1\big)\Big| \leq C_{\mathrm{Lip}} d_{\mathfrak{G}}(\mathbf{F},\widetilde{\mathbf{F}}) \text{ for all } \mathbf{F},\widetilde{\mathbf{F}} \in \mathfrak{G}\ .$$

Since $d_{\mathfrak{G}}(\mathbf{F},\mathbf{I}) \approx \big|\log(\mathbf{F}^\top\mathbf{F})\big|$, the energy potential ψ satisfies the upper estimate

$$\psi(\mathbf{F},p) \leq C_{\mathrm{upp}}(p)\big(|\mathbf{F}|{+}|\mathbf{F}^{-1}|\big)^{\gamma}.$$

This upper bound is consistent with the lower estimates also called coercivity:

$$\psi(\mathbf{F}_{\mathrm{el}},p) \geq c_1|\mathbf{F}_{\mathrm{el}}|^{r_{\mathbf{F}}}{+}c_2|p|^{r_p}{-}C_2 \text{ for all } (\mathbf{F}_{\mathrm{el}},p) \in \mathfrak{G}\times H\ . \tag{2.11}$$

We will need that the dissipation distance $D : Z \times Z \to [0,\infty]$, which is associated with the dissipation potential $\phi : \mathrm{T}Z \to [0,\infty]$, is coercive as well, namely

$$D((\mathbf{P},p),(\mathbf{I},p_*)) \geq c_3\big(|\mathbf{P}|{+}|\mathbf{P}^{-1}|\big)^{r_{\mathbf{P}}}{-}C_3 \text{ for all } (\mathbf{P},p) \in Z = \mathfrak{P}\times H\ . \tag{2.12}$$

To see that that this coercivity estimate needs a significant amount of hardening we treat the simplest example with a scalar hardening parameter $p \geq 0$ and a hardening function $h : [0,\infty) \to (0,\infty)$:

$$\widetilde{\phi}(p,\mathbf{V},\dot{p}) = \begin{cases} h'(p)\dot{p} & \text{if } \dot{p} \geq |\mathbf{V}|_{\mathfrak{p}}\ ,\\ \infty & \text{else}\ . \end{cases}$$

According to [Mie03a] we find

$$D((\mathbf{P}_0,p_0),(\mathbf{P}_1,p_1)) = \begin{cases} h(p_1){-}h(p_0) & \text{if } p_1 \geq p_0{+}d_{\mathfrak{P}}(\mathbf{P}_0,\mathbf{P}_1)\ ,\\ \infty & \text{else}\ . \end{cases}$$

Thus, assuming $p_* = 0$ and $h(0) = 0$ we obtain the lower estimate

$$D((\mathbf{P},p),(\mathbf{I},p_*)) \geq h(d_{\mathfrak{G}}(\mathbf{I},\mathbf{P})) \geq c_4\big(|\mathbf{P}|{+}|\mathbf{P}^{-1}|\big)^{r_{\mathbf{P}}} - C_4$$

only if $h(p) \geq c_5\mathrm{e}^{\gamma p}{-}C_5$ for some $c_5,\gamma > 0$, since $d_{\mathfrak{G}}(\mathbf{I},\mathbf{P})$ grows only logarithmically.

3 Existence Results

The existence results discussed in this section concern solutions without microstructure. These solutions relate to classical meso and macroscopic models for finite-strain elastoplasticity which are used for describing deep drawing or other plastic processes involving large strains. In the highly nonconvex situation we have to find assumptions on the constitutive laws which are compatible with the above geometric nonlinearities and still are good enough to prevent the formation of microstructure, which turns out to be a rather common feature in finite-strain elastoplasticity, see [OR99, CHM02, ML03a, MLG04, HH03, Mie04a, BCHH04, CT05].

We choose function spaces and functionals. The admissible deformations are supposed to lie in the set

$$\mathcal{W} = \{\, \boldsymbol{\varphi} \in \mathrm{W}^{1,r_{\boldsymbol{\varphi}}}(\mathcal{B}, \mathbb{R}^d) \mid \boldsymbol{\varphi}|_{\Gamma_{\mathrm{Dir}}} = \mathrm{id} \,\} \,.$$

For the internal variables we choose the set

$$\mathcal{Z} = \{\, (\mathbf{P}, p) \in \mathrm{L}^{r_{\mathbf{P}}}(\mathcal{B}, \mathbb{R}^{d\times d}) \times \mathrm{L}^{r_p}(\mathcal{B}, \mathbb{R}^m) \mid (\mathbf{P}(x), p(x)) \in \mathfrak{P} \times H \text{ a.e. in } \mathcal{B} \,\} \,.$$

The choice of the Lebesgue exponents $r_{\boldsymbol{\varphi}}, r_{\mathbf{F}}, r_{\mathbf{P}}$ and r_p will be a crucial step in the further analysis.

All our existence results will be based on the notion of *polyconvexity*, which means that there exists a convex and lower semi-continuous function $g : \mathbb{R}^{m_d} \to [0, \infty]$ such that $\psi(\mathbf{F}) = g(\mathcal{M}(\mathbf{F}))$ holds, where $\mathcal{M}(\mathbf{F})$ is the vector of all minors (subdeterminants) of $\mathbf{F} \in \mathbb{R}^{d\times d}$. The more general condition of *quasiconvexity* is not developed enough to handle integrands ψ which take the value $+\infty$. In fact, in the quasiconvex case the lower semi-continuity results are usually based on the upper bound $\psi(\mathbf{F}, p) \leq C(1+|\mathbf{F}|)^r$ for all $\mathbf{F} \in \mathbb{R}^{d\times d}$. This clearly contradict finite-strain elasticity where $\psi(\mathbf{F}, p) = +\infty$ for $\det \mathbf{F} \leq 0$ is imposed to prevent local interpenetration. In contrast, the multiplicative stress-control estimates (2.9) and (2.10) only lead to upper estimates on $\mathfrak{G}$.

3.1 Existence Results for the Incremental Problem

We survey the results in [Mie04b], where the incremental problem for system without any regularization is investigated. The energy functional $\mathcal{E}_0$ and the dissipation distance $\mathcal{D}$ are as defined via (2.6) with the specification of ψ and D as above. We added the subscript "$_0$" to $\mathcal{E}$ to indicate that no regularization is added.

A central rôle in this formulation is played by the *condensed energy density*

$$W^{\mathrm{cond}}((\mathbf{P}_0, p_0); \mathbf{F}) = \min\{\, \psi(\mathbf{F}, \mathbf{P}_1, p_1) + D((\mathbf{P}_0, p_0), (\mathbf{P}_1, p_1)) \mid (\mathbf{P}_1, p_1) \,\},$$

which contains the condensed information on the interplay of energy storage via ψ and energy dissipation via ϕ. Its importance derives from the fact that the minimization of $\int_{\mathcal{B}} \psi(\nabla\boldsymbol{\varphi}\mathbf{P}^{-1}, p) + D((\mathbf{P}_j, p_j), (\mathbf{P}, p))\,\mathrm{d}x$ can be done pointwise in $\boldsymbol{\mathcal{I}} = (\mathbf{P}, p)$ under the integral giving the definition of W^{cond}. Starting from Sect. 4 (see Table 1) the condensed stored energy W^{cond} is replaced by the *incremental stress potential* W, which differs from W^{cond} by a constant only.

First, for any solution process the deformation $\boldsymbol{\varphi}(t) : \mathcal{B} \mapsto \mathbb{R}^d$ must be a minimizer of the condensed functional

$$\mathcal{E}^{\rm cond}(\boldsymbol{\mathcal{I}}(t); t, \boldsymbol{\varphi}) := \textstyle\int_{\mathcal{B}} W^{\rm cond}(\boldsymbol{\mathcal{I}}(t,x); \nabla\boldsymbol{\varphi}(x))\,\mathrm{d}x - \langle \Pi_{\rm ext}(t), \boldsymbol{\varphi}\rangle \ ,$$

which follows from the stability condition (S). Hence, $W^{\rm cond}$ contains significant information on the possibility of formation of microstructure (via loss of quasiconvexity [OR99, CHM02, HH03, Mie03a, MLG04]) or failure via fracture or localization, see [ML03b, ML03a, LMD03]. In Sect. 4 and thereafter $\mathcal{E}^{\rm cond}$ is replaced by $\mathcal{E}$, which is obtained as $\mathcal{E}^{\rm cond}$ when $W^{\rm cond}$ is replaced by W. Hence, the two definitions just differ by a constant, and thus hvae the same minimizers.

Second, the incremental problem (IP) can be reduced to the following condensed incremental problem:

$$\text{(CIP)}\qquad \begin{array}{l} \text{Find iteratively } (\boldsymbol{\varphi}_j, \boldsymbol{\mathcal{I}}_j) \in \mathcal{W}\times\mathcal{Z} \text{ as follows:} \\ \boldsymbol{\varphi}_j \in \operatorname{Arg\,min}\{\, \mathcal{E}^{\rm cond}(\boldsymbol{\mathcal{I}}_{j-1}; t_j, \boldsymbol{\varphi}) \mid \boldsymbol{\varphi}\in\mathcal{W} \,\} \\ \boldsymbol{\mathcal{I}}_j(x) \in \operatorname{Arg\,min}\{\, \psi(\nabla\boldsymbol{\varphi}(x), \boldsymbol{\mathcal{I}}) + D(\boldsymbol{\mathcal{I}}_{j-1}(x), \boldsymbol{\mathcal{I}}) \mid \boldsymbol{\mathcal{I}}\in Z \,\} \end{array}$$

Thus, to guarantee existence of minimizers for (CIP) we impose the very restrictive condition, namely

$$W^{\rm cond}(\boldsymbol{\mathcal{I}}; t, \cdot) : \mathbb{R}^{d\times d} \to [0,\infty] \text{ is polyconvex.} \tag{3.13}$$

Theorem 3.3 in [Mie04b] provides the following existence result for (IP).

Theorem 3.1 *Let $\mathcal{W}$, $\mathcal{Z}$, $\mathcal{E}_0$ and $\mathcal{D}$ be defined as above. Assume that $W^{\rm cond}$ satisfies* (3.13)*. Further let the coercivity assumptions* (2.11) *and* (2.12) *be satisfied with $r_{\boldsymbol{\varphi}}, r_{\mathbf{P}}$ and r_p such that*

$$\frac{1}{r_{\mathbf{F}}} + \frac{1}{r_{\mathbf{P}}} =: \frac{1}{r_{\boldsymbol{\varphi}}} < \frac{1}{d} \ . \tag{3.14}$$

If additionally $\Pi_{\rm ext} \in \mathrm{C}^1([0,T], \mathrm{W}^{1,r_{\boldsymbol{\varphi}}}(\mathcal{B},\mathbb{R}^d)^*)$, then (IP) associated with $(\mathcal{E}_0, \mathcal{D})$ has, for each initial datum $\boldsymbol{\mathcal{I}}_0 \in \mathcal{Z}$ with $\mathcal{D}((\mathbf{I}, p_*), \boldsymbol{\mathcal{I}}_0) < \infty$ and each partition $0 = t_0 < t_1 < \cdots < t_N = T$, at least one solution $(\boldsymbol{\varphi}_j, \boldsymbol{\mathcal{I}}_j)_{j=1,\ldots,N}$ in $\mathcal{W}\times\mathcal{Z}$. Moreover, there exists a constant C (depending on the data only) such that all solutions satisfy, for $j = 1, ..., N$,

$$\|\boldsymbol{\varphi}_j\|_{1,r_{\boldsymbol{\varphi}}} + \|\mathbf{P}_j\|_{r_{\mathbf{P}}} + \|\mathbf{P}_j^{-1}\|_{r_{\mathbf{P}}} + \|p_j\|_{r_p} + \mathcal{E}_0(t_j, \boldsymbol{\varphi}_j, \boldsymbol{\mathcal{I}}_j) + \sum_{k=1}^{j} \mathcal{D}(\boldsymbol{\mathcal{I}}_{k-1}, \boldsymbol{\mathcal{I}}_k) \le C \ .$$

The proof relies on solving (CIP) with a careful bookkeeping based on the a priori estimates (2.8). The necessary coercivity of $W^{\rm cond}$ follows from those of ψ and D, after employing the invariance from $W^{\rm cond}((\mathbf{P},p);\mathbf{F}) = W^{\rm cond}((\mathbf{I},p_0);\mathbf{F}\mathbf{P}^{-1})$ and the Hölder inequality

$$|\mathbf{F}\mathbf{P}^{-1}|^{r_{\mathbf{F}}} \ge \big(|\mathbf{F}|/|\mathbf{P}|\big)^{r_{\mathbf{F}}} \ge c_r|\mathbf{F}|^{r_{\boldsymbol{\varphi}}} - d_r|\mathbf{P}|^{r_{\mathbf{P}}} \ .$$

The major drawback of the present theory is that the polyconvexity condition (3.13) is extremely difficult to check. The function $W^{\rm cond}$ is defined implicitly via ψ and D, but D itself is defined implicitly from ϕ. Hence, there are only very few cases where $W^{\rm cond}$ can be calculated explicitly. One case is in dimension $d=1$ and another case is treated in [Mie04b]. It is an isotropic situation in dimension $d=2$ using an abstract characterization of [Mie05a] for isotropic, polyconvex energy densities.

3.2 Partially Regularized Incremental Problems

The second result result concerns a model which uses a partial regularization which is based on the so-called geometric dislocation tensor

$$\mathbf{G_P}=\frac{1}{\det\mathbf{P}}\big(\,{\rm curl}\,\mathbf{P}\big)\mathbf{P}^\top\ \in\mathbb{R}^{3\times3}$$

where the "curl" of a matrix is applied row-wise. Because of our standing assumption $\det\mathbf{P}=1$ we can use a simpler form. The energy now reads

$$\mathcal{E}_{\rm curl}(t,\boldsymbol{\varphi},\mathbf{P},p)=\mathcal{E}_0(t,\boldsymbol{\varphi},\mathbf{P},p)+\textstyle\int_{\mathcal{B}}V\big(({\rm curl}\,\mathbf{P})\mathbf{P}^\top\big)\,{\rm d}x\ ,$$

where the potential $V:\mathbb{R}^{3\times3}\to[0,\infty]$ satisfies

$$V\text{ is convex and }V(\mathbf{G})\geq c_6|\mathbf{G}|^{r_{\mathbf{G}}}-C_6\text{ on }\mathbb{R}^{3\times3}. \tag{3.15}$$

In [MM06b] a general lower semi-continuity result is derived for general functionals of the form

$$\mathcal{I}(\boldsymbol{\varphi},\mathbf{P})=\int_{\mathcal{B}}U(\nabla\boldsymbol{\varphi}\mathbf{P}^{-1},\mathbf{P},({\rm curl}\,\mathbf{P})\mathbf{P}^\top)\,{\rm d}x\ .$$

Under the assumption that $U:\mathbb{R}^{3\times3}\times\mathbb{R}^{3\times3}\times\mathbb{R}^{3\times3}\to[0,\infty]$ is polyconvex in the first two arguments and convex in the third argument and that U is suitably coercive it is shown that $\mathcal{I}$ is weakly lower semicontinuous on the associated Sobolev spaces.

Consider a weakly converging sequence $(\boldsymbol{\varphi}_j,\mathbf{P}_j)\rightharpoonup(\boldsymbol{\varphi},\mathbf{P})$. Along sequences with bounded energies $\mathcal{I}(\boldsymbol{\varphi}_j,\mathbf{P}_j)\leq C$ the terms $\nabla\boldsymbol{\varphi}_j\mathbf{P}_j^{-1}$, $\mathbf{P}_j$, $\mathbf{G}_{\mathbf{P}_j}$ are controlled in suitable Lebesgue spaces. This implies a bound on ${\rm curl}\,\mathbf{P}_j$ and thus, a suitable version of the div-curl lemma can be used to show that $\mathcal{M}(\mathbf{P}_j)\rightharpoonup\mathcal{M}(\mathbf{P})$ and $\mathbf{G}_{\mathbf{P}_j}\rightharpoonup\mathbf{G_P}$. The special form of the multiplicative decomposition $\nabla\boldsymbol{\varphi}\mathbf{P}^{-1}$ together with $\det\mathbf{P}=1$ provide the minor relations

$$\mathbf{FP}^{-1}=\mathbf{F}({\rm cof}\,\mathbf{P})^\top,\ {\rm cof}(\mathbf{FP}^{-1})=({\rm cof}\,\mathbf{F})\,\mathbf{P}^\top,\ \det(\mathbf{FP}^{-1})=\det\mathbf{F}\ .$$

Hence, again applying the div-curl lemma we obtain also the convergence $\mathcal{M}(\mathbf{F}_j\mathbf{P}_j^{-1})\rightharpoonup\mathcal{M}(\mathbf{FP}^{-1})$ and the weak lower semi-continuity follows using (poly-) convexity.

This result is then applied to the incremental problem (IP) associated with $\mathcal{E}_{\rm curl}$ and $\mathcal{D}$. Again, a condensation, like in Sect. 3.1, is done for the variable $p\in H$, which does not have a derivative. We assume $\psi(\mathbf{F}_{\rm el},p)=\psi_{\rm el}(\mathbf{F}_{\rm el})+\psi_{\rm hard}(p)$ and let

$$D^{\rm cond}((\mathbf{P}_0,p_0);\mathbf{P}):=\min\{\,\psi_{\rm hard}(p)+D((\mathbf{P}_0,p_0),(\mathbf{P},p))\mid p\in H\,\}\ .$$

Then, the incremental problem involves the integrand $U(\mathbf{F}_{\rm el},\mathbf{P},\mathbf{G}) = \psi_{\rm el}(\mathbf{F}_{\rm el}) + D^{\rm cond}(\boldsymbol{\mathcal{I}}_{j-1}(x);\mathbf{P}) + V(\mathbf{G})$. Thus, the crucial assumption we have to make is that

$$D^{\rm cond}(\boldsymbol{\mathcal{I}};\cdot) : \mathbb{R}^{3\times 3} \to [0,\infty] \text{ is polyconvex.} \tag{3.16}$$

The following result is established in [MM06b].

Theorem 3.2 *Let $\mathcal{W}$, $\mathcal{Z}$, $\mathcal{E}_{\rm curl}$ and $\mathcal{D}$ be defined as above. Assume that V and $D^{\rm cond}$ satisfy* (3.15) *and* (3.16)*, respectively . Further let the coercivity assumptions* (2.11) *and* (2.12) *be satisfied with $r_{\boldsymbol{\varphi}}, r_{\mathbf{P}}, r_p$ and $r_{\mathbf{G}}$ such that*

$$\frac{1}{r_{\mathbf{F}}} + \frac{1}{r_{\mathbf{P}}} =: \frac{1}{r_{\boldsymbol{\varphi}}} < \frac{1}{d}\,, \quad \frac{1}{r_{\mathbf{G}}} + \frac{2}{r_{\mathbf{P}}} < 1\,, \text{ and } r_{\mathbf{G}} > d\,. \tag{3.17}$$

If additionally $\Pi_{\rm ext} \in \mathrm{C}^1([0,T],\mathrm{W}^{1,r_{\boldsymbol{\varphi}}}(\mathcal{B},\mathbb{R}^d)^*)$, then (IP) associated with $(\mathcal{E}_{\rm curl},\mathcal{D})$ has, for each initial datum $\boldsymbol{\mathcal{I}}_0 \in \mathcal{Z}$ with $\mathcal{D}((\mathbf{I},p_*),\boldsymbol{\mathcal{I}}_0) < \infty$ and each partition $0 = t_0 < t_1 < \cdots < t_N = T$, at least one solution $(\boldsymbol{\varphi}_j,\boldsymbol{\mathcal{I}}_j)_{j=1,\dots,N}$ in $\mathcal{W}\times\mathcal{Z}$. Moreover, there exists a constant C (depending on the data only) such that all solutions satisfy, for $j = 1,\dots,N$,

$$\|\boldsymbol{\varphi}_j\|_{1,r_{\boldsymbol{\varphi}}} + \|\mathbf{P}_j\|_{r_{\mathbf{P}}} + \|\mathbf{P}_j^{-1}\|_{r_{\mathbf{P}}} + \|p_j\|_{r_p} + \mathcal{E}_0(t_j,\boldsymbol{\varphi}_j,\boldsymbol{\mathcal{I}}_j) + \sum_{k=1}^{j} \mathcal{D}(\boldsymbol{\mathcal{I}}_{k-1},\boldsymbol{\mathcal{I}}_k) \le C\,.$$

Again the polyconvexity condition (3.16) for the condensed dissipation distance is hard to satisfy. However, we have considerably more freedom than in the case of the condensed energy potential $W^{\rm cond}$. Here the condition is based on the dissipation distance only, and we are able to take any polyconvex function $\psi_{\rm el}$ for the elastic storage. Examples are given in Section 4 of [MM06b].

However, the theory is still restrictive as we do not have good examples of dissipation distances and we do not know what type of hardening leads to polyconvexity. In the light of the example of at the end of Section 2.3 it is a natural question to ask whether the functions

$$\mathfrak{P} \ni \mathbf{P} \mapsto \exp\big(\gamma d_{\mathfrak{P}}(\mathbf{I},\mathbf{P})\big)\,,$$

if extended by $+\infty$ outside of $\mathfrak{P}$, is polyconvex for sufficiently large $\gamma > 0$. It is clear that this can only hold if $d_{\mathfrak{P}}$ is locally Lipschitz continuous with respect to the classical metric in $\mathbb{R}^{d\times d}$. Thus, sub-Riemannian or sub-Finslerian metrics are not allowed.

3.3 Strain-Gradient Plasticity

In [MM06a] a theory is developed for the case that the full gradient $(\nabla\mathbf{P},\nabla p)$ is used for regularization. For micromechanically motivated nonlocal crystal plasticity models, see [Bec06, FMAH94, Gur02, MB06, Ste96, Sve02]. This case relates to the regularized theory that was developed for other rate-independent material models like shape-memory materials, damage, brittle fracture, magnetostriction or piezoelectricity. We refer to the survey [Mie06a] in this volume.

In the present theory the incremental problem will be used as a tool to construct piecewise constant solutions for partitions with smaller and smaller step sizes. We are then able to extract a subsequence which converges to a solution of the time-continuous problem (S) and (E) as derived in Section 2.2. The analysis follows closely the abstract approach for general rate-independent systems on topological spaces as developed in [MM05, Mie05b, FM06].

We only treat the simplest case and consider the energy functional

$$\mathcal{E}_{\mathrm{reg}}(t,\boldsymbol{\varphi},\mathbf{P},p)=\mathcal{E}_0(t,\boldsymbol{\varphi},\mathbf{P},p)+\int_{\mathcal{B}} c_1|\nabla\mathbf{P}|^{r_1}+c_2|\nabla p|^{r_2}\,\mathrm{d}x\ ,$$

where $c_1,c_2>0$ and $r_1,r_2>1$. The dissipation distance $\mathcal{D}$ is kept as above.

For the admissible deformations $\boldsymbol{\varphi}$ we keep the function space $\mathcal{W}\subset\mathrm{W}^{1,r_\varphi}(\mathcal{B},\mathbb{R}^d)$ equipped with the weak topology. For the internal variables we now set $\mathcal{Z}_{\mathrm{reg}}=\mathcal{Z}_{\mathbf{P}}\times\mathcal{Z}_p$ with

$$\begin{aligned}&\mathcal{Z}_{\mathbf{P}}:=\{\,\mathbf{P}\in\mathrm{W}^{1,r_1}(\mathcal{B},\mathbb{R}^{d\times d})\mid\mathbf{P}(x)\in\mathfrak{P}\text{ a.e. on }\mathcal{B}\,\}\text{ and}\\&\mathcal{Z}_p:=\{\,p\in\mathrm{W}^{1,r_2}(\mathcal{B},\mathbb{R}^m)\mid p(x)\in H\text{ a.e. on }\mathcal{B}\,\}\ ,\end{aligned}$$

where $\mathcal{Z}_{\mathrm{reg}}$ carries the weak topology of $\mathrm{W}^{1,r_1}(\mathcal{B},\mathbb{R}^{d\times d})\times\mathrm{W}^{1,r_1}(\mathcal{B},\mathbb{R}^m)$.

Using polyconvexity of $\mathbf{F}\mapsto\psi(\cdot,p)$ and the above coercivity assumptions it is possible to show that the incremental problem (IP) associated with $(\mathcal{E}_{\mathrm{reg}},\mathcal{D})$ has at least one solution $((\boldsymbol{\varphi}_j^k,\boldsymbol{\mathcal{I}}_j^k))_{j=1,\ldots,N_k}$, where we already assumed that we have a sequence of partitions indexed by $k\in\mathbb{N}$ such that the fineness $\phi_k=\max\{t_j^k-t_{j-1}^k\mid j=1,\ldots,N_k\}$ tends to 0. We define the piecewise constant interpolants $(\overline{q}_k,\overline{\boldsymbol{\mathcal{I}}}_k):[0,T]\to\mathcal{W}\times\mathcal{Z}_{\mathrm{reg}}$ with

$$\begin{aligned}&(\overline{q}_k(t),\overline{\boldsymbol{\mathcal{I}}}_k(t))=(\boldsymbol{\varphi}_{j-1}^k,\boldsymbol{\mathcal{I}}_{j-1}^k)\text{ for }t\in[t_{j-1}^k,t_j^k)\\&\text{and }(\overline{q}_k(T),\overline{\boldsymbol{\mathcal{I}}}_k(T))=(\boldsymbol{\varphi}_{N_k}^k,\boldsymbol{\mathcal{I}}_{N_k}^k).\end{aligned}$$

According to (2.7) these piecewise constant solutions satisfy the stability conditions (S) on each point of the partition, i.e., $(\overline{q}_k(t_j^k),\overline{\boldsymbol{\mathcal{I}}}_k(t_j^k))\in\mathcal{S}(t_j^k)$ with

$$\mathcal{S}(t):=\{\,(\boldsymbol{\varphi},\boldsymbol{\mathcal{I}})\mid\forall\,(\widetilde{\boldsymbol{\varphi}},\widetilde{\boldsymbol{\mathcal{I}}})\colon\mathcal{E}_{\mathrm{reg}}(t,\boldsymbol{\varphi},\boldsymbol{\mathcal{I}})\le\mathcal{E}_{\mathrm{reg}}(t,\widetilde{\boldsymbol{\varphi}},\widetilde{\boldsymbol{\mathcal{I}}})+\mathcal{D}(\boldsymbol{\mathcal{I}},\widetilde{\boldsymbol{\mathcal{I}}})\,\}\ .$$

Moreover, the energy estimate (2.8) provides the energy bounds

$$\begin{aligned}&\mathcal{E}_{\mathrm{reg}}(t_j^k,\overline{\boldsymbol{\varphi}}_k(t_j^k),\overline{\boldsymbol{\mathcal{I}}}_k(t_j^k))+\mathrm{Diss}_{\mathcal{D}}(\overline{\boldsymbol{\mathcal{I}}}_k,[0,t_j^k])\\&\le\mathcal{E}_{\mathrm{reg}}(0,\boldsymbol{\varphi}_0,\boldsymbol{\mathcal{I}}_0)+\int_0^{t_j^k}\partial_s\mathcal{E}_{\mathrm{reg}}(s,\overline{\boldsymbol{\varphi}}_k(s),\overline{\boldsymbol{\mathcal{I}}}_k(s))\,\mathrm{d}s\ .\end{aligned}\tag{3.18}$$

They give rise to the bounds

$$\begin{aligned}&\|(\overline{\boldsymbol{\varphi}}_k,\overline{\boldsymbol{\mathcal{I}}}_k)\|_{\mathrm{L}^\infty([0,T],\mathrm{W}^{1,r_\varphi}\times\mathrm{W}^{1,r_1}\times\mathrm{W}^{1,r_2})}\le C\ ,\\&\sup_{t\in[0,T]}\mathcal{E}_{\mathrm{reg}}(t_j^k,\overline{\boldsymbol{\varphi}}_k(t_j^k),\overline{\boldsymbol{\mathcal{I}}}_k(t_j^k))\le C,\quad\mathrm{Diss}_{\mathcal{D}}(\overline{\boldsymbol{\mathcal{I}}}_k,[0,T])\le C\ .\end{aligned}$$

Thus, by using a suitable version of Helly's selection principle (cf., [MM05]) it is possible to extract a subsequence and to find a limit process $(\boldsymbol{\varphi},\boldsymbol{\mathcal{I}}) : [0,T] \to \mathcal{W}\times\mathcal{Z}_{\mathrm{reg}}$, which is a candidate for an energetic solution.

Using weak lower semi-continuity the energy bound (3.18) easily supplies the upper energy estimate

$$\mathcal{E}_{\mathrm{reg}}(t,\boldsymbol{\varphi}(t),\boldsymbol{\mathcal{I}}(t))+\mathrm{Diss}_{\mathcal{D}}(\boldsymbol{\mathcal{I}},[0,t]) \leq \mathcal{E}_{\mathrm{reg}}(0,\boldsymbol{\varphi}_0,\boldsymbol{\mathcal{I}}_0)+\int_0^t \partial_s\mathcal{E}_{\mathrm{reg}}(s,\boldsymbol{\varphi}(s),\boldsymbol{\mathcal{I}}(s))\,\mathrm{d}s\,.$$

The crucial step in the convergence proof is to show that the sets $\mathcal{S}(t)$ of stable states are sequentially closed in the weak Banach space topology. This step is easy if $\mathcal{D}$ is weakly continuous but it also works in more realistic cases with hardening, which is irreversible, see [MM06a]. If this step is done we know that the limit process satisfies (S) and, moreover, a general abstract proposition yields the lower energy estimate and hence (E) holds as well.

We summarize the result as follows.

Theorem 3.3 *Let $\mathcal{W}$, $\mathcal{Z}_{\mathrm{reg}}$, $\mathcal{E}_{\mathrm{reg}}$ and $\mathcal{D}$ be given as above with ψ and D satisfying the coercivity estimates* (2.11) *and* (2.12) *with $\frac{1}{r_{\mathbf{F}}}+\frac{1}{r_{\mathbf{P}}}=\frac{1}{r_{\varphi}}<\frac{1}{d}$. Moreover, assume $\Pi_{\mathrm{ext}}\in\mathrm{C}^1([0,T],\mathrm{W}^{1,r_\varphi}(\mathcal{B},\mathbb{R}^d)^*)$. Then, for each stable initial state $(\boldsymbol{\varphi}_0,\boldsymbol{\mathcal{I}}_0)\in\mathcal{S}(0)$ the energetic formulation (S) and (E) has at least one solution $(\boldsymbol{\varphi},\boldsymbol{\mathcal{I}}) : [0,T]\to\mathcal{W}\times\mathcal{Z}_{\mathrm{reg}}$. All solutions satisfy*

$$(\boldsymbol{\varphi},\mathbf{P},p)\in\mathrm{L}^\infty([0,T],\mathrm{W}^{1,r_\varphi}(\mathcal{B},\mathbb{R}^d)\times\mathrm{W}^{1,r_1}(\mathcal{B},\mathbb{R}^{d\times d})\times\mathrm{W}^{1,r_2}(\mathcal{B},\mathbb{R}^m))$$

and $\mathrm{Diss}_{\mathcal{D}}((\mathbf{P},p),[0,T])<\infty$.

3.4 Time-Dependent Boundary Conditions

The existence results of Sections 3.1 to 3.3 rely on the fact that the space $\mathcal{W}$ of admissible deformations is independent of time. For many applications one needs to generalize this assumption. For the incremental problem (IP) it is not too difficult to work with $\mathcal{W}(t)$, however for the energetic formulation it is not clear how to define the power $\partial_t\mathcal{E}(t,q)$ of the external loadings that are due to changes of $\mathcal{W}(t)$.

The usual way to implement time-dependent Dirichlet data is to substract a sufficiently smooth function that has the correct boundary value and then try to find the homogeneous part. In the case of small strain, when working with $\mathbf{u}: x\mapsto\boldsymbol{\varphi}(x)-x$ this means $\mathbf{u}(t,x)=\mathbf{u}_{\mathrm{Dir}}(t,x)+\mathbf{v}(t,x)$ with $\mathbf{v}(t,\cdot)|_{\Gamma_{\mathrm{Dir}}}=0$. We let $\mathcal{W}=\{\,\mathbf{v}\in\mathrm{W}^{1,p}(\mathcal{B},\mathbb{R}^d)\mid \mathbf{v}|_{\Gamma_{\mathrm{Dir}}=0}\,\}$ and define the shifted energy $\widetilde{\mathcal{E}}(t,\mathbf{v},\boldsymbol{\mathcal{I}})=\mathcal{E}(t,\mathbf{u}_{\mathrm{Dir}}(t)+\mathbf{v},\boldsymbol{\mathcal{I}})$. The power of the external loading now takes the form

$$\begin{aligned}\partial_t\widetilde{\mathcal{E}}(t,\mathbf{v},\boldsymbol{\mathcal{I}}) &= \int_{\mathcal{B}}\partial_{\mathbf{F}}\psi(\nabla(\mathbf{u}_{\mathrm{Dir}}(t)+\mathbf{v}),\boldsymbol{\mathcal{I}}):\nabla\dot{\mathbf{u}}_{\mathrm{Dir}}(t)\,\mathrm{d}x\\ &\quad -\langle\dot{\Pi}_{\mathrm{ext}}(t),\mathbf{u}_{\mathrm{Dir}}(t)+\mathbf{v}\rangle-\langle\Pi_{\mathrm{ext}}(t),\dot{\mathbf{u}}_{\mathrm{Dir}}(t)\rangle\,.\end{aligned}$$

However, in the case of finite-strain elasticity we cannot guarantee that the integrand $\partial_{\mathbf{F}}\psi$ lies in $\mathrm{L}^1(\mathcal{B})$, since we cannot control the Piola-Kirchoff stress $\mathbb{P} = \partial_{\mathbf{F}}\psi$ by ψ itself.

In the case of finite-strain elasticity the stored energy density ψ takes the value $+\infty$ and $\partial_{\mathbf{F}}\psi(\mathbf{F},\boldsymbol{\mathcal{I}})$ exists only on $\mathfrak{G}$. In order to use the multiplicative stress control (2.9) for the Kirchhoff stress $\mathbf{K}$ we assume that time-dependent Dirichlet data $\boldsymbol{\varphi}_{\mathrm{Dir}}$ are given. We then decompose the desired solution $\boldsymbol{\varphi}$ via composition of functions

$$\boldsymbol{\varphi}(t,x) = \boldsymbol{\varphi}_{\mathrm{Dir}}(t,\boldsymbol{\xi}(t,x)) = (\boldsymbol{\varphi}_{\mathrm{Dir}}(t,\cdot)\circ\boldsymbol{\xi}(t,\cdot))(x)\;.$$

Here, we assume that $\boldsymbol{\varphi}_{\mathrm{Dir}}$ can be extended such that $\boldsymbol{\varphi}_{\mathrm{Dir}} \in \mathrm{C}^2([0,T]\times\mathbb{R}^d,\mathbb{R}^d)$ and that $\nabla_x\boldsymbol{\varphi}_{\mathrm{Dir}}$ and $(\nabla_x\boldsymbol{\varphi}_{\mathrm{Dir}})^{-1}$ are bounded on $[0,T]\times\mathbb{R}^d$. The set of admissible deformations is now $\mathcal{W} = \{\,\boldsymbol{\xi}\in \mathrm{W}^{1,p}(\mathcal{B},\mathbb{R}^d)\mid \boldsymbol{\xi}|_{\Gamma_{\mathrm{Dir}}} = \mathrm{id}\,\}$ with $p>d$ and the shifted energy is $\widehat{\mathcal{E}}(t,\boldsymbol{\xi},\boldsymbol{\mathcal{I}}) = \mathcal{E}(t,\boldsymbol{\varphi}_{\mathrm{Dir}}(t)\circ\boldsymbol{\xi},\boldsymbol{\mathcal{I}})$. Using the classical chain rule formula

$$\nabla_x(\boldsymbol{\varphi}_{\mathrm{Dir}}(t)\circ\boldsymbol{\xi})(x) = \nabla_y\boldsymbol{\varphi}_{\mathrm{Dir}}(t,\boldsymbol{\xi}(x))\nabla_x\boldsymbol{\xi}(x)$$

and the definition of $\mathbf{K}$ in Section 2.3 we find the expression for the power

$$\begin{aligned}\partial_t\widehat{\mathcal{E}}(t,\boldsymbol{\xi},\boldsymbol{\mathcal{I}}) \;=\;& \textstyle\int_{\mathcal{B}}\mathbf{K}(\nabla\boldsymbol{\varphi}_{\mathrm{Dir}}\nabla\boldsymbol{\xi},\boldsymbol{\mathcal{I}}){:}(\nabla\boldsymbol{\varphi}_{\mathrm{Dir}})^{-1}\nabla\dot{\boldsymbol{\varphi}}_{\mathrm{Dir}}\,\mathrm{d}x\\ &-\langle\dot{\Pi}_{\mathrm{ext}}(t),\boldsymbol{\varphi}_{\mathrm{Dir}}\circ\boldsymbol{\xi}\rangle - \langle\Pi_{\mathrm{ext}}(t),\dot{\boldsymbol{\varphi}}_{\mathrm{Dir}}\circ\boldsymbol{\xi}\rangle\;.\end{aligned}$$

Here, for $\widetilde{\mathcal{E}}(t,q)<\infty$ we may conclude via (2.9) that $\mathbf{K}\in\mathrm{L}^1(\mathcal{B},\mathbb{R}^{d\times d})$ while $(\nabla\boldsymbol{\varphi}_{\mathrm{Dir}})^{-1}\nabla\dot{\boldsymbol{\varphi}}_{\mathrm{Dir}}$ lies in $\mathrm{C}^0(\overline{\mathcal{B}},\mathbb{R}^{d\times d})$. Hence, the right-hand side is indeed well defined and the power control

$$|\partial_t\widetilde{\mathcal{E}}(t,\boldsymbol{\xi},\boldsymbol{\mathcal{I}})| \le c_1^E\big(\widetilde{\mathcal{E}}(t,\boldsymbol{\xi},\boldsymbol{\mathcal{I}})+c_0^E\big)$$

can be established easily. We refer to Section 5 in [FM06] for more details concerning the full existence result for energetic solutions in the case of time-dependent Dirichlet data.

In [KM06] very similar ideas are used to derive formulas for the energy-release rate in crack propagation for the case of finite-strain elasticity. Also a very restricted case of temperature dependence can be treated by this method of energy control, see [Mie06b] and Sect. 5.4 in [Mie06a].

4 Modeling of Microstructure via Relaxation

In principle, the time incremental problem (IP) and the energetic formulation (S) & (E) introduced in Sect. 2.2 is a very flexible tool to treat the relaxation as well. We refer to [Mie03b, Mie04a, MRS06b, MO06, MT06] for some recent developments. However, the analytical methods are not yet adapted to the specific nonlinearities involved in finite-strain elastoplasticity. In particular, there is no theory which combines the theory of gradient Young measure with finite-strain plasticity. Thus, the evolutionary theory for gradient Young measures used in models for shape-memory alloys in [KMR05] cannot be generalized to the present situation. Despite of the lacking mathematical tools in this area, the following sections show that the algorithmical approach for these problems has advanced considerably over the last decade.

4.1 Incremental Stability of Standard Dissipative Solids

As pointed out in [ML03b, ML03a, MLG04] a key advantage of the variational formulation outlined briefly in Table 1 is the opportunity to analyze the incremental stability of inelastic solids in terms of terminologies used in finite elasticity. In the following we define the material stability of standard dissipative solids based on global weak convexity properties of the incremental stress potential.

The existence of the constitutive minimization problem allows the introduction of an incremental minimization formulation of the boundary-value problem of finite inelasticity for standard dissipative solids. Now consider a functional $\mathcal{E}$ of the current deformation field $\boldsymbol{\varphi}_{n+1}$at the right boundary of the increment $[t_n, t_{n+1}]$:

$$\mathcal{E}(\boldsymbol{\varphi}_{n+1}) = \int_{\mathcal{B}} W(\mathbf{F}_{n+1}) \ \mathrm{d}x - [\, \Pi_{\mathrm{ext}}(\boldsymbol{\varphi}_{n+1}) - \Pi_{\mathrm{ext}}(\boldsymbol{\varphi}_n) \,] \ , \tag{4.19}$$

with the global load potential function $\Pi_{\mathrm{ext}}(\boldsymbol{\varphi}) = \int_{\mathcal{B}} \boldsymbol{\varphi} \cdot \boldsymbol{\gamma} \ \mathrm{d}x + \int_{\partial\mathcal{B}_t} \boldsymbol{\varphi} \cdot \mathbf{t} \ \mathrm{d}x$ of dead body forces $\boldsymbol{\gamma}(x,t)$ in $\mathcal{B}$ and surface tractions $\mathbf{t}(x,t)$ on $\partial\mathcal{B}_t$. As outlined in Sect. 3.1, see also [ML03b, ML03a, MLG04], the current deformation map of inelastic standard dissipative materials can then be determined by a *principle of minimum incremental energy for standard dissipative solids*

$$\mathcal{E}(\boldsymbol{\varphi}^*_{n+1}) = \inf_{\boldsymbol{\varphi}_{n+1} \in \mathcal{W}} \mathcal{E}(\boldsymbol{\varphi}_{n+1}) \ , \tag{4.20}$$

subject to the essential boundary conditions of a prescribed deformation $\bar{\boldsymbol{\varphi}}$ on $\partial\mathcal{B}_{\boldsymbol{\varphi}}$, written in the form $\boldsymbol{\varphi}_{n+1} \in \mathcal{W} := \{ \ \boldsymbol{\varphi} \in \mathrm{W}^{1,p}(\mathcal{B}) \mid \boldsymbol{\varphi}(x) = \bar{\boldsymbol{\varphi}}(x) \text{ on } \partial\mathcal{B}_{\boldsymbol{\varphi}} \ \}$. As usual, we consider a decomposition of the surface into a part where the deformation is prescribed and a part where the tractions are given, i.e. $\partial\mathcal{B} = \partial\mathcal{B}_{\boldsymbol{\varphi}} \cup \partial\mathcal{B}_t$ and $\partial\mathcal{B}_{\boldsymbol{\varphi}} \cap \partial\mathcal{B}_t = \emptyset$. The minimization problem (4.20) governs the response of the inelastic solid in the finite increment $[t_n, t_{n+1}]$ in a structure identical to the principle of minimum potential energy in finite elasticity.

4.1.1 Quasiconvexity of the Incremental Stress Potential

Extending results of the existence theory in finite elasticity as summarized in [Bal77, Cia88, Dac89, MH94, Šil97] to the incremental response of standard dissipative solids in the finite step $[t_n, t_{n+1}]$, we consider the *sequentially weakly lower semicontinuity* (s.w.l.s.) of the functional (4.19) as the key property for the *existence of sufficiently regular minimizers* of the variational problem (4.20). The internal part of the functional (4.19) is sequentially weakly lower semicontinuous, if the incremental stress

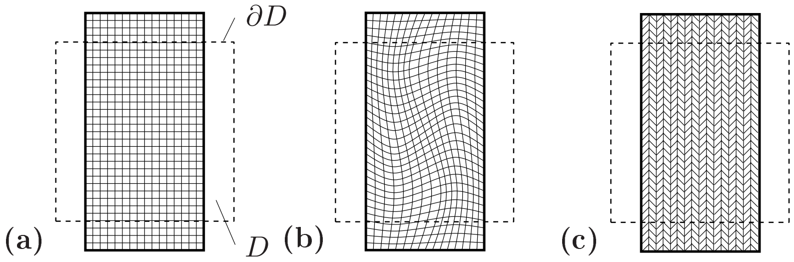


Figure 3: Interpretation of incremental energetic stability conditions of an inelastic material. A given homogeneous deformation state $\mathbf{F}_{n+1}$ of the material such as the pure shear mode of Figure **(a)** is stable if superimposed fine-scale fluctuation patterns **(b)** (*quasiconvexity*) with support on the boundary ∂D or first-order laminates **(c)** (*rank-one convexity*) increase the averaged incremental stress potential on D

potential defined by the constitutive minimization problem is quasiconvex and also it satisfies some technical growth condition, see for example [Dac89, AF84, Šil97]. We regard the *quasiconvexity* introduced in [Mor52] of the incremental stress potential W as the fundamental criterion for the incremental material stability of the inelastic solid. W is said to be quasiconvex at $\mathbf{F}_{n+1}$ if condition

$$W(\mathbf{F}_{n+1}) \leq \inf_{\mathbf{w}\in\mathcal{W}_0} \frac{1}{|D|} \int_D W(\mathbf{F}_{n+1} + \nabla\mathbf{w}(\mathbf{y}))\,\mathrm{d}x \;, \tag{4.21}$$

holds with $\mathbf{y} \in D$ subject to the constraint $\mathbf{w} \in \mathcal{W}_0 := \{\mathbf{w} \in \mathrm{W}^{1,\infty}(D) | \mathbf{w} = \mathbf{0} \text{ on } \partial D\}$ providing a support on ∂D. Here, $D \subset \mathbb{R}^3$ is an arbitrarily chosen part of the inelastic solid. The condition states that for all fluctuations $\mathbf{w}$ on D with support on ∂D the homogeneous deformation given by $\mathbf{F}_{n+1}$ provides an absolute minimizer of the incremental potential in D. Thus the condition rules out internal buckling, the development of local fine-scale microstructures and phase decomposition of a homogeneous local deformation state. This mechanical interpretation is visualized in Fig. 3. The material is stable if the superimposed fluctuation field of Fig. 3(b) with $\mathbf{w} = \mathbf{0}$ on ∂D yields a higher energy level than the homogeneous deformation $\mathbf{F}_{n+1}$ of Fig. 3(a).

The well-motivated concept of quasiconvexity is based on a global integral condition in space which is hard to verify in practice. The central difficulty is to find the fluctuation field $\mathbf{w} \in \mathcal{W}_0$ on D that minimizes the integral in (4.21). However, recall that weak convexity conditions are related via

$$\text{convexity} \Rightarrow \text{polyconvexity} \Rightarrow \text{quasiconvexity} \Rightarrow \text{rank-one convexity} \;, \tag{4.22}$$

and that the slightly weaker rank-one convexity condition is considered as a close approximation of the quasiconvexity condition, see for example [Dac89]. In what follows, we focus on the rank-one convexity as a criterion for material stability.

4.1.2 Rank-one Convexity of the Incremental Stress Potential

The definition of *rank-one convexity* can be traced back to the work of Corall and Graves, see for example [Šil97]. The incremental stress potential W is said to be *rank-one convex* at $\mathbf{F}_{n+1}$ if the condition

$$W(\mathbf{F}_{n+1}) \leq \inf_{\xi,\mathbf{F}^+,\mathbf{F}^-} \{\, \xi W(\mathbf{F}^+) + (1-\xi) W(\mathbf{F}^-) \,\} , \tag{4.23}$$

holds for the *laminate deformations* $\mathbf{F}^+$ and $\mathbf{F}^-$ which satisfy the conditions

$$\mathbf{F}_{n+1} = \xi \mathbf{F}^+ + (1-\xi)\mathbf{F}^- \quad \text{and} \quad \text{rank}[\mathbf{F}^+ - \mathbf{F}^-] \leq 1 , \tag{4.24}$$

in terms of the volume fraction $\xi \in [0,1]$. Condition $(4.24)_1$ states that the volume average of the micro-deformations $\mathbf{F}^\pm$ yields the macroscopic homogeneous deformation $\mathbf{F}_{n+1}$. The compatibility of the micro-phases $(\pm)$ along their interface is ensured by $(4.24)_2$. The rank-one convexity condition (4.23) rules out the development of local fine-scale microstructures in the form of first-order laminates defined by a rank-one deformation tensor. The material is stable if the superimposed first-order laminate-type fluctuation field of Fig. 3(c) yields a higher energy level than the homogeneous deformation $\mathbf{F}_{n+1}$ of Fig. 3(a). A qualitative picture of a non-convex, unstable incremental response is given in Fig. 4. Observe carefully, that (4.23) is a *global stability criterion* that needs the knowledge about the global range of instability between $\mathbf{F}^-$ and $\mathbf{F}^+$. The material stability cannot be directly decided in terms of a given local deformation $\mathbf{F}_{n+1}$, but needs the rank-one convex hull construction governed by $\mathbf{F}^-$ and $\mathbf{F}^+$. The *local form of the rank-one convexity condition* is the classical *Legendre-Hadamard* or *ellipticity condition*

$$(\mathbf{M} \otimes \mathbf{N}) : \partial^2_{\mathbf{FF}} W(\mathbf{F}_{n+1}) : (\mathbf{M} \otimes \mathbf{N}) \geq 0 , \tag{4.25}$$

in terms of the consistent tangent modulus for arbitrary unit vectors $\mathbf{M}$ and $\mathbf{N}$, see [Had03, TN65]. As shown in [ML03b, ML03a], classical conditions of material stability of elastic-plastic solids outlined in [Tho61, Hil62, Ric76] are consistent with this local convexity condition, which is often motivated by considering wave propagation in solids. As shown in Fig. 4, the associated range of instability is different from the one predicted by the global condition (4.23). Recall that both conditions are mathematical definitions related to the existence of regular solutions of the variational problem (4.20). The question whether the global or local conditions (4.23) and (4.25) are relevant depends on the physical ability of an inelastic solid material to develop deformation microstructures in the associated unstable ranges. This can only be clarified by experimental investigations.

In what follows we rewrite the rank-one convexity condition (4.23) for two-dimensional problems. To this end, we introduce the ansatz

$$\mathbf{F}^\pm := \mathbf{F}_{n+1}\mathbf{L}^\pm \quad \text{with} \quad \left\{ \begin{array}{lcll} \mathbf{L}^+ := & \mathit{1} & +(1-\xi) d\, \mathbf{M} \otimes \mathbf{N} \, , \\ \mathbf{L}^- := & \mathit{1} & - \quad \xi d\, \mathbf{M} \otimes \mathbf{N} \, , \end{array} \right. \tag{4.26}$$

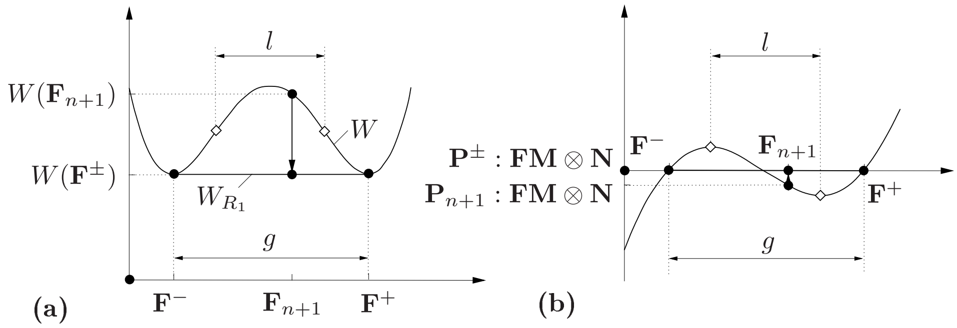


Figure 4: Qualitative representation of a non-convex incremental stress potential and its convexification. l and g characterize the ranges where the *local* and the *global* convexity criterion are not satisfied, respectively. **(a)** At $\mathbf{F}_{n+1}$ the stress potential W is not rank-one convex (dashed). As a consequence, the macroscopic deformation state $\mathbf{F}_{n+1}$ is not stable and decomposes into micro-phases $\mathbf{F}^{\pm}$ which determine the rank-one convex envelope (solid). **(b)** The relaxed stress-strain relation characterizes a snap-through behavior between the micro-phases $\mathbf{F}^{\pm}$ due to the constant slope of the rank-one convex envelope

for the two deformation phases that satisfies the conditions (4.24). It models a first-order laminate in terms of the two Lagrangian unit vectors $\mathbf{M}$ and $\mathbf{N}$, which correspond with those used in the Hadamard condition (4.25). For two-dimensional problems, these vectors can be parameterized by two angles φ and χ, i.e. $\mathbf{M}(\varphi) = [cos\varphi \;\; \sin\varphi]^T$ and $\mathbf{N}(\chi) = [\cos\chi \;\; \sin\chi]^T$. The scalar d describes the intensity of the bifurcation on the micro-scale. ξ is the volume fraction of the phase $(+)$ and can be understood as a probability measure in the sense of [You69]. Hence, for a two-dimensional description of the rank-one laminate, deformations microstructures are characterized by *four micro-variables* $\mathbf{q} = [\xi, d, \varphi, \chi]^T \in \mathcal{Q}$, which are constrained to lie in the admissible domain $\mathcal{Q} := \{\mathbf{q} \mid 0 \le \xi \le 1\,,\; d \ge 0\,,\; 0 \le \varphi \le \pi\,,\; 0 \le \chi \le \pi\}$. With this notation at hand, we write the global rank-one convexity condition (4.23) for two-dimensional problems as the minimization problem

$$W(\mathbf{F}_{n+1}) \le \inf_{\mathbf{q}\in\mathcal{Q}} \{\bar{W}^h(\mathbf{F}_{n+1}, \mathbf{q})\} \;, \tag{4.27}$$

in terms of the function

$$\bar{W}^h(\mathbf{F}_{n+1}, \mathbf{q}) = \xi W(\mathbf{F}^+(\mathbf{F}_{n+1}, \mathbf{q})) + (1-\xi) W(\mathbf{F}^-(\mathbf{F}_{n+1}, \mathbf{q})) \;, \tag{4.28}$$

that represents the the volume average of the potentials in the two deformation phases. Figure 4(a) provides a visual demonstration for a non-convex incremental stress potential W. The incremental stress potential $W(\mathbf{F}_{n+1})$ is greater than the interpolation of the potentials $W(\mathbf{F}^+)$ and $W(\mathbf{F}^-)$ of the phases. As a consequence, the homogeneous deformation state is not stable and decomposes into the micro-deformations $\mathbf{F}^{\pm}$ which minimize the function $\bar{W}^h$. In a typical incremental analysis of an inelastic solid, the accompanying check of incremental rank-one convexity in

$[t_n, t_{n+1}]$ needs the solution of the local minimization problem (4.27)

$$\inf_{\mathbf{q}\in\mathcal{Q}} \{\bar{W}^h(\mathbf{F}_{n+1}, \mathbf{q})\} \left\{ \begin{array}{ll} = W(\mathbf{F}_{n+1}) : & \text{rank-one convex at } \mathbf{F}_{n+1} \\ < W(\mathbf{F}_{n+1}) : & \text{not rank-one convex at } \mathbf{F}_{n+1} \end{array} \right. , \tag{4.29}$$

for the four variables $\mathbf{q}$ defined before. The necessary condition of the minimization problem

$$\bar{W}^h_{,\mathbf{q}} = \mathbf{0} \; , \tag{4.30}$$

is a nonlinear equation for the determination of the micro-variables $\mathbf{q}$. Note that $\bar{W}^h$ is not convex and for the solution of (4.30) the Newton iteration cannot directly be applied. We refer to [ML03b, ML03a, MLG04] for solution procedures.

4.2 Relaxation of a Non-Convex Constitutive Response

As pointed out in the recent papers [LMD03, ML03b, ML03a, MLG04], the incremental variational formulation for the constitutive response opens up the opportunity to resolve the developing microstructure in non-stable standard dissipative solids by a relaxation of the associated non-convex incremental variational problem. If the above outlined material stability analysis detects a non-convex incremental stress potential W, an energy-minimizing *deformation microstructure* is assumed to develop such as indicated in Fig. 3. A *relaxation* is associated with a *convexification* of the non-convex function W by constructing its *convex envelopes* W_Q. The convexification is concerned with the determination of a developing microstructure. This section develops a framework for a first-order rank-one relaxation of standard dissipative solids.

4.2.1 Quasi-Convexified Relaxed Incremental Variational Problem

If material instabilities are detected at a point $\mathbf{X} \in \mathcal{B}$ of the solid by a failure of conditions (4.23) or (4.27), we face a non-convexity of the incremental potential W in some region of the inelastic solid. If the incremental potential function W is not quasiconvex, the internal part of the functional (4.19) is assumed to be not sequentially weakly lower semicontinuous. Then the existence of solutions of (4.20) is not ensured. In other words, the minimum of the incremental boundary-value problem (4.20) is not attained. Following [Dac89, AF84] we consider the *relaxed energy functional*

$$\mathcal{E}_Q(\boldsymbol{\varphi}_{n+1}) = \int_{\mathcal{B}} W_Q(\mathbf{F}_{n+1}) \; \mathrm{d}x - [\, \Pi_{\text{ext}}(\boldsymbol{\varphi}_{n+1}) - \Pi_{\text{ext}}(\boldsymbol{\varphi}_n) \,] \; , \tag{4.31}$$

where the internal part of the relaxed energy functional is obtained by replacing the non-convex integrand W in (4.19) by its *quasiconvex envelope* W_Q. The current deformation field of the elastic-plastic solid is then determined by the relaxed incremental variational principle

$$\mathcal{E}_Q(\boldsymbol{\varphi}^*_{n+1}) = \inf_{\boldsymbol{\varphi}_{n+1}\in\mathcal{W}} \mathcal{E}_Q(\boldsymbol{\varphi}_{n+1}) \; , \tag{4.32}$$

that *minimizes the relaxed incremental potential energy* $\mathcal{E}_Q$. The quasiconvexified incremental stress potential W_Q is defined by the minimization problem

$$W_Q(\mathbf{F}_{n+1}) = \inf_{\mathbf{w}\in\mathcal{W}_0} \frac{1}{|D|} \int_D W(\mathbf{F}_{n+1} + \nabla\mathbf{w}(\mathbf{y}))\,\mathrm{d}x \;, \tag{4.33}$$

with respect to the microscopic fluctuation field $\mathbf{w}$ that constitutes the development of a deformation microstructure, subject to a boundary condition providing a support on ∂D. The first and second derivatives of the relaxed potential W_Q function define relaxed stresses and tangent moduli

$$\bar{\mathbb{P}}_{n+1} := \partial_{\mathbf{F}} W_Q(\mathbf{F}_{n+1}) \quad \text{and} \quad \bar{\mathbb{C}}_{n+1} := \partial^2_{\mathbf{FF}} W_Q(\mathbf{F}_{n+1}) \;. \tag{4.34}$$

The relaxed problem (4.32) is considered to be a *well-posed problem* as close as possible to the unstable problem (4.20). The minimization problem (4.33) determines a micro-fluctuation field $\mathbf{w}$ as indicated in Fig. 3(b). However, as already mentioned the basic difficulty is the detection of relevant functions $\mathbf{w}$ which define the minimizing microstructure.

4.2.2 Rank-One-Convexified Relaxed Incremental Variational Problem

A failure of rank-one convexity conditions (4.23) or (4.27) indicates the instability of the homogeneous deformation state $\mathbf{F}_{n+1}$ and the development of a pattern of first- and higher-order laminates as indicated in Fig. 3(c). We consider the *relaxed energy functional*

$$\mathcal{E}_R(\boldsymbol{\varphi}_{n+1}) = \int_{\mathcal{B}} W_R(\mathbf{F}_{n+1}) \;\mathrm{d}x - [\,\Pi_{\mathrm{ext}}(\boldsymbol{\varphi}_{n+1}) - \Pi_{\mathrm{ext}}(\boldsymbol{\varphi}_n)\,] \;, \tag{4.35}$$

where the internal part of the relaxed energy functional is obtained by replacing the non-convex integrand W in (4.19) by its *rank-one-convex envelope* W_R, which is considered to be close to the quasi-convex envelope W_Q. The current deformation field of the elastic-plastic solid is then determined by the relaxed incremental variational principle

$$\mathcal{E}_R(\boldsymbol{\varphi}^*_{n+1}) = \inf_{\boldsymbol{\varphi}_{n+1}\in\mathcal{W}} \mathcal{E}_R(\boldsymbol{\varphi}_{n+1}) \;, \tag{4.36}$$

that *minimizes the relaxed incremental potential energy* $\mathcal{E}_R$ for the admissible deformation field. In [KS86] a construction was proposed to characterize the rank-one convexification based on a recursion formula. Starting with $W_{R_0}(\mathbf{F}_{n+1}) = W(\mathbf{F}_{n+1})$, one computes the functions

$$W_{R_k}(\mathbf{F}_{n+1}) = \inf_{\xi^+,\xi^-,\mathbf{F}^+,\mathbf{F}^-} \{\xi^+ W_{R_{k-1}}(\mathbf{F}^+) + \xi^- W_{R_{k-1}}(\mathbf{F}^-)\} \quad \text{with} \quad k \geq 1 \;, \tag{4.37}$$

for the scales $k = 1, 2, 3....$ After an infinite number of steps $k \to \infty$ the exact rank-one convexified incremental stress potential

$$W_R(\mathbf{F}_{n+1}) = \lim_{k\to\infty} W_{R_k}(\mathbf{F}_{n+1}) \;, \tag{4.38}$$

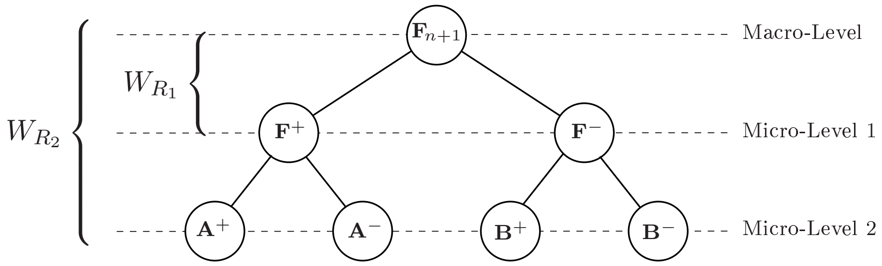


Figure 5: Rank-one convexification and development of sequential laminates. The rank-one convexification $W_{R_k}(\mathbf{F}_{n+1})$ based on Kohn-Strang's recursion formula implies the development of a sequential laminate. Starting from the homogeneous deformation state $\mathbf{F}_{n+1}$ any phase of level $k-1$ decomposes into two phases $(+)$ and $(-)$ of level k. As a consequence, a typical binary tree structure emerges

is obtained. Similar to (4.34), relaxed stresses and tangent moduli are defined as $\bar{\mathbb{P}}_{n+1} := \partial_{\mathbf{F}} W_R(\mathbf{F}_{n+1})$ and $\bar{\mathbb{C}}_{n+1} := \partial^2_{\mathbf{FF}} W_R(\mathbf{F}_{n+1})$. According to recursive approach by [KS86] any phase of order $k-1$ decomposes into two phases $(+)$ and $(-)$ of order k and minimize the average of the corresponding incremental stress potentials. The developing micro-phases form a *sequential laminate*.

Figure 5 shows the typical binary tree structure of a rank-2 laminate. The unstable macroscopic deformation state $\mathbf{F}_{n+1}$ decomposes into two micro-phases $\mathbf{F}^+$ and $\mathbf{F}^-$ of micro-level 1 which again split into two pairs of micro-phases $\mathbf{A}^+$, $\mathbf{A}^-$ and $\mathbf{B}^+$, $\mathbf{B}^-$ of micro-level 2. The rank-one convexified potential W_{R_2} then consists of the volume average of the stress potentials W at the root of the tree, i.e. $W_{R_2}(\mathbf{F}_{n+1}) = \xi^{\mathbf{F}^+}[\xi^{\mathbf{A}^+}W(\mathbf{A}^+)+\xi^{\mathbf{A}^-}W(\mathbf{A}^-)]+\xi^{\mathbf{F}^-}[\xi^{\mathbf{B}^+}W(\mathbf{B}^+)+\xi^{\mathbf{B}^-}W(\mathbf{B}^-)]$. In the context of subgrain dislocation structures in single crystal plasticity, [OR99, ORS00] relax the incremental constitutive description of the material based on the explicit construction of microstructures by recursive lamination and their subsequent equilibration. However, they applied, based on physical arguments, a strong approximation by freezing the orientation of the laminates and the volume fractions during the deformation process. Such a strong assumption has also been applied by [ML03b, ML03a] for the analysis of microstructure development in strain-softening von Mises plasticity. In contrast to these approaches, in [MLG04, AFO03] a rank-one convexification has been proposed that determines both the developing orientation of the laminates as well as the volume fraction.

4.2.3 First-Order Rank-One-Convexified Incremental Problem

We approximate the exact rank-one convexification procedure outlined above by a *two-phase analysis* that takes into account only the first micro-level of Fig. 5. Hence, an unstable macro-deformation $\mathbf{F}_{n+1}$ decomposes into the two phases $\mathbf{F}^+$ and $\mathbf{F}^-$ modeled by ansatz (4.26). Then the first-order rank-one convexification of the nonconvex function W is obtained for two-dimensional problems by the minimization problem

$$W_{R_1}(\mathbf{F}_{n+1}) = \inf_{\mathbf{q}\in\mathcal{Q}} \bar{W}^h(\mathbf{F}_{n+1}, \mathbf{q}) \ , \tag{4.39}$$

for the function $\bar{W}^h$ defined in (4.28) with respect to the set of micro-variables $\mathbf{q}$. A problem similar to (4.39) was solved in [LMD03] for a one-dimensional strain-softening elastic-plastic bar. The solution of the minimization problem (4.39) yields solutions of ξ, d, φ, χ, which in the two-dimensional context determine two stable phases. The relaxed stresses and moduli are obtained by evaluation of derivatives of the function (4.28) with respect to $\mathbf{F}$. The first derivative of (4.39) with respect to the deformation $\mathbf{F}_{n+1}$ at the solution point $\mathbf{q}^*$ reads

$$\partial_{\mathbf{F}} W_{R_1} = \bar{W}^h_{,\mathbf{F}} + [\,\bar{W}^h_{,\mathbf{q}}\,][\,\mathbf{q}_{,\mathbf{F}}\,] \ . \tag{4.40}$$

Here, the last term vanishes due to the necessary condition (4.30) of the minimization problem. Thus we identify the macro-stresses

$$\bar{\mathbb{P}}_{n+1} = \bar{W}^h_{,\mathbf{F}} \ . \tag{4.41}$$

The second derivative of the potential reads

$$\partial^2_{\mathbf{FF}} W_{R_1} = \bar{W}^h_{,\mathbf{FF}} + [\,\bar{W}^h_{,\mathbf{Fq}}\,][\,\mathbf{q}_{,\mathbf{F}}\,] \ . \tag{4.42}$$

Here, the sensitivity of the fluctuation with respect to the macro-deformation is obtained by taking the linearization of (4.30), i.e. $\mathbf{q}_{,\mathbf{F}} = -[\bar{W}^h_{,\mathbf{qq}}]^{-1}[\bar{W}^h_{,\mathbf{qF}}]$. Insertion into (4.42) finally specifies the relaxed moduli to

$$\bar{\mathbb{C}}_{n+1} = \bar{W}^h_{,\mathbf{FF}} - [\,\bar{W}^h_{,\mathbf{Fq}}\,][\,\bar{W}^h_{,\mathbf{qq}}\,]^{-1}[\,\bar{W}^h_{,\mathbf{qF}}\,] \ . \tag{4.43}$$

Observe that the relaxed moduli consist of the volume average of the moduli of the phases and a softening part. The latter is the consequence of the flexibility of the rank-one laminate due to the phase decay. The algorithm of first-order rank-one convexification is summarized in Table 2.

5 Relaxation of Strain Softening Isotropic Plasticity

The relaxation technique outlined in Sect. 4 is applied to the treatment of shearband localizations in strain-softening isotropic elastoplasticity. The softening response of the model causes localization phenomena which is interpreted as microstructure developments on multiple scales associated with non-convex incremental stress potentials. The strain softening inelastic materials with non-convex incremental stress potentials have been investigated in the context of one dimensional elastic-plastic bar in [LMD03], and in isochoric damage mechanics in [GM06].

The main goals of the numerical investigations are the analysis of the *developing microstructures* and the demonstration of the *mesh-invariance* of the relaxation technique proposed. We refer to [ML03b, ML03a] for details of the relaxation algorithm. The elastic energy storage function has the following form

$$\psi(\mathbf{F}_{\rm el}, \alpha) = \frac{\mu}{2}[\,\|\mathbf{F}_{\rm el}\|^2 - 3\,] + \frac{\mu^2}{\lambda}[\,J^{-\lambda/\mu} - 1\,] + \frac{1}{2}h\alpha^2 \ , \tag{5.44}$$

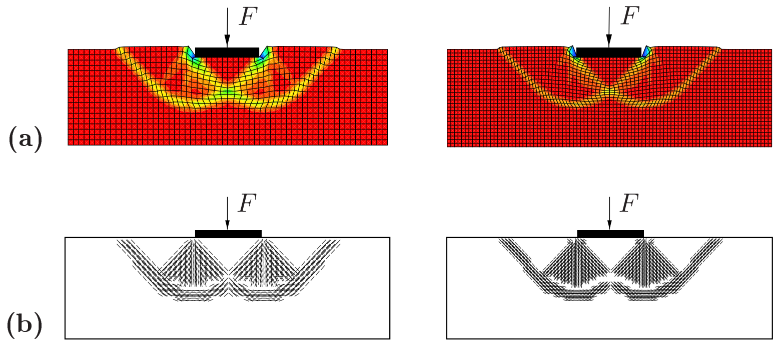


Figure 6: Localization of Indentation Test at Plane Strain. Comparison of the 30×18 and the 45×27 element meshes. **(a)** Deformed meshes with equivalent plastic strains, **(b)** Relevant localization directions

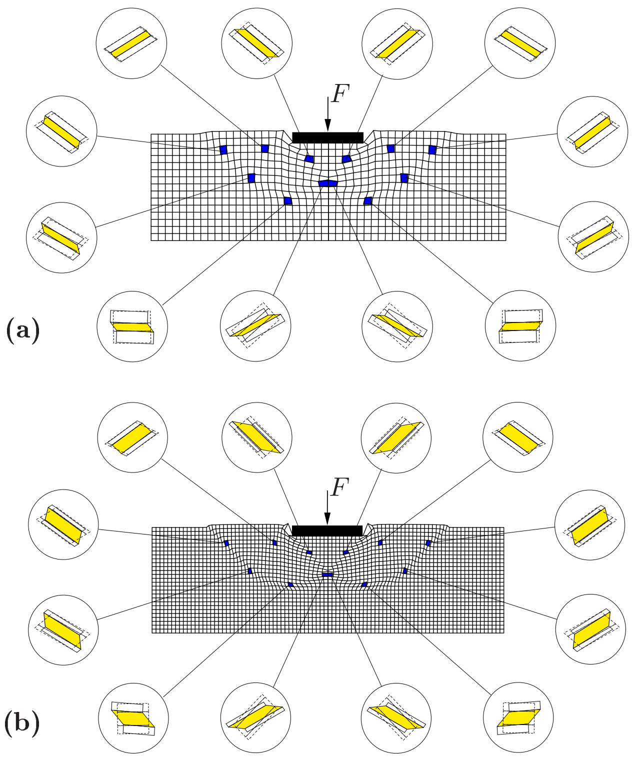


Figure 7: Indentation Test at Plane Strain. Deformed mesh with zoomed-out microstructures of shaded elements at the center Gauss point for the **(a)** 25×15 and **(b)** 45×27 element mesh

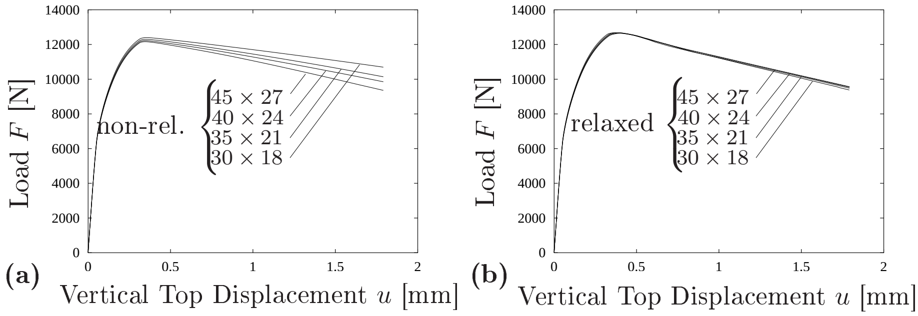


Figure 8: Localization of Indentation Test at Plane Strain. Load displacement curves for different finite element meshes **(a)** with non-relaxed formulation and **(b)** proposed relaxation technique

with $J := \det \mathbf{F}_{\text{el}} = \det \mathbf{F}$, the shear modulus $\mu > 0$, the Lame constant $\lambda > 0$ and the softening modulus $h < 0$. The level set function is given as

$$\mathbb{E} = \{ (\boldsymbol{\Sigma}, \beta) \mid \|\boldsymbol{\Sigma}\| + \sqrt{\frac{2}{3}}\beta \leq c \} , \tag{5.45}$$

where $\boldsymbol{\varphi}$ is the Mandel stress, β is the conjugate force to the hardening variable α and c is a material parameter. Then the dissipation function for the isotropic von Mises plasticity with softening can be formulated as

$$\phi(\mathbf{L}_{\text{pl}}, \dot{\alpha}) = \sup_{(\boldsymbol{\Sigma},\beta)\in\mathbb{E}} \{\boldsymbol{\Sigma} : \mathbf{L}_{\text{pl}} + \beta\dot{\alpha}\} , \tag{5.46}$$

in terms of the plastic velocity gradient $\mathbf{L}_{\text{pl}} := \dot{\mathbf{P}}\mathbf{P}^{-1}$ and the rate of hardening variable $\dot{\alpha}$.

Here, we approximate the minimization problem (4.39) by introduction of an a priori length scale δ representing the width of micro-shearband. Then, in the finite element context the volume fraction ξ at each integration point is described as a function of the length scale δ and a characteristic geometric parameter g of the finite element. A further simplification to the minimization problem is obtained by fixing the laminate orientation angle χ to the critical direction θ_{cr} obtained from the acoustic tensor

$$\mathbf{Q}(\alpha) := \mathbf{L}(\alpha) \cdot \partial^2_{\mathbf{FF}} W(\mathbf{F}_{n+1}) \cdot \mathbf{L}(\alpha) , \tag{5.47}$$

where $\mathbf{L}(\alpha) = [\cos(\alpha)\ \sin(\alpha)]^T$ is a unit vector to describe the localization direction. The material stability is controlled by the following minimization problem for the determinant of the acoustic tensor

$$\min_{\alpha}\{\det[\mathbf{Q}(\alpha)]\} \begin{cases} > 0 : & \text{stable at } \mathbf{F}_{n+1} \\ \leq 0 : & \text{unstable at } \mathbf{F}_{n+1} \end{cases} , \tag{5.48}$$

and if the determinant becomes zero or negative then the critical angle θ_{cr} and the laminate orientation χ are determined as

$$\chi = \theta_{\text{cr}} = \arg\{\min_{\alpha}[\det \mathbf{Q}(\alpha)]\} . \tag{5.49}$$

Furthermore we consider $\mathbf{M} \cdot \mathbf{N} = 0$ which characterizes a shear band type failure. Then, the approximated relaxed energy is obtained by a minimization with respect to one scalar variable d,

$$W_{R_1}(\mathbf{F}_{n+1}) = \inf_d \left[\xi W^+(\mathbf{F}_{n+1}, d) + (1-\xi) W^-(\mathbf{F}_{n+1}, d) \right] . \tag{5.50}$$

Having computed W_{R_1} the relaxed stresses $\bar{\mathbb{P}}$ and the relaxed moduli $\bar{\mathbb{C}}$ can be computed from (4.41) and (4.43), respectively.

As a representative example, we consider next a plane strain indentation test where a localization in the form of curved shear bands are observed experimentally. The equivalent plastic strains and the formation of shear bands with corresponding localization directions are plotted in Fig. 6. In Fig. 7 the development of microstructures is visualized at the selected integration points for two discretizations. In order to prove the mesh objectivity of the proposed relaxation algorithm load-deflection curves are plotted in Fig. 8 for four different mesh densities. The non-relaxed formulation in Fig. 8(a) shows a clear mesh dependency whereas the proposed relaxation algorithm in Fig. 8(b) exhibits no mesh dependency in the post-critical regime.

6 Relaxation of Non-Convex Single-Slip Plasticity

We now point out details of the first-order rank-one convexification analysis introduced in Sect. 4 for the model problem of single slip plasticity. Different from the strain softening example discussed in Sect. 5, the non-convexity appears in the single-slip plasticity as a result of geometric constraints related with the orientation of a slip-system. The model problem of single slip crystal plasticity has already been investigated in several works, see [CHM02, BCHH04, Mie04a, CT05, CO05, MLG04].

The main goals of the numerical investigations are the analysis of the *developing microstructures* and the demonstration of the *mesh-invariance* of the relaxation technique proposed. We refer to [MLG04] for details of the relaxation algorithm based on first-order rank-one convexification. As a concrete form, we apply a compressible Neo-Hookean material

$$\psi(\mathbf{F}_{\mathrm{el}}) = \frac{\mu}{2}[\|\mathbf{F}_{\mathrm{el}}\|^2 - 3] + \frac{\kappa}{4}[J^2 - 2(1 + 2\frac{\mu}{\kappa}) \ln J - 1] , \tag{6.51}$$

with $J := \det\mathbf{F}_{\mathrm{el}} = \det\mathbf{F}$. $\kappa > 0$ and $\mu > 0$ denote the bulk and the shear moduli, respectively. The dissipation function for the linear hardening model of single-slip plasticity is

$$\phi(\mathbf{L}_{\mathrm{pl}}) = [\tau_0 + h\gamma] \, |\mathbf{L}_{\mathrm{pl}} : (\mathbf{S} \otimes \mathbf{T})| , \tag{6.52}$$

in terms of the Schmid stress τ associated with the slip system of single-slip plasticity and the linear hardening modulus h. The slip system is described by the slip direction $\mathbf{S}$ and the slip normal $\mathbf{T}$ with $\mathbf{S} \cdot \mathbf{T} = 0$.

Here, a key contribution is the derivation of a *semi-analytical solution* that reduces for two-dimensional problems the independent micro-variables from four in $\mathbf{q}$ to just

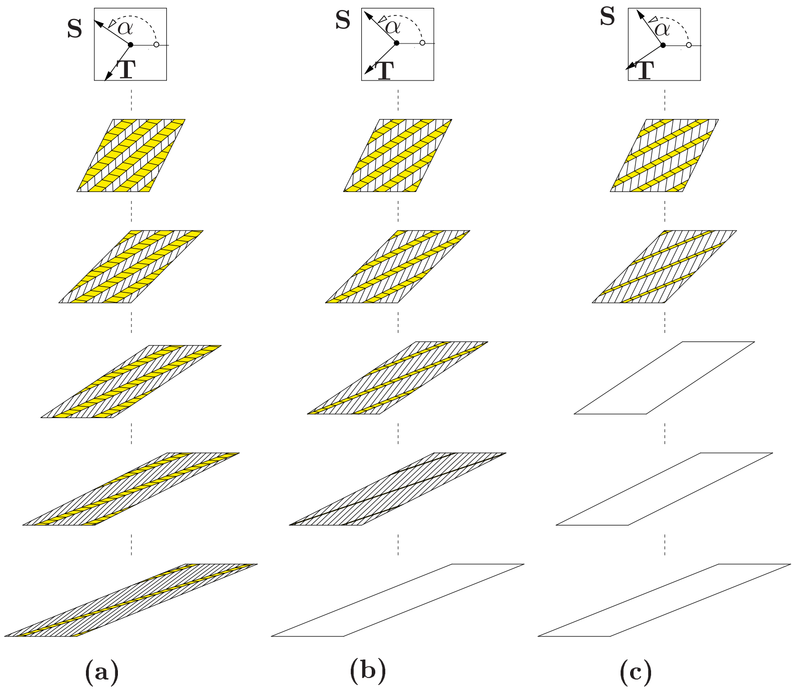


Figure 9: Simple shear test. Comparison of evolution of microstructures for simple shear test with three different slip system **(a)** $\alpha = 145°$, **(b)** $\alpha = 135°$, **(c)** $\alpha = 125°$. After loss of material stability microstructures develop which are modeled as first-order rank-one laminates

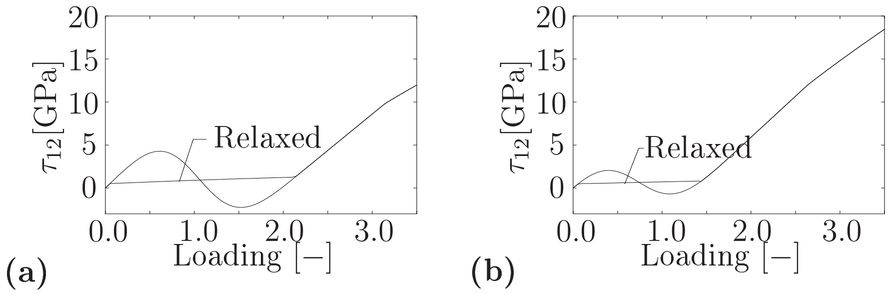


Figure 10: Simple shear test. Comparison of relaxed and non-relaxed Kirchhoff stress components for plane strain tension test and two different slip systems. **(a)** $\alpha = 135°$, **(b)** $\alpha = 125°$. The shape of the governing stress coordinate τ_{12} represents a snap-through behavior within the non-convex range. After recovery of the stable homogeneous state the relaxed and the non-relaxed stress responses coincide again

one variable. Recall the necessary conditions (4.30) of the minimization problem of relaxation

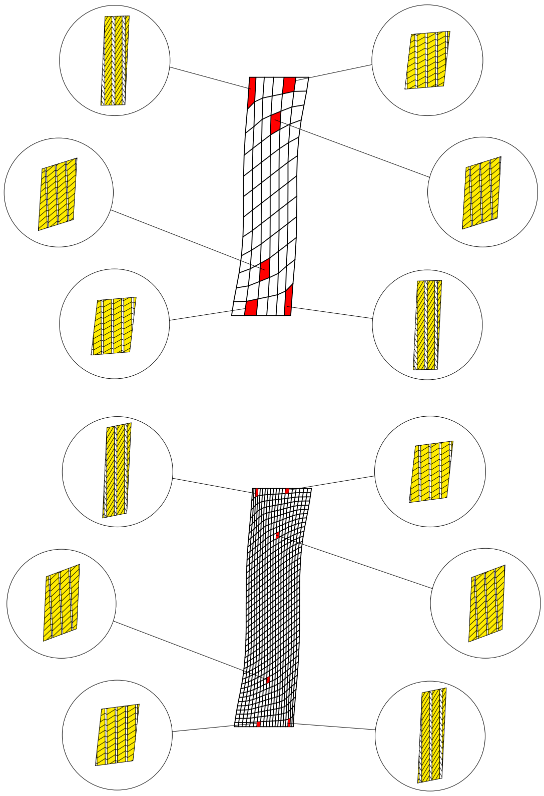
$$\left.\begin{array}{lcll} \bar{W}^h_{,\xi} & = & W^+ - W^- - d\,[\xi\mathbb{P}^+ + (1-\xi)\mathbb{P}^-] : (\mathbf{FM}\otimes\mathbf{N}) & = 0 \\ \bar{W}^h_{,d} & = & \xi(1-\xi)\quad [\mathbb{P}^+ - \mathbb{P}^-] : (\mathbf{FM}\quad\otimes\mathbf{N}) & = 0 \\ \bar{W}^h_{,\varphi} & = & \xi(1-\xi)\,d\,[\mathbb{P}^+ - \mathbb{P}^-] : (\mathbf{FM}_{,\varphi}\otimes\mathbf{N}) & = 0 \\ \bar{W}^h_{,\chi} & = & \xi(1-\xi)\,d\,[\mathbb{P}^+ - \mathbb{P}^-] : (\mathbf{FM}\quad\otimes\mathbf{N}_{,\chi}) & = 0 \end{array}\right\} , \qquad (6.53)$$

in terms of the four micro-variables $\mathbf{q} := [\xi, d, \varphi, \chi]^T$. Note that first two conditions in (6.53) are the physical and the configurational force equilibrium conditions on the interface between two phases. In the sequel, we will evaluate these conditions and derive a semi-analytical solution for the minimizing laminate $\mathbf{F}^{\pm}$. The plastic deformation $\mathbf{P}^{\pm}$ and the hardening variable in the phases $(\pm)$ are denoted

$$\mathbf{P}^{\pm} = \mathbf{P}^{\star}(\mathbf{1} \pm \Delta\gamma^{\pm}\mathbf{S}\otimes\mathbf{T}) \quad \text{and} \quad \gamma^{\pm} = \gamma^{\star} + \Delta\gamma^{\pm} , \qquad (6.54)$$

where $\Delta\gamma^{\pm} = (\gamma - \gamma_n)^{\pm}$ are the incremental plastic arc lengths. $\mathbf{P}^{\star}$ and $\gamma^{\star}$ are the plastic deformation and $\gamma^{\star}$ the hardening variable of the last stable homogeneous state, respectively. Equation (6.54) points out the cause of the phase decay for the model problem of single slip plasticity that results from the *bifurcation of the plastic deformation* starting from $\mathbf{P}^{\star}$ with $\Delta\gamma^{\pm}$. The equilibrium of the Schmidt stresses $\tau^+ = \tau^-$ yields the identity $\Delta\gamma^+ = \Delta\gamma^- = \Delta\gamma$ of the incremental slips. If one postulates the preservation of the volumetric deformation $\det[\mathbf{F}^+] = \det[\mathbf{F}^-] = \det[\mathbf{F}]$ it turns out that the Lagrangian laminate vectors are orthogonal, i.e. $\mathbf{N}\cdot\mathbf{M} = 0$. This result allows for the parameterization of these vectors in terms of the vectors of the slip system $\mathbf{N} = \cos\theta\ \mathbf{S} - \sin\theta\ \mathbf{T}$ and $\mathbf{M} = \sin\theta\ \mathbf{S} + \cos\theta\ \mathbf{T}$ where θ is an in-plane orientation angle. Exploitation of these results leads to the identification of the inclination angle and a formula for the micro-intensity

$$\tan\theta = -\mathbf{P}^{\star} : \mathbf{S}\otimes\mathbf{T} \quad \text{and} \quad d = \frac{2\Delta\gamma}{\cos^2\theta\ (1 + \Delta\gamma^2)} \ . \qquad (6.55)$$



(a) **(b)**

Figure 11: Rectangular specimen in plane strain tension. Visualization of microstructure developments at selected Gauss points for discretizations with **(a)** 6×12 and **(b)** 20×40 elements

Insertion of the above obtained results in the necessary conditions $(6.53)_{1,4}$ yields an expression for the volume fraction

$$\xi = \frac{1}{2} + d^{-1}\left[\frac{c_{NM}}{c_{MM}} + \tan\theta\right] , \tag{6.56}$$

where we have introduced the abbreviation $c_{XY} = \mathbf{X}\cdot\mathbf{C}\cdot\mathbf{Y}$. The incremental plastic multiplier $\Delta\gamma$ can be determined by algebraic manipulations as

$$\Delta\gamma = \frac{2d + E}{\cos^2\theta\; d^2 + F} , \tag{6.57}$$

in terms of the coefficients $E = -4(h\gamma^\star + c)/(\mu c_{MM})$ and $F = [4h/\mu + 4\cos^2\theta\,(c_{NN} - c_{NM}^2/c_{MM})]/c_{MM}$. Note, that the incremental slip is only a function of the micro-intensity.

As a consequence, insertion of (6.57) into $(6.55)_2$ leads to a polynomial of degree five that $p(d)$ which *only depends on the micro-intensity*. The solutions of the polynomial $p(d)$ is the relevant micro-intensity d^* that minimizes the volume average of the stress-potentials in the two micro-phases

$$d^* = \arg\{\inf_{d\in\mathcal{D}}[\bar{W}^h]\} \quad \text{with} \quad \mathcal{D}\in\{d|p(d)=0\} . \tag{6.58}$$

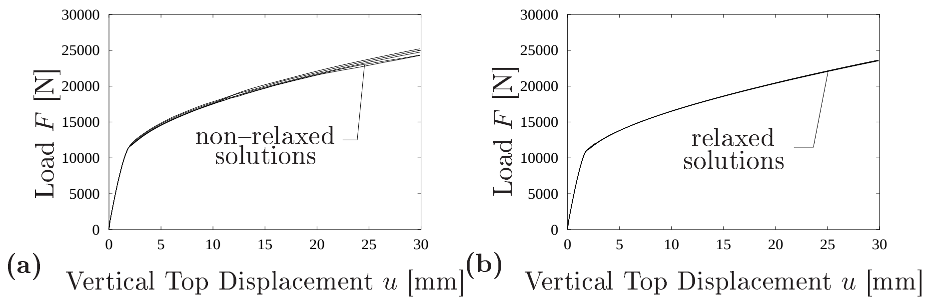


Figure 12: Rectangular specimen in tension. Load-displacement curves for five different finite element meshes in terms of **(a)** the non-relaxed (non-objective) formulation (the finer the mesh the softer the response) **(b)** the proposed relaxation technique

Having computed d, the volume fraction ξ, the relaxed stresses $\bar{\mathbb{P}}$ and the relaxed moduli $\bar{\mathbb{C}}$ can be computed from (6.56), (4.41) and (4.43), respectively.

First, we investigate a homogeneous simple shearing with different slip systems shown in Fig. 9. Because of a specific choice of the orientation of the slip-systems the material stability of the homogeneous deformation can be lost and microstructures may arise. The development of the first-order rank-one laminate type microstructures is plotted in Fig. 9 for various level of deformation. During the macro deformation the plastic slip-systems start to rotate and align to the principal loading mode. The stronger the blocking of the principal deformation the longer the non-convex range. The shear component of the Kirchhoff stress for the relaxed and the non–relaxed solutions are plotted in Fig. 10 where the range of the non-convex domain is clearly dependent on the chosen slip system orientation.

Next, a plane strain tension test is considered where the slip direction vector is taken to be 10° counterclockwise from the horizontal. In Fig. 11 the development of microstructures is visualized for two different mesh densities. The specific orientation of the slip system causes the non-convex incremental potential which is relaxed by the proposed algorithm in terms of first-order laminates. In order to prove the mesh objectivity of the proposed relaxation algorithm load-deflection curves are plotted in Fig. 12 for different mesh densities. Although there is no softening in the model, the non-relaxed formulation shows mesh dependency due to non-convexity in the problem whereas the proposed relaxation algorithm exhibits no mesh dependency.

Final example is concerned with a rectangular specimen in shear where the slip direction is chosen to be 135° counterclockwise from the horizontal. In Fig. 13, the development of microstructures is visualized by considering two different levels of deformation where the evolution of volume fractions and laminate orientations can be seen.

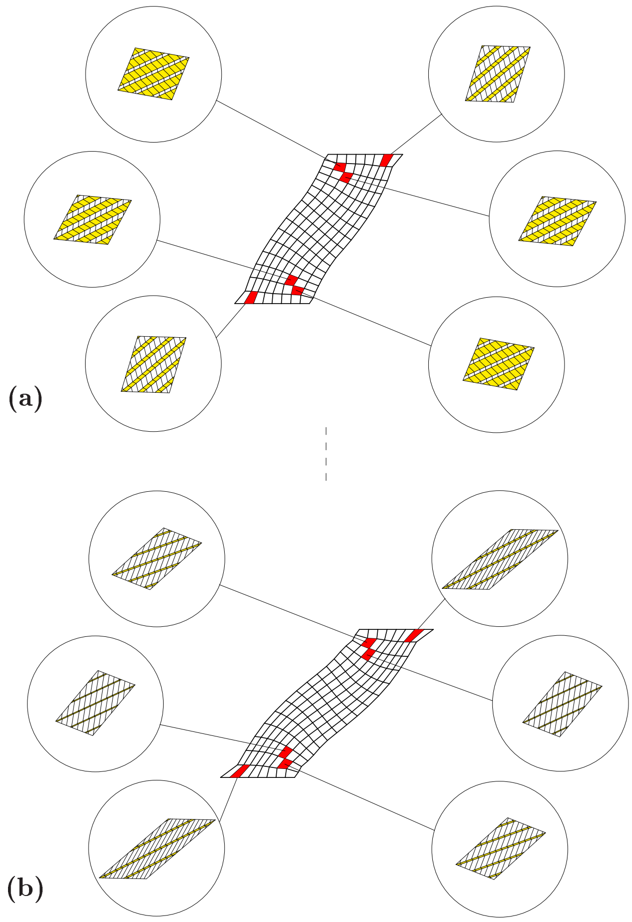


Figure 13: Rectangular specimen in shear. Visualization of microstructure developments at selected Gauss points at **(a)** $u = 20mm$ **(b)** $u = 30mm$

7 Conclusions

The energetic formulation for finite-strain elastoplasticity has been proved as a very flexible mathematical tool that links the heavily used time-incremental minimization problem to a suitable weak time-continuous problem. Moreover, the theory of the the calculus of variations can be used to provide existence results for the incremental problem as well as for the time-continuous one. For the latter case we still need to assume spatial regularizations to prevent the formation of microstructure. At present the global existence theory has proved to be successful in the simplest situations, but further developments is needed to explore the capability of the method for providing classical solutions, i.e., without microstructure. Moreover, it will be essential to derive reliable and efficient numerical algorithms in the spirit of [MR05, MRS06b].

The energetic formulation has the major drawback that the stability condition (S) is a *global* condition, whereas a *local* condition would be more physical and better for numerical purposes. First results to understand rate-independent systems as limits of systems with small viscosity are presented in [EM06], but this theory is restricted to finite-dimensional Hilbert spaces. Generalizations to infinite dimensions including abstract metric spaces are developed in [MRS06a], but there applicability in elastoplasticity is still out of reach.

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Table 1: Overview: Minimization Principles for Standard Dissipative Solids

(M) Constitutive Model. $\mathbf{F} \in GL_+(3)$ at $x \in \mathcal{B}$ is the local deformation and $\boldsymbol{\mathcal{I}} \in Z$ a generalized vector of internal variables. Set of local material equations has the structure

$$\begin{array}{ll} \textit{stresses} & \mathbb{P} = \partial_{\mathbf{F}} \psi(\mathbf{F}, \boldsymbol{\mathcal{I}}) \\ \textit{evolution equation} & \mathbf{0} \in \partial_{\boldsymbol{\mathcal{I}}} \psi(\mathbf{F}, \boldsymbol{\mathcal{I}}) + \partial_{\dot{\boldsymbol{\mathcal{I}}}} \phi(\dot{\boldsymbol{\mathcal{I}}}, \boldsymbol{\mathcal{I}}) \ , \ \boldsymbol{\mathcal{I}}(0) = \boldsymbol{\mathcal{I}}_0 \end{array}$$

defined in terms of an energy storage and a dissipation function ψ, ϕ.

(C) Incremental Variational Formulation of Constitutive Model. In a finite time increment $[t_n, t_{n+1}]$, the *minimization problem of the constitutive response*

$$\begin{array}{ll} \textit{stresses} & \mathbb{P}_{n+1} = \partial_{\mathbf{F}} W(\mathbf{F}_{n+1}) \\ \textit{stress potential} & W(\mathbf{F}_{n+1}) = \inf_{\boldsymbol{\mathcal{I}}} \int_{t_n}^{t_{n+1}} [\, \dot{\psi} + \phi \,] \, \mathrm{d}t \ , \ \boldsymbol{\mathcal{I}}(t_n) = \boldsymbol{\mathcal{I}}_n \end{array}$$

determines the current internal state $\boldsymbol{\mathcal{I}}_{n+1} \in Z$ and provides a potential for the stresses at time t_{n+1}.

(S) Stability of Incremental Constitutive Response. In $[t_n, t_{n+1}]$ the *material is locally stable if the incremental stress potential W is quasi–convex*

$$\begin{array}{ll} \textit{stable response} & W(\mathbf{F}_{n+1}) \leq \inf_{\mathbf{w}} \frac{1}{|D|} \int_D W(\mathbf{F}_{n+1} + \nabla \mathbf{w}(\mathbf{y})) \, \mathrm{d}x \end{array}$$

for all possible fluctuations $\mathbf{w}(\mathbf{y})$ on the domain D.

(R) Microstructure Development in Non–Stable Materials. For an unstable non–convex response, the *incremental minimization problem of convexification*

$$\begin{array}{ll} \textit{macro–stresses} & \mathbb{P}_{Qn+1} = \partial_{\mathbf{F}} W_Q(\mathbf{F}_{n+1}) \\ \textit{relaxation} & W_Q(\mathbf{F}_{n+1}) = \inf_{\mathbf{w}} \frac{1}{|D|} \int_D W(\mathbf{F}_{n+1} + \nabla \mathbf{w}(\mathbf{y})) \, \mathrm{d}x \end{array}$$

provides a relaxed quasi–convex hull W_Q of W and determines the current microstructure fluctuation field $\mathbf{w}(\mathbf{y})$.

Table 2: First–Order Rank–One Convexification of Incremental Response

1. Database $\{\mathbf{F}_{n+1}, \boldsymbol{\mathcal{I}}_n^+, \boldsymbol{\mathcal{I}}_n^-\}$ and starting value $\mathbf{q}_0 := \{\xi, d, \mathbf{N}, \mathbf{M}\}_0$ given.

2. Set micro–deformation phases

$$\mathbf{F}^{\pm} := \mathbf{F}_{n+1}\mathbf{L}^{\pm} \quad \text{with} \quad \left\{ \begin{array}{lcl} \mathbf{L}^+ := & \mathbf{1} & +(1-\xi)d\,\mathbf{M}\otimes\mathbf{N}\,, \\ \mathbf{L}^- := & \mathbf{1} & - \quad \xi d\,\mathbf{M}\otimes\mathbf{N}\,. \end{array} \right.$$

3. Evaluate the potential $\bar{W}^h(\mathbf{F}_{n+1}, \mathbf{q}) = \xi W(\mathbf{F}^+) + (1-\xi)W(\mathbf{F}^-)$

   and its derivatives $\bar{W}^h_{,\mathbf{F}}$, $\bar{W}^h_{,\mathbf{q}}$, $\bar{W}^h_{,\mathbf{FF}}$, $\bar{W}^h_{,\mathbf{qq}}$, $\bar{W}^h_{,\mathbf{q}\bar{\mathbf{F}}}$.

4. Convergence check: If $(\| \bar{W}^h_{,\mathbf{q}} \| \leq tol)$ go to 6.

5. Newton update of micro–variables $\mathbf{q} \Leftarrow \mathbf{q} - [\,\bar{W}^h_{,\mathbf{qq}}\,]^{-1}[\,\bar{W}^h_{,\mathbf{q}}\,]$.

6. Set relaxed macro–stresses and tangent macro–moduli

$$\bar{\mathbb{P}}_{n+1} = \bar{W}^h_{,\bar{\mathbf{F}}} \quad \text{and} \quad \bar{\mathbb{C}}_{n+1} = \bar{W}^h_{,\bar{\mathbf{F}}\bar{\mathbf{F}}} - [\,\bar{W}^h_{,\bar{\mathbf{F}}\mathbf{q}}\,][\,\bar{W}^h_{,\mathbf{qq}}\,]^{-1}[\,\bar{W}^h_{,\mathbf{q}\bar{\mathbf{F}}}\,]\,.$$