Homoclinic and Heteroclinic Solutions in Two–Phase Flow

Alexander Mielke
Institut für Angewandte Mathematik, Universität Hannover
mielke@ifam.uni-hannover.de

1. Introduction

We consider travelling waves in a system of two fluid layers of infinite extent which are placed between the rigid bottom and top of a channel under the action of gravity. Both fluids are assumed to be irrotational, inviscid, and of constant density \( \rho_1 \neq \rho_2 \). Our aim is to give a rigorous approach to the existence of solitary waves and of bores. Both of these solutions types look at infinity like parallel flows, however the first attains the same limits upstreams and downstreams whereas the second is a front–like solution connecting parallel flows of different heights. These solutions are also called bores and are observed in some rivers when waves coming from the ocean travel upstreams.

There are several approaches to this problem. In the case of no surface tension Turner\(^{14}\) obtained the two–fluid model by considering stratified fluids with smooth density profiles converging to a piecewise constant one. The stratified fluid model (Long–Yih equation) is a semilinear elliptic problem\(^9\) and can be treated using variational methods, and thus global results can be derived\(^1\).

Surface wave problems and interface problems are described by quasilinear equations giving rise to more delicate phenomena like surface singularities as for the Stokes wave of extreme height. We use the spatial center manifold approach in the form of\(^{12}\), which is based on the original ideas in \(^9\). Thus, we are restricted to a local theory, however, all difficulties arising from the quasilinearity are circumvented by this approach.

Depending on the densities \( \rho_i \), upstream velocities \( u_{i\infty} \) and the heights \( h_i \) of the fluids we define the dimensionless elevation number \( E = h_2^2/h_1^2 - \rho_2 u_{2\infty}^2/(\rho_1 u_{1\infty}^2) \), which tells us whether bifurcating solitary waves are waves of elevation (\( E > 0 \)) or waves of depression (\( E < 0 \)). Here we analyze the unfolding of the case \( E \approx 0 \) which leads to a scenario where the growing branch of solitary waves obtains waves with larger and larger plateaus, see\(^{15}\) for physical observations of this effect. Suitable translates of these plateau–like solitary waves converge on compact intervals to heteroclinic solutions (bores).

Here we have restricted ourselves to the case of zero surface tension, however the method applies equally well in cases with surface tension\(^2\), \(^11\). Also the influence of localized perturbations travelling with the same frame speed can be analyzed, see\(^{12}\).
2. The basic equations

At the inflow \((x \to -\infty)\) the fluid layers have height \(h_1\) and \(h_2\) and inflow velocities \(u_{1\infty}\) and \(u_{2\infty}\), respectively. The interface between the layers is given by the function \(y = Y(x)\). Taking \(h_1 + h_2, \rho_1, u_{i\infty}\), and \(\rho_1 u_{i\infty}^2\) as reference quantities for length, density, velocities \((u_i, v_i)\) and the pressure, we obtain the following equations in dimensionless form:

\[
(x, y) \in S_i: \begin{cases}
  u_{ix} + v_{iy} = 0, \\
  u_{iy} - v_{ix} = 0,
\end{cases}
\]

for \(i = 1, 2\);

\[
y = 0: v_1 = 0; \quad y = 1: v_2 = 0;
\]

\[
y = Y(x): \begin{cases}
  \frac{u_1}{u_2} = Y', \quad \frac{1}{2}(u_1^2 + v_1^2) + \lambda Y + p = C_1 = \text{const}, \\
  \frac{u_2}{u_2} = Y', \quad \frac{2}{\rho_2}(u_2^2 + v_2^2) + \frac{\rho_2}{\rho_1} \lambda Y + p = C_2 = \text{const}.
\end{cases}
\]

(1)

The fluid layers occupy the regions \(S_1\) and \(S_2\) given by \(0 < y < Y(x)\) and \(Y(x) < y < 1\), respectively. The first two equations are mass conservation and irrotationality. On the interface \(y = Y\) we have the kinematic constraint and Bernoulli’s law for both fluids (the interface is a streamline). From the inflow conditions \((u_i, v_i) \to (1, 0)\) and \(Y \to h\) for \(x \to -\infty\) we find the constants \(C_1 = 1/2 + \lambda h\) and \(C_2 = \rho/2 + \rho_2 \lambda h / \rho_1\).

The coupling between the layers occurs through the pressure \(p\) which can be easily eliminated. Here and further on we use the non-dimensionlal parameters

\[
\lambda = \frac{g(h_1 + h_2)}{u_{1\infty}^2}, \quad \rho = \frac{\rho_2 u_{2\infty}^2}{\rho_1 u_{1\infty}^2}, \quad h = \frac{h_1}{h_1 + h_2}, \quad \mu = \frac{\rho_1 - \rho_2}{\rho_1} \lambda.
\]

Often one is interested in waves travelling through fluid layers in rest at infinity. Then, in the moving frame we have \(u_{1\infty} = u_{2\infty}\). In any case we have \((u_i, v_i) \to (1, 0)\) for \(x \to -\infty\).

We want to transform the system such that it can be written as an abstract differential equation in the form

\[
\frac{d}{dx} \varphi = L_\mu \varphi + N(\varphi), \quad \varphi \in X,
\]

(2)

where \(N(\varphi) = O(\|\varphi\|^2)\). Therefore we introduce the stream function \(\psi\) through \((u_i, v_i) = (\psi_y, -\psi_x)\) and \(\psi(x, 0) = 0, \psi(x, y) \to y\) for \(x \to \infty\). The stream function \(\psi\) is continuous but not differentiable across the interface, where \(\psi(x, Y(x)) = h\).

Following we transform the velocities according to \(U_i(x, \psi(x, y)) = (u_1^2(x, y)^2 + v_1^2(x, y)^2 - 1)/2\) and \(V_i(x, \psi(x, y)) = v_i(x, y)/u_i(x, y)\). Using \(u_i = R_i(U_i, V_i) = \sqrt{(1 + 2U_i)/(1 + V_i^2)}\) we find

\[
(x, \psi) \in \bar{S}_i: \begin{cases}
  \frac{\partial}{\partial x} \begin{pmatrix} U_i \\ V_i \end{pmatrix} = \begin{pmatrix} V_i R_i & -R_i^3 \\ 1/R_i & V_i R_i \end{pmatrix} \frac{\partial}{\partial \psi} \begin{pmatrix} u_i \\ v_i \end{pmatrix}, \\
  \psi = 0: V_1 = 0, \quad \psi = 1: V_2 = 0,
\end{cases}
\]

\[
\psi = h: V_1 = V_2 = Y', \quad U_1 - \rho U_2 + \mu[Y - h] = 0,
\]

2
where \( \tilde{S}_1 = \mathbb{R} \times (0, h) \) and \( \tilde{S}_2 = \mathbb{R} \times (h, 1) \). Additionally, we have the relations

\[
Y = \int_0^h \frac{1}{S_1} \, d\psi = 1 - \int_h^1 \frac{1}{S_2} \, d\psi
\]  
(3)

which are a consequence of \( Y(x) = \int_0^x \frac{1}{S(\psi)} \, d\psi \) \( \frac{1}{S_1} \, d\psi = \int_0^1 \frac{1}{S_2} \, d\psi \) and the analogous consideration for \( y \in \text{int} \). For

According to (3) this is a nonlinear condition, since

\[
\text{condition reads}
\]

Again following we introduce the variable \( B = U_1(h) - \rho U_2(h) \), and the interfacial condition reads

\[
B + \mu(Y - h) = 0.
\]  
(4)

According to (3) this is a nonlinear condition, since \( Y \) has to be expressed through \((U, V)\). We differentiate (4) and use \( Y' = V_1(h) \) in order to obtain \( B' = dB/dx = -\mu V_1(h) \). With \( \varphi = (U_1, V_1, U_2, V_2, B)^T \) the problem takes the form (2) where the basic phase space is \( X = L^2(0, h)^2 \times L^2(h, 1)^2 \times \mathbb{R} \),

\[
D(L_\mu) = \{ \varphi \in H^1(0, h)^2 \times H^1(h, 1)^2 \times \mathbb{R} : V_1(0) = V_2(1) = 0, V_1(h) = V_2(h),
B = U_1(h) - \rho U_2(h) \},
\]

\[
L_\mu = \begin{pmatrix}
U_1 \\
V_1 \\
U_2 \\
V_2 \\
B
\end{pmatrix}
\]

and

\[
N(\varphi) = \begin{pmatrix}
V_1 R_1 \partial_\psi U_1 - (R_1^2 - 1) \partial_\psi V_1 \\
V_1 R_1 \partial_\psi V_1 - (R_1^2 - 1) \partial_\psi U_1 \\
V_2 R_2 \partial_\psi U_2 - (R_2^2 - 1) \partial_\psi V_2 \\
V_2 R_2 \partial_\psi V_2 - (R_2^2 - 1) \partial_\psi U_2 \\
-\mu V_1(h)
\end{pmatrix}
\]

Here \( N \) is a smooth (analytic) mapping from \( D(L_\mu) \) into \( X \), which vanishes quadratically for \( \varphi \to 0 \). For later use we derive the following spectral properties of \( L_\mu \) in dependence of \( \mu > 0 \).

**Theorem 2.1** (a) The spectrum of \( L_\mu \) consists of discrete eigenvalues. They are exactly the solutions of the dispersion relation

\[
F_\mu(\sigma) = [\mu - \sigma \cot(\sigma h) - \rho \sigma \cot(\sigma(1 - h))]\sigma^2.
\]

(b) For all \( \mu \) the operator \( L_\mu \) has a two-fold eigenvalue \( 0 \). For \( \mu < \mu_0 := 1/h + \rho/(1 - h) \), there are no further eigenvalues on the imaginary axis. For \( \mu \geq \mu_0 \) there is a pair of purely imaginary eigenvalues \( \pm i \omega(\mu) \) with \( \omega(\mu_0) = 0, \frac{d\omega}{d\mu} > 0 \), and \( \omega(\mu)/\mu \to 1/(1 + \rho) \) for \( \mu \to \infty \).

(c) For all \( \mu \) the estimate \( \| (L_\mu + is)^{-1} \|_{X \to X} = \mathcal{O}(1/|s|), \ s \in \mathbb{R} \), holds.

**PROOF:** The eigenvalue problem reduces to an ordinary differential equation. The homogeneous problem \( \sigma \varphi = L_\mu \varphi \) gives \( \sigma U_i = -\partial_\psi V_i \) and \( \sigma V_i = \partial_\psi U_i \). With \( V_1(0) = V_2(1) = 0 \) and \( V_1(h) = V_2(h) \) this leads to

\[
(U_1, V_1, U_2, V_2) = c_0 \begin{pmatrix}
-\cos \sigma \psi \\
\sin \sigma \psi \\
\cos \sigma(1 - \psi) \\
\sin \sigma(1 - \psi)
\end{pmatrix}
\]

\[
\sin \sigma \psi, \sin \sigma(1 - \psi) \sin \sigma(1 - \psi), \sin \sigma(1 - \psi)
\]

\[
\begin{pmatrix}
-\cos \sigma \psi \\
\sin \sigma \psi \\
\cos \sigma(1 - \psi) \\
\sin \sigma(1 - \psi)
\end{pmatrix}
\]

\[
\begin{pmatrix}
-\cos \sigma \psi \\
\sin \sigma \psi \\
\cos \sigma(1 - \psi) \\
\sin \sigma(1 - \psi)
\end{pmatrix}
\]
From \(-\mu V_1(h) = \sigma B = \sigma[U_1(h)-\rho U_2(h)]\), we find that \(c_0\) has to be 0 unless \(F_\mu(\sigma) = 0\), and part (a) is proved. Part (b) is a simple discussion of the zeros of \(F_\mu\).

To establish the resolvent estimate, we consider a general \(\eta = (f_1, g_1, f_2, g_2), \alpha \in X\) and \(s \in \mathbb{R}\). If \(F_\mu(is) \neq 0\) the resolvent equation \((L_\mu + is)\phi = \eta\) is solvable. It reads

\[
\begin{align*}
-\partial_\psi V_j + isU_j &= f_j, \quad \partial_\psi U_j + isV_j = g_j, \quad j = 1, 2; \\
-\mu V_1(h) + isB &= \alpha, \quad V_1(0) = V_2(1) = 0, \quad V_1(h) = V_2(2), \quad B = U_1(h) - \rho U_2(h).
\end{align*}
\]

Using simple integrations by part we find

\[
\begin{align*}
J_0^h(|f_1|^2 + |g_1|^2)d\psi &= J_0^h(|\partial_\psi U_1|^2 + |\partial_\psi V_1|^2 + s^2|U_1|^2 + s^2|V_1|^2)d\psi + 2is\text{Im}(V_1(h)\overline{U_1(h)}), \\
J_h^1(|f_2|^2 + |g_2|^2)d\psi &= J_h^1(|\partial_\psi U_2|^2 + |\partial_\psi V_2|^2 + s^2|U_2|^2 + s^2|V_2|^2)d\psi - 2is\text{Im}(V_2(h)\overline{U_2(h)}).
\end{align*}
\]

Using \(V_1(h) = V_2(h)\) and \(B = U_1(h) - \rho U_2(h)\) leads to

\[
\begin{align*}
\|&(f_1, g_1, \sqrt{\rho} f_2, \sqrt{\rho} g_2)\|^2 = \|\partial_\psi(U_1, V_1, \sqrt{\rho} U_2, \sqrt{\rho} V_2)\|^2 \\
&\quad + s^2\|U_1, V_1, \sqrt{\rho} U_2, \sqrt{\rho} V_2\|^2 + 2is\text{Im}(V_1(h)\overline{B})).
\end{align*}
\]

Moreover, we have \(|sB| = |\alpha + \mu V_1(h)| \leq |\alpha| + \mu|V_1(h)|\) and \(|V_1(h)|^2 \leq \delta\|\partial_\psi V_1\|^2 + \|V_1\|^2/\delta\) for any \(\delta > 0\). This allows the estimate

\[
s^2|B|^2 - 2is\text{Im}(V_1(h)\overline{B}) \leq 2s^2|B|^2 + |V_1(h)|^2 \leq 4\alpha^2 + (5\mu^2 + 1)\delta\|\partial_\psi V_1\|^2 + (5\mu^2 + 1)|V_1|^2/\delta.
\]

Choosing \(\delta = 1/(5\mu^2 + 1)\) and inserting the result into (5) gives

\[
\begin{align*}
\min\{1, \rho\}(s^2 - (5\mu^2 + 1)^2)\|\phi\|^2 & \leq (s^2 - (5\mu^2 + 1)^2) \left[\|U_1, V_1, \sqrt{\rho} U_2, \sqrt{\rho} V_2\|^2 + |B|^2\right] \\
& \leq \|(f_1, g_1, \sqrt{\rho} f_2, \sqrt{\rho} g_2)\|^2 + 4\alpha^2 \leq \max\{4, \rho\}\|\eta\|^2,
\end{align*}
\]

which is the content of part (c).

\(\square\)

3. Reduction by first integrals

As indicated in Theorem 1, the operator \(L_\mu\) has a double zero eigenvalue. It corresponds to the two-dimensional family of equilibria given by

\[
\phi = (U_1, V_1, U_2, V_2, B)^T = (\alpha, 0, \gamma, 0, 0)^T, \alpha, \gamma \in \mathbb{R}.
\]

These are parallel flows with constant speeds \(R_1 = \sqrt{1+2\alpha}\) and \(R_2 = \sqrt{1+2\gamma}\) in the lower and upper layer, respectively. From this we find \(Y = \int_0^h \frac{d\psi}{R_1} = h/\sqrt{1+2\alpha}\) and the height of both layers is \(\int_0^h 1/R_1d\psi + \int_h^1 1/R_2d\psi = h/\sqrt{1+2\alpha} + (1-h)/\sqrt{1+2\gamma}\).

Since all these solutions can be rescaled to the solution \(\alpha, \gamma = 0\), we see that this family is generated artificially. In fact, one 0 eigenvalue is due to the transformation
from \((x, y)\) into \((x, \psi)\) and the other stems from differentiating (4). We have the following two conserved quantities for (2):

\[
J_1(\varphi) = B + \mu \int_{0}^{h} \frac{1}{R_1} d\psi, \quad J_2(\varphi) = \int_{0}^{h} \frac{1}{R_2} d\psi + \int_{h}^{1} \frac{1}{R_2} d\psi.
\]

(6)

From (4) we know \(J_1(\varphi) = \mu h\) and \(J_2\) is the channel height \(y(x, 1) = J_2(\varphi(x))\) which equals to 1 by our scaling (cf. (3)).

Additionally there is a third integral \(J_3\) which derives from the variational structure of the problem and invariance with respect to translations in \(x\)-direction. In terms of the variables \((u_i, v_i)\) and \(Y\) it reads

\[
J_3(u_1, v_2, u_2, v_2, Y) = \int_{0}^{Y} \frac{1}{2}(u_1^2 - v_1^2)dy + \int_{Y}^{1} \frac{1}{2}(u_2^2 - v_2^2)dy + (C_1 - C_2)Y - \frac{h}{2}Y^2.
\]

(Taking the \(x\)-derivative of \(J_3\) along a solutions of (1) easily shows \(dJ_3/dx = 0\).) In terms of \((U_i, V_i)\) and \(Y = \int_{0}^{h} \frac{1}{R_1} d\psi\) the integral \(J_3\) can be expressed as

\[
J_3(\varphi) = \int_{0}^{h} \frac{h}{2}(1-V_i^2)d\psi + \int_{h}^{1} \frac{h}{2}(1-V_i^2)d\psi + (C_1 - C_2)Y - \frac{h}{2}Y^2.
\]

(7)

In\(^6\) \(J_3\) is called the flow–force per cross–section, and in\(^{13}\), where the case with of capillary surface waves was treated, it was observed that functions like \(J_3\) can be interpreted as a Hamiltonian function when a properly chosen sympletic structure is employed, see\(^5\),\(^7\) for surface waves and \(^8\) for interfacial waves. In\(^{13}\) a general theory for elliptic variational problems is developed which allows to reduce the Hamiltonian structure to the center manifold of finite dimension. Although we do not emphasize the Hamiltonian structure in this paper, the function \(J_3\) will still play a major role in our discussion in Section 5.

We now restrict our problem (2) to cut out the artificial double zero eigenvalue. Without loss of generality we restrict our solutions to lie in the manifold \(\mathcal{M}_\mu = \{ \varphi \in D(L_\mu) : J_1(\varphi) = \mu h, \ J_2(\varphi) = 1 \}\), which has codimension 2 and is invariant with respect to (2). To describe the reduced flow in \(\mathcal{M}_\mu\) we project \(\mathcal{M}_\mu\) locally onto its tangent space at \(\varphi = 0\). To find a suitable projection we analyze the kernel of \(L_\mu\) further. Here we restrict ourselves to one interesting case, namely \(\mu \approx \mu_0 = \frac{1}{h} + \frac{h}{1-h}\).

For \(\mu = \mu_0\) we know that \(\sigma = 0\) is a four–fold eigenvalue and bifurcations should occur for \(\mu\) passing through \(\mu_0\). The generalized kernel of \(L_{\mu_0}\) is spanned by

\[
\varphi_1 = \begin{pmatrix} -1/h \\ 0 \\ 1/(1-h) \\ 0 \\ -\mu_0 \end{pmatrix}, \ \varphi_2 = \begin{pmatrix} 0 \\ \psi/h \\ 0 \\ (1-\psi)/(1-h) \\ 0 \end{pmatrix}, \ \varphi_3 = \begin{pmatrix} 3\psi^2/h - h \\ 0 \\ \kappa(\psi) \\ 0 \\ 0 \end{pmatrix}, \ \varphi_4 = \varphi_3 - \frac{1}{3\rho} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -\rho \end{pmatrix},
\]

where \(\Delta = h + \rho(1-h)\) and \(\kappa(\psi) = 2\Delta/\rho+1-h-3(1-\psi)^2/(1-h)\). We have \(L_{\mu_0} \varphi_1 = 0, \ L_{\mu_0} \varphi_2 = \varphi_1, \text{ and } L_{\mu_0} \varphi_3,4 = \varphi_2\). Using the standard scalar product \(\langle \cdot, \cdot \rangle\) in \(X\) the
adjoint $L^*$ of $L_{\mu_0}$ is given by

$$D(L^*) = \{ (U_1, V_1, U_2, V_2, B) \in H^1(0, h)^2 \times H^1(h, 1)^2 \times \mathbb{R} : V_1(0) = V_2(1) = 0, \\
V_2(h) = \rho V_1(h), \quad \mu_0 B + U_1(h) - \rho U_2(h) = 0 \},$$

$$L^*(U_1, \ldots, B)^T = (-\partial_\psi V_1, \partial_\psi U_1, -\partial_\psi V_2, \partial_\psi U_2, V_1(0))^T.$$ The generalized kernel of $L^*$ is spanned by

$$\eta_1 = \frac{3}{\Delta} \begin{pmatrix} -\mu_0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \eta_2 = c_2 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \eta_3 = \frac{3}{\Delta} \begin{pmatrix} \psi/h \\ 0 \\ \kappa_0(\psi) \\ 0 \end{pmatrix}, \quad \eta_4 = \frac{1}{\Delta} \begin{pmatrix} \kappa_1(\psi) \\ 0 \\ \kappa_2(\psi) \\ 0 \end{pmatrix} - c_4 (\eta_1 + \eta_2),$$

where $\kappa_0(\psi) = \rho (1 - \psi)/(1 - h), \quad \kappa_1(\psi) = 3 \psi^2/h - 3 \Delta - \rho (1 - h), \quad \kappa_2(\psi) = \rho (1 - h) - 3 \rho (1 - \psi)^2/(1 - h), \quad c_2 = 3 \rho/((1 - h)\Delta), \quad$ and $c_4 = (2h^3 + 2 \rho (1 - h)^3)/(3 \Delta)$. We have $L^* \eta_{1,2} = 0, L^* \eta_3 = \eta_1 + \eta_2, L^* \eta_4 = \eta_3,$ and $\langle \varphi_1, \eta_j \rangle = 1$ for $i + j = 5$ and 0 else. Moreover, $\eta_1$ and $\eta_2$ are chosen such that

$$D_{\varphi} J_1(0)[\tilde{\varphi}] = \frac{3}{\Delta} \langle \eta_1, \tilde{\varphi} \rangle \quad \text{and} \quad D_{\varphi} J_2(0)[\tilde{\varphi}] = \frac{(1 - h) \Delta}{\rho} \langle \eta_2, \tilde{\varphi} \rangle$$

for all $\tilde{\varphi}$. Thus, the tangent space $X_0$ of $\mathcal{M}_{\mu_0}$ at $\varphi = 0$ is the orthogonal complement of span$\{\eta_1, \eta_2\}$. We define the projection $Q_0 : X \to X_0; \varphi \to \varphi - \langle \varphi, \eta_1 \rangle \varphi_4 - \langle \varphi, \eta_2 \rangle \varphi_3$ and decompose $\varphi \in X$ into $\varphi = \varphi_0 + \nu_3 \varphi_3 + \nu_4 \varphi_4$, where $\varphi_0 = Q_0 \varphi \in X_0$. Then,

$$J_1(\varphi_0 + \nu_3 \varphi_3 + \nu_4 \varphi_4) - \mu h = 0, \quad J_2(\varphi_0 + \nu_3 \varphi_3 + \nu_4 \varphi_4) - 1 = 0 \quad (8)$$

can be solved locally (\varphi_0, \nu - \nu_0 small) for $\nu_j = \nu_j(\mu, \varphi) \in \mathbb{R}$ by the implicit function theorem. Thus, $\varphi_0$ serves as coordinate in the tangent space $X_0$ and the correction $\nu_3(\mu, \varphi_0) \varphi_3 + \nu_4(\mu, \varphi_0) \varphi_4$ takes into account the curvature of $\mathcal{M}_\mu$.

To derive the differential equation for $\varphi_0$ we simply apply $Q_0$ to (2). Since $Q_0 L_{\mu_0}(\varphi_0 + \nu_3 \varphi_3 + \nu_4 \varphi_4) = Q_0 [L_{\mu_0} \varphi_0 + (\nu_3 + \nu_4) \varphi_2]$ and $Q_0 \varphi_2 = \varphi_2$ we find

$$\frac{d}{dx} \varphi_0 = \mathcal{L} \varphi_0 + \mathcal{N}(\mu, \varphi_0) \quad (9)$$

where $\mathcal{N}(\mu, \varphi) = (\nu_3 + \nu_4) \varphi_2 + Q_0 [(L_{\mu_0} - L_{\mu_0})(\varphi_0 + \nu_3 \varphi_3 + \nu_4 \varphi_4) + N(\varphi_0 + \nu_3 \varphi_3 + \nu_4 \varphi_4)]$ $\nu_j = \nu_j(\mu, \varphi_0)$ and $\mathcal{L} = Q_0 L_{\mu_0}|_{X_0} = L_{\mu_0}|_{X_0}$. Again $\mathcal{N}$ is a smooth (analytic) mapping from a neighborhood of $(\mu_0, 0)$ in $\mathbb{R} \times D(\mathcal{L})$ into $X_0$, where $D(\mathcal{L}) = D(L_{\mu_0}) \cap X_0$.

4. Reduction onto the center manifold

We define the center space projection $Q_1 : X_0 \to X_0; \varphi_0 \mapsto \varphi_0 - \langle \varphi_0, \eta_3 \rangle \varphi_2 - \langle \varphi_0, \eta_4 \rangle \varphi_1$ and the splitting $\varphi_0 = a \varphi_1 + b \varphi_2 + \Phi, \Phi \in X_1 = Q_1 X_0$, which transfers (9) into

$$\frac{d}{dx} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} + f_1(\mu, a, b, \Phi), \quad \frac{d}{dx} \Phi = \mathcal{L} \Phi + f_2(\mu, a, b, \Phi), \quad (10)$$
where \( \mathcal{L}_1 = \mathcal{L}|_{X_1} = Q_1 \mathcal{L}|_{X_1} \) and

\[
\begin{align*}
f_1 &= \begin{pmatrix} \langle \eta_1, \mathcal{N}(\mu, a \varphi_1 + b \varphi_2 + \Phi) \rangle \\ \langle \eta_3, \mathcal{N}(\mu, a \varphi_1 + b \varphi_2 + \Phi) \rangle \end{pmatrix}, \\
f_2 &= Q_1 \mathcal{N}(\mu, a \varphi_1 + b \varphi_2 + \Phi)
\end{align*}
\]

According to Theorem 2.1(c) the operator \( \mathcal{L}_1 \) has no eigenvalues on the imaginary axis, and satisfies \( \|(\mathcal{L}_1 + is)^{-1}\|_{X_1 \to X_1} \leq C/(1 + |s|) \), for all \( s \in \mathbb{R} \). Hence, the reduction theorem in\(^\text{12}\) is applicable, and there exists a local center manifold \( M_C \) which contains all small bounded solutions and can be written as graph over the center space (here \( (a, b) \in \mathbb{R}^2 \)).

**Theorem 4.1** For each \( k \in \mathbb{N} \) there is an \( \varepsilon > 0 \), a neighborhood \( \mathcal{O}_1 \subset D(\mathcal{L}_1) = X_1 \cap D(\mathcal{L}_{\mu_0}) \) and a reduction function \( \mathcal{H} = \mathcal{H}(\mu, a, b) \in \mathcal{C}^k((\mu_0 - \varepsilon, \mu_0 + \varepsilon) \times (-\varepsilon, \varepsilon)^2, \mathcal{O}_1) \), such that the reduced system

\[
\begin{align*}
d \begin{pmatrix} a \\ b \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} + g(\mu, a, b), \\
\Phi &= \mathcal{H}(\mu, a, b),
\end{align*}
\]

with \( g(\mu, a, b) = f_1(\mu, a, b, \mathcal{H}(\mu, a, b)) \), is locally equivalent to (10) in the sense that every small bounded solution of one equation is also a solution of the other equation.

We remark that the problem has a reflection symmetry \( x \to -x \). For the differential equation (2) this gives reversibility with respect to the involution

\[
T : X \to X; (U_1, V_1, U_2, V_2, B)^T \mapsto (U_1, -V_1, U_2, -V_2, B)^T.
\]

This means \( T L_\mu = -L_\mu T \) and \( N(T \varphi) = -T N(\varphi) \). As a consequence \( \varphi = \varphi(x) \) is a solution if and only if \( \tilde{\varphi}(x) = T \varphi(-x) \) is one. The reversibility is inherited onto the reduced problem (10) is reversible, i.e., with \( T_0(a, b)^T = (a, -b)^T \) we have \( g(\mu, T_0(a, b)) = -T_0 g(\mu, a, b) \).

To calculate the coefficients of the leading nonlinear terms of \( g \), we first expand the functions \( \nu_i \) with respect to \( \varphi_0 = a \varphi_1 + b \varphi_2 + \Phi \) and \( \delta = \mu - \mu_0 \):

\[
\begin{align*}
\nu_3 &= \frac{9 \rho}{2 \Delta h(1-h)^2} a^2 + \frac{3 \rho(5-h)}{2 \Delta h^2(1-h)^3} a^3 + \text{h.o.t.}, \\
\nu_4 &= -\frac{3}{\Delta} \delta a = \frac{9 \rho \mu_0}{2 \Delta h} a^2 - \frac{15 \rho \mu_0}{2 \Delta h^2} a^3 + \text{h.o.t.},
\end{align*}
\]

with h.o.t. \( = \mathcal{O}(a^4 + b^2 + \|\Phi\|^2_{D(L)} + |a| \|\Phi\|_{D(L)} + |b| \|\Phi\|_{D(L)} + |b|) \). This implies

\[
\begin{align*}
f_1 &= \begin{pmatrix} |b| \mathcal{O}(|\delta| + |a| + b^2 + \|\Phi\|^2_{D(L)}) \\ -\frac{3}{\Delta} \delta a - E_0 a^2 + G_0 a^3 + \text{h.o.t.} \end{pmatrix}, \\
f_2 &= \mathcal{O}(|a|^3 + b^2 + \|\Phi\|^2 + |a| \|\Phi\|_{D(L)} + |\delta| a^2 + |b| + \|\Phi\|_{D(L)})
\end{align*}
\]

where \( E_0 = \frac{9}{2 \Delta} \left( \frac{1}{h^2} - \frac{\rho}{(1-h)^2} \right) \), \( G_0 = \frac{3}{2 \Delta h} \left( \frac{\rho(5-h)}{1-h} \right) - 5 \mu_0 \).

The function \( f_2 \) does not contain a term of order \( a^2 \) because of

\[
\begin{align*}
f_2(\mu, a, 0, 0) &= Q_1 \left\{ (\nu_3 + \nu_4) \varphi_2 + Q_0 \left[ (L_\mu - L_{\mu_0}) \varphi + N(a \varphi_1 + \nu_3 \varphi_3 + \nu_4 \varphi_4) \right] \right\} \\
&= Q_1 Q_0 N(a \varphi_1 + \mathcal{O}(a^2)) = \mathcal{O}(|a|^3),
\end{align*}
\]
and \(N(a \varphi_1) = 0\) for all \(a\). Thus, the reduction function \(\Phi = \mathcal{H}(\mu, a, b)\) satisfies the estimate \(\|\mathcal{H}(\mu, a, b)\|_{D(L)} = O(|a|^3 + b^2 + |\delta|(a^2 + |b|))\), and insertion of \(\mathcal{H}\) into \((f_1, f_2)\) yields

\[
g(\mu, a, b) = \left( \frac{b}{3}(\mu_0 - \mu) a - E_0 a^2 + G_0 a^3 + O(a^4 + b^2 + |\mu - \mu_0| a^2) \right).
\]

The reduced system (11) can be rewritten as a second–order equation by solving \(a' = b + g_1(\mu, a, b)\) with respect to \(b = a' + \text{h.o.t.}\) and inserting this into \(b' = g_2(\mu, a, b)\):

\[
a'' - \sigma^2(\mu)a + E(\mu)a^2 - G(\mu)a^3 + M(\mu, a, a') = 0,
\]

where \(M(\mu, a, a') = M(\mu, a, -a') = O(a^4 + a^2)\) and

\[
\sigma^2 = \frac{3}{\Lambda}(\mu_0 - \mu) + O(|\mu - \mu_0|^2), \quad E = E_0 + O(|\mu - \mu_0|), \quad G = G_0 - c(\mu_0)E_0 + O(|\mu - \mu_0|).
\]

5. Homoclinic and heteroclinic solutions

It is well–known that in the case \(E_0 \neq 0\) equation (12) has a bifurcation of homoclinic solutions for \(\mu_0 - \mu > 0\), which have the expansion

\[
a(\mu, x) = \frac{\sigma^2(\mu)}{E_0 \mu} \frac{3}{1 + \cosh(\sigma(\mu)x)} + O((\mu_0 - \mu)^2e^{-\sigma(\mu)|x|})
\]

for \(\mu \to \mu_0\), uniformly in \(x \in \mathbb{R}\), see\(^{9,10}\). Using (3), the interface \(Y\) satisfies

\[
Y(x) = h - \int_0^h U_1 d\Psi + O(\|\varphi\|^2_{D(L)}) = h + \frac{2(\mu_0 - \mu)}{(1 - h^2)(1 + \cosh(\sigma(\mu)x))} + O(\ldots).
\]

We find that \(E_0 > 0\) yields elevation waves \((Y > h)\) and \(E_0 < 0\) yields depression waves \((Y < h)\), which explains the name elevation number for \(E_0\), see\(^{3,4}\).

The case of \(E_0\) very small gives rise to new phenomena, especially the existence of heteroclinic solutions, so–called bores. We now consider \(\sigma = \sigma(\mu)\) and \(E = E(\mu)\) as two independent small parameters. This can be achieved when, in addition to \(\mu\), also \(\rho\) (or \(h\)) is taken as a control parameter. Of course, then also \(G\) and \(M\) depend on \(\sigma\) and \(E\). Note that \(E_0 = 0\) implies \(\rho h^2 = (1 - h)^2\) and hence \(G(\mu_0, E_0) = 6/[\Delta h^3(1 - h)] > 0\).

We are only interested in the case \(\mu_0 - \mu > 0\) and define the scalings

\[
t = \sigma x, \quad z = \sqrt{G(\mu, E)}a/\sigma, \quad \alpha = \sqrt{G(\mu, E)}E/\sigma.
\]

Hence, \(\alpha \in \mathbb{R}\) measures the relative size between the elevation number \(E\) and the closeness to criticality \(\mu_0 - \mu = \Delta \sigma^2/3 \approx 0\). For \(z = z(t)\) we obtain the equation

\[
\ddot{z} - z + \alpha z^2 - z^3 + \dot{M}(\sigma, \alpha, z, \dot{z}) = 0,
\]

with \(\dot{M}(\sigma, \alpha, z, \dot{z}) = \dot{M}(\sigma, \alpha, z, -\dot{z}) = O(\sigma(z^4 + \dot{z}^2)).\)
In the limit $\sigma = 0$ this equation can be discussed explicitly. It has the first integral
\[ \tilde{f}(\alpha, z, \dot{z}) = \frac{1}{2}z^2 - \frac{1}{2}z^2 + \frac{a}{3}z^3 - \frac{1}{4}z^4, \]
and all equilibria lie on the $z$-axis. For $\alpha \in [0, 3/\sqrt{2}]$ there is one equilibrium ($z = 0$), for $\alpha > 3/\sqrt{2}$ there are three (from now on we only treat the case $\alpha \geq 0$, since $\alpha < 0$ can be handled by changing $z$ to $-z$). Moreover, for $\alpha > 3/\sqrt{2}$ there are solutions which are homoclinic to the origin:
\[
z_{\text{hom}}(t) = \frac{72}{36\sqrt{2}e^{-t} + 24\alpha + (2\alpha^2 - 9)\sqrt{2}e^t} = \frac{3}{2\alpha + \sqrt{\alpha^2 - 9/2}\cosh(t + c)},
\]
where $c = \log 6 - \frac{1}{2}\log(2\alpha^2 - 9)$. We have shifted $z_{\text{hom}}$ such that it converges to the heteroclinic solution $z_{\text{het}}(t) = \sqrt{2}/(1 + e^{-t})$ for $\alpha \to 3/\sqrt{2}$.

The persistence of the homoclinic and heteroclinic solution for small $\sigma > 0$ follows by considering the conserved quantity $J_3(\varphi)$ as expressed in (7). We define the restriction of $J_3$ to the center manifold $M_C$,
\[ j_3(a, b) = J_3(a\varphi_1 + b\varphi_2 + \mathcal{H}(\mu, a, b) + \nu_3(\ldots)\varphi_3 + \nu_4(\ldots)\varphi_4) \]
where $\nu_k(\ldots) = \nu_k(\mu, a\varphi_1 + b\varphi_2 + \mathcal{H}(\mu, a, b))$. Obviously, $j_3$ is constant on solutions of the reduced problem (11) and even in $b$. Moreover, scaling $(\mu, a, b)$ as above shows that $\tilde{f}$ is exactly the scaled limit of $j_3$. Hence, the persistence of the phase portrait for small $\sigma > 0$ is trivial as all solutions are level curves of $j_3$.

**Remark:** It is shown in [11] that the case with surface tension leads to a similar equation, where $a''$ is be replaced by $-\delta_1(\beta)a''$. Here $\beta$ is the dimensionless Bond number measuring the relative strength of the surface tension. For $\beta > \beta_0$, $\delta_1$ is positive and $\delta_1(\beta) < 0$ for $\beta < 0$. In the latter case homoclinic bifurcation occurs for $\mu_0 - \mu < 0$, and for $E_0$ small, a scaling similar to (13) yields $z'' - z + 2z^3 + 3z^4 + \mathcal{O}(\sigma) = 0$. The difference to our case is the plus sign in front of the cubic term, which leads to coexistence of elevation and depression waves for open sets in the parameter space, see [11] for details.

We discuss the above results in the original dimensionless parameter space. Recalling $\sigma^2(\mu) = \frac{3}{8}(\mu_0 - \mu) + \mathcal{O}(|\mu - \mu_0|^2)$ and $E = \alpha\sigma\sqrt{G(\mu_0, 0)} + \mathcal{O}(|\mu - \mu_0|)$ the existence domain of solitray waves in a neighborhood of $(\mu, E) = (\mu_0, 0)$ is given by $\mu \in (\mu_0, \mu_0 + \Gamma(E))$, where $\Gamma$ has the expansion
\[ \Gamma(E) = \frac{A^{3/2}(1-h)}{81}E^2 + \mathcal{O}(|E|^3) \quad \text{for } E \to 0. \]

For $E > 0$ the solitary waves are waves of elevations and for $E < 0$ waves of depression.

Moreover, we have proved, at least, locally for small elevation number $E$, a conjecture of C. Amick and R. Turner [3]. Taking $E$ as small but fixed and letting $\mu$ vary on $(-\infty, \mu_0]$ we find a branch of bifurcating solitary waves (homoclinic solutions).
In[3] it is shown that this branch is an unbounded connected continuum in the space $H^1(\mathbb{R}; X) \cup C^1_{bdd}(\mathbb{R}; X)$. The conjecture is that the solutions remain bounded in $C^1_{bdd}$ while the $H^1$-norm blows up due to broadening of the plateau[15]. Our local analysis easily shows that the width of the plateau grows like $\log(\Gamma(E) - \mu)$ for $\mu \to \Gamma(E)$, which implies the blowup of the $H^1$ norm.

References