Differential, energetic and metric formulations for rate-independent processes

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C.I.M.E. Summer School on
NONLINEAR PDEs and APPLICATIONS
Cetraro, 23–28 June 2008

Contents

What is rate independence?

Input-output system on the (process) time interval $[0, T]$:

Inputs: Initial datum q_0

Loading/forcing $\ell \in \mathrm{C}^0([0, T]; Y)$

Output: $q \in \mathrm{C}^0([0, T; Q)$

Input-output operator $q(\cdot) = \mathcal{H}(q_0, \ell(\cdot))$

Rate independence = invariance under time rescalings:

For all $\alpha \in \mathrm{Diff}_+([0, T]; [0, T])$: $\mathcal{H}(q_0, \ell \circ \alpha) = \mathcal{H}(q_0, \ell) \circ \alpha$

Differential equation $F(\dot{q}(t), q(t), \ell(t)) = 0$ rate-independent

$\iff$ $F(\lambda v, q, \ell) = F(v, q, \ell)$ for all (v, q, ℓ) and $\lambda > 0$

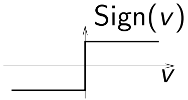
1.1 Introduction

$F(\dot q, q, \ell(t)) = 0$ rate-independent, if $F(\cdot, q, \ell)$ 0-homogeneous.

[ψ is q-homog., if $\psi(\lambda v) = \lambda^q \psi(v)$]

$\Longrightarrow$ Rate independence implies **non-smoothness**

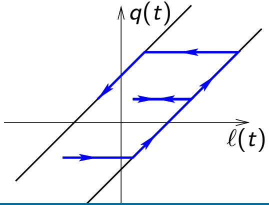
Simplest example: $q \in \mathcal{Q} = \mathbb{R}$

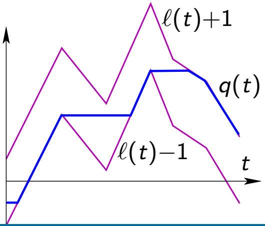


$$\boxed{0 \in \mathrm{Sign}(\dot q) + q - \ell(t)}$$

Observations:

- $|q(t) - \ell(t)| \leq 1$
- $|q(t) - \ell(t)| < 1 \implies \dot q(t) = 0$
- $\dot q(t) > 0 \implies q(t) = \ell(t) - 1$
- $\dot q(t) < 0 \implies q(t) = \ell(t) + 1$





1.2 Abstract setting

More general structure:

State space $\mathcal{Q} \ni q$ state of the system

Today: smooth manifold or Banach space
(later also: topological space or distance space $(\mathcal{Q}, d)$)

$\mathcal{E} : [0, T] \times \mathcal{Q} \to \mathbb{R}_\infty \stackrel{\text{def}}{=} \mathbb{R} \cup \{\infty\}$ energy functional

$\mathcal{R} : \mathrm{T}\mathcal{Q} \to [0, \infty]$ dissipation potential

Force balance

$$\text{(DI)} \qquad 0 \in \underbrace{\partial_v \mathcal{R}(q(t), \dot{q}(t))}_{\ni \text{ friction force}} + \underbrace{\mathrm{D}\mathcal{E}(t, q(t))}_{\text{potential force}} \quad \subset \mathrm{T}^*_q \mathcal{Q}$$

BLAU**Force balance**

$$\text{(DI)} \qquad 0 \in \underbrace{\partial_v \mathcal{R}(q(t), \dot q(t))}_{\ni \text{ friction force}} + \underbrace{\mathrm{D}\mathcal{E}(t, q(t))}_{\text{potential force}} \quad \subset \mathrm{T}^*_q \mathcal{Q}$$

Includes gradient flows:

$$\mathcal{R}(q, v) = \frac{1}{2}\langle G(q)v, v\rangle \quad \rightsquigarrow \quad \partial_v \mathcal{R} = G(q)v$$

Force balance: $G(q)\dot q = -\mathrm{D}\mathcal{E}(t, q)$

Rate independent system $(\mathcal{Q}, \mathcal{E}, \mathcal{R})$

if additionally $\mathcal{R}(q, \lambda v) = \lambda \mathcal{R}(q, v)$ (1-homogeneity)

Special *doubly nonlinear differential inclusions*
(Colli & Visintin 1990, Colli 1992)

1.2 Abstract setting

Classical model:

Linearized elastoplasticity (Moreau 1974/76)

$q = (u, z) \in \mathcal{U} \times \mathcal{Z} = \mathcal{Q}$ with

$u : \Omega \to \mathbb{R}^d$ displacement ($\overset{\mathsf{s}}{\nabla} u = \frac{1}{2}(\nabla u + \nabla u^{\mathsf{T}})$ strain)

$z = e_{\text{plast}} : \Omega \to \mathbb{S}_d = \mathbb{R}^{d\times d}_{\text{sym}}$ plastic strain

$\mathcal{U} \times \mathcal{Z} = \mathrm{H}^1_{\Gamma_{\text{Dir}}}(\Omega; \mathbb{R}^d) \times \mathrm{L}^2(\Omega; \mathbb{S}_d)$

$\mathcal{E}(t, u, z) = \int_\Omega \frac{1}{2}(\overset{\mathsf{s}}{\nabla} u - z){:}\boldsymbol{C}{:}(\overset{\mathsf{s}}{\nabla} - z) + \frac{h}{2}|z|^2 \,\mathrm{d}x - \langle \ell(t), u\rangle$

$\mathcal{R}(q, \dot q) = \int_\Omega \sigma_{\text{yield}} |\dot z| \,\mathrm{d}x = \sigma_{\text{yield}} \|\dot z\|_{\mathrm{L}^1}$

Force balance (elastic equilibrium & plastic flow rule)

$$0 \in \begin{pmatrix} 0 \\ \sigma_{\text{yield}} \operatorname{Sign}(\dot z) \end{pmatrix} + \begin{pmatrix} -\operatorname{div}\big(\boldsymbol{C}{:}(\overset{\mathsf{s}}{\nabla} u - z)\big) - \ell(t) \\ \boldsymbol{C}{:}(z - \overset{\mathsf{s}}{\nabla} u) + hz \end{pmatrix}$$

For absolutely continuous solutions (i.e. $q \in \mathrm{W}^{1,1}([0,T];\mathcal{Q})$) many formulations are equivalent.

However, to derive weak forms it is good to have other formulations.

$$\partial_v \mathcal{R}(q,v) = \{\, \eta \in \mathrm{T}^*_q\mathcal{Q} \mid \forall\, \widehat{v} \in \mathrm{T}_q\mathcal{Q}\colon \mathcal{R}(q,\widehat{v}) \geq \mathcal{R}(q,v) + \langle \eta, \widehat{v}-v\rangle \,\}$$

Thus, (DI) is equivalent to the *evolutionary variational inequality*

$$\text{(EVI)} \qquad \begin{cases} \forall_{\text{a.a.}}\, t \in [0,T]\ \forall\, \widehat{v} \in \mathrm{T}_q\mathcal{Q}\colon \\ \langle \mathrm{D}\mathcal{E}(t,q), \widehat{v}-\dot{q}\rangle + \mathcal{R}(q,\widehat{v}) - \mathcal{R}(q,\dot{q}) \geq 0 \end{cases}$$

Rate independence makes the structure of subdifferentials special.

Lemma (Subdifferentials of 1-homogeneous functions)

B Banach space, $\Psi : B \to \mathbb{R}_\infty$ lsc, convex, 1-homogeneous. Then,

(i) $\partial\Psi(v) \subset \partial\Psi(0)$ *for all* $v \in B$;

(ii) $\partial\Psi(v) = \{\, \eta \in \partial\Psi(0) \mid \Psi(v) = \langle \eta, v \rangle \,\}$.

Thus, (DI) $0 \in \partial_v \mathcal{R}(q(t), \dot q(t)) + \mathrm{D}\mathcal{E}(t, q(t))$

is equivalent to $(\mathrm{S})_{\mathrm{loc}}$ and $(\mathrm{E})_{\mathrm{loc}}$:

$(\mathrm{S})_{\mathrm{loc}}$: $0 \in \partial_v \mathcal{R}(q(t), 0) + \mathrm{D}\mathcal{E}(t, q(t))$ (purely static !!)

$(\mathrm{E})_{\mathrm{loc}}$: $0 = \mathcal{R}(q(t), \dot q(t)) + \langle \mathrm{D}\mathcal{E}(t, q(t)), \dot q(t)) \rangle$.

Here $(\mathrm{S})_{\mathrm{loc}}$ is equivalent to the static variational inequality

$\langle \mathrm{D}\mathcal{E}(t, q(t)), \widehat{v}) \rangle + \mathcal{R}(q(t), \widehat{v}) \geq 0$ for all $\widehat{v} \in \mathrm{T}_q\mathcal{Q}$.

Assuming the chain rule

$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t,q(t)) = \langle \mathrm{D}\mathcal{E}(t,q(t)), \dot{q}(t)\rangle + \partial_t\mathcal{E}(t,q(t))$

$(\mathrm{E})_{\mathrm{loc}}$ is equivalent to the global energy balance (E).

$$(\mathrm{E})\quad \underbrace{\mathcal{E}(t,q(t))}_{\text{present energy}} + \underbrace{\int_0^t \mathcal{R}(q(s),\dot{q}(s))\mathrm{d}s}_{\text{dissipated energy}} = \underbrace{\mathcal{E}(0,q(0))}_{\text{initial energy}} + \underbrace{\int_0^t \partial_s\mathcal{E}(s,q(s))\,\mathrm{d}s}_{\text{work of external forces}}$$

$\partial_t\mathcal{E}(t,q)$ power of external forces

E.g.: $\mathcal{E}(t,q) = \Phi(q) - \langle \ell(t), q\rangle \quad \rightsquigarrow \quad \partial_t\mathcal{E}(t,q) = -\langle \dot{\ell}, q\rangle$

General assumptions throughout these lectures:

$$(\mathcal{E}1) \quad \begin{cases} \text{Sublevels of } \mathcal{E}(t,\cdot) : \mathcal{Q} \to \mathbb{R}_\infty \\ \text{are (weakly sequentially) compact.} \end{cases}$$

$$(\mathcal{E}2) \quad \begin{cases} \exists\, C_E > 0 \ \forall\, (t_*, q_*) : \\ \mathcal{E}(t_*, q_*) < \infty \implies \mathcal{E}(\cdot, q_*) \in \mathrm{C}^1([0,T]) \\ \text{and } |\partial_t \mathcal{E}(t, q_*)| \leq C_E \mathcal{E}(t, q_*) \text{ for all } t. \end{cases}$$

Gronwall and $(\mathcal{E}2)$ yield

$\mathcal{E}(t, q_*) \leq \mathrm{e}^{C_E|t-s|}\mathcal{E}(s, q_*)$

$|\partial_t \mathcal{E}(t, q)| \leq C_E \mathrm{e}^{C_E|t-s|}\mathcal{E}(s, q)$

(E) $\mathcal{E}(t,q(t)) + \int_0^t \mathcal{R}(q(s),\dot{q}(s))\mathrm{d}s = \mathcal{E}(0,q(0)) + \int_0^t \partial_s\mathcal{E}(s,q(s))\,\mathrm{d}s$

($\mathcal{E}2$) $|\partial_t\mathcal{E}(t,q_*)| \leq C_E\mathcal{E}(t,q_*)$

Using the abbreviations

$e(t) = \mathcal{E}(t,q(t))$ and $\delta(t) = \int_0^t \mathcal{R}(q(s),\dot{q}(s))\,\mathrm{d}s \geq 0$

and ($\mathcal{E}2$) in the energy balance (E) leads to

$$e(t) + \underbrace{\delta(t)}_{\geq 0} \leq e(0) + \int_0^t C_E e(s)\,\mathrm{d}s$$

- Gronwall implies $e(t) \leq \mathrm{e}^{C_E t} e(0)$.
- Inserting again gives

$$\delta(t) \leq e(0) + \int_0^t C_E \mathrm{e}^{C_E s} e(0)\,\mathrm{d}s - \underbrace{e(t)}_{\geq 0} \leq \mathrm{e}^{C_E t} e(0).$$

We assume **coercivities in Banach spaces**

$\mathcal{Q} = \mathcal{Y} \times \mathcal{Z}$ with $\mathcal{Z} \subset \widetilde{Z}$

$\mathcal{E}(t,q) \geq c\|q\|_{\mathcal{Q}} - C$ with $c, C > 0$

$\mathcal{R}((y,z),(\dot{y},\dot{z})) \geq c\|\dot{z}\|_{\widetilde{Z}}$

A priori bounds for solutions (and for suitable approximations)

$$\left.\begin{array}{rcl} \|y(t)\|_{\mathcal{Y}} + \|z(t)\|_{\mathcal{Z}} & \leq & C\mathrm{e}^{C_E t} e(0) + C \\ \int_0^t \|\dot{z}(s)\|_{\widetilde{Z}}\,\mathrm{d}s & \leq & C\mathrm{e}^{C_E t} e(0) \end{array}\right\} \text{ for all } t \in [0,T]$$

The L^1 bound for the derivative $\dot{z}$ is too weak.

It is actually a BV bound

$$\mathrm{Var}_{\widetilde{Z}}(z,[0,T]) \overset{\text{def}}{=} \sup_{\text{all part.}} \sum_1^N \|z(t_j) - z(t_{j-1})\|_{\widetilde{Z}}$$

For $z \in \mathrm{W}^{1,1}([0,T];\widetilde{Z})$ we have $\mathrm{Var}_{\widetilde{Z}}(z,[0,T]) = \int_0^T \|\dot{z}(s)\|_{\widetilde{Z}}\,\mathrm{d}s$

Convexity improves the a priori estimates on the derivative!

Additional assumptions (rather strong, only this subsection)

(Conv1) $\mathcal{Q}$, X Banach spaces, $\mathcal{Q} \subset X$

(Conv2) $\exists\, \alpha > 0$: $\mathcal{E}(t,(1-\theta)q_0+\theta q_1)$
$$\leq (1-\theta)\mathcal{E}(t,q_0)+\theta\mathcal{E}(t,q_1) - \alpha\theta(1-\theta)\|q_1-q_0\|_X^2$$

(Conv3) $|\partial_t\mathcal{E}(t,q_1) - \partial_t\mathcal{E}(t,q_0)| \leq C_{tq}\|q_1-q_0\|_X$

(Conv4) $\mathcal{R}(q,v) = \Psi(v)$ (no q-dependence, no coercivity)

Lemma (Lipschitz bound)

$(\mathcal{E}1),(\mathcal{E}2)$, (Conv1–4) hold and $q:[0,T] \to \mathcal{Q}$ solves (DI).
Then, $\|\dot q(t)\|_X \leq C_{tq}/\alpha$ a.e. in $[0,T]$.

Sketch of proof: $(\mathrm{S})_{\mathrm{loc}}$ and (Conv2+4) yield

$$\forall\, \widehat{q} \in \mathcal{Q}\colon\ \mathcal{E}(t, q(t)) + \tfrac{\alpha}{2}\|\widehat{q}-q(t)\|_X^2 \leq \mathcal{E}(t, \widehat{q}) + \Psi(\widehat{q}-q(t)).$$

($q(t)$ minimizes the right-hand side, which is uniformly convex)

Letting $\widehat{q} = q(t+h)$ we find

$$\frac{\alpha}{2}\|q(t+h)-q(t)\|_X^2 \leq \mathcal{E}(t, q(t+h)) - e(t) + \Psi(q(t+h)-q(t))$$

$$\leq e(t+h) - e(t) + \int_t^{t+h} \Psi(\dot{q}(s))\,\mathrm{ds} - \int_t^{t+h} \partial_s\mathcal{E}(s, q(t+h))\,\mathrm{ds}$$

$$\overset{(\mathrm{E})}{=} \int_t^{t+h} \partial_s\mathcal{E}(s, q(s)) - \partial_s\mathcal{E}(s, q(t+h))\,\mathrm{ds}$$

$$\overset{(\mathrm{Conv3})}{\leq} C_{tq} \int_t^{t+h} \|q(s)-q(t+h)\|_X\,\mathrm{ds}$$

M. & Rossi '07: $\|\dot{q}(t)\|_X \leq C_{tq}/\alpha$ a.e. in $[0, T]$

Rate-independent system $(\mathcal{Q}, \mathcal{E}, \mathcal{R})$

(DI) $0 \in \partial_v \mathcal{R}(q(t), \dot{q}(t)) + \mathrm{D}\mathcal{E}(t, q(t))$

General estimates for different solutions ($\mathrm{W}^{1,1}$)

- $\mathcal{E}$ coercive in $\mathcal{Q}$: $\mathcal{E}(t, q) \geq c\|q\|_{\mathcal{Q}} - C$.

  $q \in \mathrm{L}^\infty([0, T], \mathcal{Q})$

- $\mathcal{R}$ coercive on $\widetilde{Z}$: $\mathcal{R}(y, z, \dot{y}, \dot{z}) \geq c\|\dot{z}\|_{\widetilde{Z}}$

  $z \in \mathrm{BV}([0, T]; \widetilde{Z})$

- $\mathcal{E}(t, \cdot)$ uniformly convex w.r.t. $X \supset \mathcal{Q}$

  $q \in \mathrm{C}^{\mathrm{Lip}}([0, T]; X)$

Typical simple examples like elastoplasticity:

- $\mathcal{Q}$ Hilbert space
- $A : \mathcal{Q} \to \mathcal{Q}^*$ bounded, $A = A^* \geq \alpha J_{\text{Riesz}}$ $(\alpha > 0)$
- $\mathcal{E}(t, q) = \frac{1}{2}\langle Aq, q\rangle - \langle \ell(t), q\rangle$
- $\mathcal{R}(q, v) = \Psi(v)$ with $\Psi : \mathcal{Q} \to [0, \infty]$ lsc and convex

Theorem (Moreau 1968, Brezis 1973)

$\ell \in \mathrm{C}^1([0,T]; \mathcal{Q}^*)$, $q_0 \in \mathcal{Q}$ with $0 \in \partial\Psi(0) + Aq_0 - \ell(0)$. Then, there exists a unique solution $q \in \mathrm{C}^{Lip}([0,T]; \mathcal{Q})$ of the RIS $(\mathcal{Q}, \mathcal{E}, \Psi)$ with $q(0) = q_0$.

Generalizations giving also

Existence, uniqueness, Lipschitz continuity in t and initial data:

- M.&Theil'04:

$\mathcal{Q}, \Psi$ as above, $\mathcal{E}$ nonquadratic:

$\mathcal{E} \in \mathrm{C}^{2,\mathrm{Lip}}([0,T] \times \mathcal{Q})$ with $(\mathcal{E}2)$ and

$\mathrm{D}_q^2 \mathcal{E}(t,q) \geq \alpha J_{\mathrm{Riesz}}$ $(\alpha > 0)$

- Brokate&Krejči&Schnabel'04:

$\mathcal{E}$ quadratic but $\mathcal{R} : \mathcal{Q} \times \mathcal{Q} \to [0,\infty]$ general

- M.&Rossi'07:

$\mathcal{E}$ nonquadratic, $\mathcal{R}$ general

Joint convexity condition

$\langle \mathrm{D}_q^2 \mathcal{E}(t,q)\widehat{v}, \widehat{v}\rangle + \mathrm{D}_q \mathcal{R}(q,\widehat{v})[\widehat{v}] \geq \alpha \|\widehat{v}\|_{\mathcal{Q}}^2$

2.1 Incremental minimization

Approximative solutions occur via

solutions via **incremental minimization problems**:

Partition $\Pi = \{0 = t_0 < t_1 < \cdots < t_N = T\}$, q_0 initial datum

$$\textbf{(IMP)} \quad q_k \in \operatorname*{Argmin}_{q\in\mathcal{Q}} \mathcal{E}(t_k, q) + (t_k - t_{k-1})\mathcal{R}\Big(q_{k-1}, \tfrac{1}{t_k - t_{k-1}}(q - q_{k-1})\Big)$$

Rate independence makes (IMP) independent of $t_k - t_{k-1}$

$\widehat{q}_\Pi : [0, T] \to \mathcal{Q}$ piecewise affine interpolant

$\overline{q}_\Pi : [0, T] \to \mathcal{Q}$ left-continuous, piecewise constant interpolant

$\underline{q}_\Pi : [0, T] \to \mathcal{Q}$ right-continuous, piecewise constant interpolant

For $t \in [0, T] \backslash \{t_1, t_2, ..., t_N\}$: $0 \in \partial_v \mathcal{R}(\underline{q}_\Pi, \dot{\widehat{q}}_\Pi) + \mathrm{D}\mathcal{E}(\overline{t}_\Pi, \overline{q}_\Pi)$

As above we obtain a priori estimates indepedent of Π:

For all Π: $\int_0^T \mathcal{R}(\underline{q}_\Pi(t), \dot{\widehat{q}}_\Pi(t))\,\mathrm{d}t \leq C$

Coercivity of $\mathcal{R}$ gives $\mathrm{Var}_{\mathcal{Z}}(\widehat{z}_\Pi, [0,T]) \leq C/c$

Helly's selection principle (later more) provides

- a subsequence $(\widehat{z}_{\Pi_k})_{k\in\mathbb{N}}$ and
- a limit $z : [0,T] \to \mathcal{Z}$ such that

$\mathrm{Var}(z,[0,T]) \leq C/c$ and $\forall\, t \in [0,T]$: $\widehat{z}_{\Pi_k}(t) \rightharpoonup z(t)$ in $\mathcal{Z}$

But z may have **jumps** !!

Any reasonably theory needs

$$\underbrace{\int_0^T \mathcal{R}(z(s), \dot{z}(s))\,\mathrm{d}s}_{\text{not defined}} \leq \liminf_{k\to\infty} \int_0^T \mathcal{R}(\widehat{z}_{\Pi_k}(s), \dot{\widehat{z}}_{\Pi_k}(s))\,\mathrm{d}s$$

. . . more on this in Lecture 2.

2.2 Dissipation distance

$(\mathcal{Q},\mathcal{R})$ space with infinitesimal (Finsler) **metric**

$\mathcal{R} : \mathrm{T}Q \to [0,\infty],\ (q,v) \mapsto \mathcal{R}(q,v)$ e.g. $\mathcal{R}(q,v) = \sqrt{\langle G(q)v,v\rangle}$

$(\mathcal{Q},\mathcal{D})$ space with [esq]-distance

$\mathcal{D} : \mathcal{Q}\times\mathcal{Q} \to [0,\infty]; (q_0,q_1) \mapsto \mathcal{D}(q_0,q_1)$

To avoid confusion between "metric" and "metric":

$(\mathcal{Q},\mathcal{D})$ is called a **[esq]-distance space**

$\mathcal{D}$ is mostly called **dissipation distance**

Natural definition (geodesic dissipation length)

$$\mathcal{D}(q_0, q_1) = \inf \Big\{ \int_0^1 \mathcal{R}(\widetilde{q}(s), \dot{\widetilde{q}}(s))\,\mathrm{d}s \;\Big|\; \widetilde{q}(0) = q_0,\ \widetilde{q}(1) = q_1,\ \widetilde{q} \in \mathrm{W}^{1,1}([0,1]; \mathcal{Q}) \Big\}$$

Mathematically, $\mathcal{D}$ is an **extended, semi-quasi-distance**

- $\mathcal{D} : \mathcal{Q} \times \mathcal{Q} \to [0, \infty]$ — extended
- $\mathcal{D}(q_0, q_1) \geq 0$ and $\mathcal{D}(q_0, q_0) = 0$ — semi

  NOT $\mathcal{D}(q_0, q_1) = 0 \;\Rightarrow\; q_0 = q_1$
- possibly unsymmetric ($\mathcal{D}(q_0, q_1) \neq \mathcal{D}(q_1, q_0)$) — quasi-
- $\mathcal{D}(q_0, q_2) \leq \mathcal{D}(q_0, q_1) + \mathcal{D}(q_1, q_2)$ triangle ineq. — **distance**

In mechanics we often have $\mathcal{Q} = \mathcal{Y} \times \mathcal{Z} \ni (y, z) = q$

$\mathcal{Y}$ dissipation free (e.g. displacement, electric field)

$\mathcal{Z}$ dissipative internal variable (e.g., plastic tensor, magnetization)

$\mathcal{D}((y_0, z_0), (y_1, z_1)) = \widehat{\mathcal{D}}(z_0, z_1)$

For notational simplicity: $\mathcal{D} \triangleq \widehat{\mathcal{D}}$

Main abstract assumptions on $\mathcal{D} : \mathcal{Z} \times \mathcal{Z} \to [0, \infty]$ throughout these lectures:

$$(\mathcal{D}1) \quad \begin{cases} \mathcal{D} \text{ is an extended quasi-metric} \\ \bullet \quad \mathcal{D}(z_0, z_1) = 0 \iff z_0 = z_1, \\ \bullet \quad \mathcal{D}(z_0, z_2) \leq \mathcal{D}(z_0, z_1) + \mathcal{D}(z_1, z_2); \end{cases}$$

$(\mathcal{D}2)$ $\mathcal{D}$ is (weakly seq.) lower semi-continuous.

WIAS

Dissipation along a process $z : [0,T] \to \mathcal{Z}$

$\mathrm{Diss}_{\mathcal{D}}(z,[0,t]) \overset{\text{def}}{=} \sup \{ \sum_1^N \mathcal{D}(z(t_{j-1}), z(t_j)) \mid \text{all partit.} \}$

Lemma (Rossi& M.&Savaré '08)

$\mathcal{Z}$ reflexive Banach space and
$\mathcal{R} : \mathcal{Z} \times \mathcal{Z} \to [0,\infty]$ (s,w)-lsc

(1) If $\mathcal{R}(z,v) \le c\|v\|$, then for all $z \in \mathrm{W}^{1,1}([0,T],\mathcal{Z})$:

$$\int_0^T \mathcal{R}(q(s),\dot z(s))\,\mathrm{d}s = \mathrm{Diss}_{\mathcal{D}}(z,[0,T]).$$

(2) If for all $t \in [0,T]$ we have $z_k(t) \to z(t)$, then

$$\mathrm{Diss}_{\mathcal{D}}(z,[0,T]) \le \liminf_{k\to\infty} \mathrm{Diss}_{\mathcal{D}}(z_k,[0,T]).$$

Rate-indpendent system $(\mathcal{Q},\mathcal{E},\mathcal{D})$ or $(\mathcal{Q},\mathcal{E},\mathcal{R})$

Differential solution (DI) $0 \in \partial_v \mathcal{R}(q(t),\dot{q}(t)) + \mathrm{D}\mathcal{E}(t,q(t))$

Definition (Local solution)

$q:[0,T] \to \mathcal{Q}$ (defined everywhere!!) is called **local solution** of the RIS $(\mathcal{Q},\mathcal{E},\mathcal{R})$, if

$(\mathrm{S})_{\mathrm{loc}}$ $0 \in \partial_v \mathcal{R}(q(t),0) + \mathrm{D}\mathcal{E}(t,q(t))$ a.e. in $[0,T]$;

(UE) For all r,t with $0 \leq r < t \leq T$ we have
the upper energy estimate
$$\mathcal{E}(t,q(t)) + \mathrm{Diss}_{\mathcal{D}}(q,[r,t]) \leq \mathcal{E}(r,q(r)) + \int_r^t \partial_s \mathcal{E}(s,q(s))\,\mathrm{d}s$$

No derivative $\dot{q}$ needed any more! [Dal Maso et al @ SISSA]

WIAS

Definition (Energetic solution)

$\boldsymbol{q} : [0,T] \to \mathcal{Q}$ is **energetic solution** of RIS $(\mathcal{Q},\mathcal{E},\mathcal{D})$, if for all $t \in [0,T]$ we have **global stability (S)** and **energy balance (E)**:

(S) $\quad \forall\, \widetilde{q} \in \mathcal{Q}: \ \mathcal{E}(t,q(t)) \leq \mathcal{E}(t,\widetilde{q}) + \mathcal{D}(q(t),\widetilde{q}),$

(E) $\ \mathcal{E}(t,q(t)) + \mathrm{Diss}_{\mathcal{D}}(q,[0,t]) = \mathcal{E}(0,q(0)) + \int_0^t \partial_s \mathcal{E}(s,q(s))\,\mathrm{d}s.$

Very general notion:

- needs no linear structure $\rightsquigarrow$ topological spaces
- no derivatives occur: $\dot{q}$, $\partial_v \mathcal{R}$, $\mathrm{D}\mathcal{E}$
- very general existence theory
- very robust under perturbations (Γ-convergence)

M. & Theil '99, survey in Handboof of Diff. Eqns.II, 2005.

(see Lecture 3+4)

2.3 Other solution concepts

Further assumption $\mathcal{Q} \subset H \triangleq H^* \subset \mathcal{Q}^*$ with H Hilbert space:

Add small viscosity $\mathcal{R}_{\text{visc}}(q,v) = \frac{\varepsilon}{2}\|v\|_H^2$

$0 \in \partial_v \mathcal{R}(q_\varepsilon, \dot{q}_\varepsilon) + \varepsilon \dot{q}_\varepsilon + \mathrm{D}\mathcal{E}(t, q_\varepsilon)$

Existence of viscous approximations (Colli & Visintin '90,'92)

$q_\varepsilon \in \mathrm{H}^1([0,T],H) \times \mathrm{L}^\infty([0,T],\mathcal{Q})$

Definition (H-approximable solution, [Dal Maso et al. @ SISSA])

$q : [0,T] \to \mathcal{Q}$ is ***H*-approximable solution** of RIS $(\mathcal{Q},\mathcal{E},\mathcal{R})$, if $\exists\, \varepsilon_k \to 0\ \forall\, t \in [0,T]$: $q_{\varepsilon_k}(t) \rightharpoonup q(t)$ in $\mathcal{Q}$.

- Physically desirable approach (as rate independ. never is perfect)
- What is a good choice of $\mathcal{R}_{\text{visc}}$ or H, resp.?
- Missing: direct characterization of limits via (P)DEs

To study the difference between the solutions concepts concerning JUMPS, consider a nonconvex problem.

(For strictly convex problems formulations are usually equivalent.)

Example (M. & Rossi & Savaré July'08)

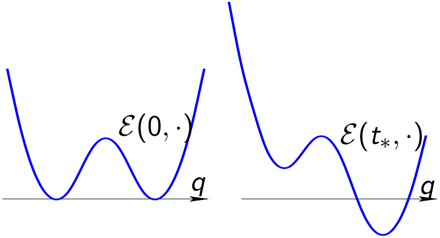
$\mathcal{Q} = \mathbb{R}$;

$$\mathcal{E}(t,q) = \Phi(q) - tq \text{ with } \Phi(q) = \begin{cases} \frac{1}{2}(q+4)^2 & \text{for } q \le -2, \\ 4 - \frac{1}{2}q^2 & \text{for } |q| \le 2, \\ \frac{1}{2}(q-4)^2 & \text{for } q \ge 2; \end{cases}$$

$\mathcal{R}(q,v) = |v|$

$\Rightarrow \quad \mathcal{D}(q_0,q_1) = |q_1 - q_0|$.

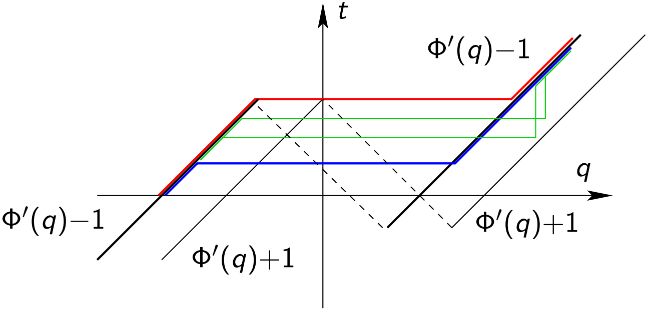
Initial state $q(0) = -4$.



(DI) $0 \in \mathrm{Sign}(\dot q) + \Phi'(q) - t$

$\dot q > 0 \quad \Rightarrow 0 = 1 + \Phi'(q) - t$

$\Rightarrow$ either $q(t) = t - 5 \le -2$ or $q(t) = t + 3 \ge 2$

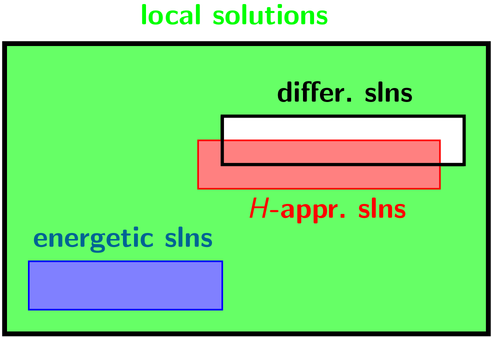


Energetic slns jump as early as possible.

Approx. slns jumps as late as possible.

Local slns. have many choices (2D family).

Diffential solution:	$(S)_{loc}$ & (E)
Local solution:	$(S)_{loc}$ & (UE)
Energetic solution:	$(S)_{glob}$ & (E)
H-approximable solutions:	Cluster point



Main assumptions:

$(\mathcal{Q}, \mathcal{E}, \mathcal{D})$ rate-independent system

$\mathcal{Q}$ (good) topological space (countable union of compact sets and each compact subset is separable and metrizable.

$\mathcal{E} : [0, T] \times \mathcal{Q} \to \mathbb{R}_\infty$ energy-storage potential

$(\mathcal{E}1)$ Sublevels of $\mathcal{E}(t, \cdot)$ compact.

$$(\mathcal{E}2) \quad \begin{cases} \exists\, C_E > 0\ \forall\, (t_*, q_*) : \\ \mathcal{E}(t_*, q_*) < \infty \ \Rightarrow\ |\partial_t \mathcal{E}(t, q_*)| \le C_E \mathcal{E}(t, q_*) \text{ for all } t. \end{cases}$$

$\mathcal{D} : \mathcal{Z} \times \mathcal{Z} \to [0, \infty]$ disspation distance

$$(\mathcal{D}1) \quad \begin{cases} \mathcal{D} \text{ is an extended quasi-metric:} \\ \mathcal{D}(z_0, z_1) = 0 \Leftrightarrow z_0 = z_1 \text{ and } \mathcal{D}(z_0, z_2) \le \mathcal{D}(z_0, z_1) + \mathcal{D}(z_1, z_2); \end{cases}$$

$(\mathcal{D}2)$ $\mathcal{D}$ is (weakly seq.) lower semi-continuous.

The energetic formulation is closely linked to

(IMP) Incremental Minimization Problem

Time discretization $0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T$

Given initial state $q_0 \in \mathcal{Q}$ find iteratively

$$q_k \in \operatorname*{Argmin}_{q \in \mathcal{Q}} \mathcal{E}(t_k, q) + \mathcal{D}(q_{k-1}, q).$$

Old algorithm: $q_k \in \operatorname{Argmin} \mathcal{E}(t_k, q) + \mathcal{R}(q_{k-1}, q - q_{k-1})$

No triangle inequality:

$$\mathcal{R}(q_0, q_2 - q_0) \overset{???}{\leq} \mathcal{R}(q_0, q_1 - q_0) + \mathcal{R}(q_{???}, q_2 - q_1)$$

The following result essentially relies on the **triangle inequality** for $\mathcal{D}$.

(IMP) Incremental Minimization Problem

Time discretization $0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T$

Given initial state $q_0 \in \mathcal{Q}$ find iteratively

$$q_k \in \operatorname*{Argmin}_{q\in\mathcal{Q}} \mathcal{E}(t_k, q) + \mathcal{D}(q_{k-1}, q).$$

Theorem

If $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$ satisfies $(\mathcal{E}1) - (\mathcal{D}2)$, then

(a) (IMP) always has a solution $(q_k)_{k=1,..,N}$;

(b) each q_k is stable, i.e., $\mathcal{E}(t_k, q_k) \leq \mathcal{E}(t_k, \widetilde{q}) + \mathcal{D}(q_k, \widetilde{q})$ for all $\widetilde{q}$;

(c) there is a two-sided energy estimate

$\int_{t_{k\text{-}1}}^{t_k} \partial_s \mathcal{E}(s, q_k)\,\mathrm{d}s \leq \mathcal{E}(t_k, q_k) + \mathcal{D}(q_{k\text{-}1}, q_k) - \mathcal{E}(t_{k\text{-}1}, q_{k\text{-}1}) \leq \int_{t_{k\text{-}1}}^{t_k} \partial_s \mathcal{E}(s, q_{k\text{-}1})\,\mathrm{d}s$

WIAS

- (IMP), which uses global minimization, is closely linked to $(S)_{\text{glob}}$ & (E)
- (IMP) was first used by engineers for decades. (S) & (E) was developed later as limit of (IMP).

Proof.

(a) **Existence** follows directly from Weierstraß' principle (Functionals with compact sublevels attain their infimum.)
(b) **Stability** of $q_k \in \operatorname{Argmin} \mathcal{E}(t_k, q) + \mathcal{D}(q_{k-1}, q)$:
For $\widetilde{q}$ arbitrary we have

$$\begin{aligned} \mathcal{E}(t_k, q_k) &\leq_{[q_k \text{ minim.}]} \mathcal{E}(t_k, \widetilde{q}) + \mathcal{D}(q_{k-1}, \widetilde{q}) - \mathcal{D}(q_{k-1}, q_k) \\ &\leq_{[\text{triangle ineq.}]} \mathcal{E}(t_k, \widetilde{q}) + \mathcal{D}(q_k, \widetilde{q}) \end{aligned}$$

- (IMP), which uses global minimization, is closely linked to $(\mathrm{S})_{\mathrm{glob}}$ & (E)
- (IMP) was first used by engineers for decades. (S) & (E) was developed later as limit of (IMP).

Proof.

(c)

Upper energy estimate, where $e_k = \mathcal{E}(t_k, q_k)$, $\delta_k = \mathcal{D}(q_{k\text{-}1}, q_k)$:

$$\begin{aligned} e_k + \delta_k - e_{k-1} &\leq_{[q_k \text{minim.}]} \mathcal{E}(t_k, q_{k-1}) + \mathcal{D}(q_{k-1}, q_{k-1}) - e_{k-1} \\ &= \int_{t_{k-1}}^{t_k} \partial_s \mathcal{E}(s, q_{k-1})\,\mathrm{d}s \end{aligned}$$

Lower energy estimate follows from stability alone:

$$\begin{aligned} e_k + \delta_k - e_{k\text{-}1} &= \mathcal{E}(t_{k-1}, q_k) + \mathcal{D}(q_{k\text{-}1}, q_k) - e_{k\text{-}1} + \int_{t_{k\text{-}1}}^{t_k} \partial_s \mathcal{E}(s, q_k)\mathrm{d}s \\ &\geq_{[q_{k-1} \text{ stable}]} e_{k-1} + 0 - e_{k-1} - \int_{t_{k-1}}^{t_k} \partial_s \mathcal{E}(s, q_k)\,\mathrm{d}s \end{aligned}$$

Upper incremental energy estimate gives

$$e_k + \delta_k \leq e_{k\text{-}1} + \int_{t_{k-1}}^{t_k} \partial_s \mathcal{E}(s, q_{k-1})\,\mathrm{d}s$$
$$\leq_{(\mathcal{E}2)} e_{k\text{-}1} + \int_{t_{k-1}}^{t_k} C_E\, \mathrm{e}^{C_E(s-t_{k\text{-}1})} e_{k-1}\,\mathrm{d}s \;=\; \mathrm{e}^{C_E(t_k-t_{k\text{-}1})} e_{k-1}$$

Piecewise constant, right-continuous interpolant

$\underline{q}_\Pi : [0,T] \to \mathcal{Q}$, $\underline{q}_\Pi(t) = q_{k-1}$ for $t \in [t_{k-1}, t_k[$

A priori bounds (as in the time-continuous case)

$\mathcal{E}(t, \underline{q}_\Pi(t)) \leq \mathrm{e}^{C_E t} \mathcal{E}(0, q_0)$

$\mathrm{Diss}_{\mathcal{D}}(\underline{q}_\Pi, [0,t]) \leq \mathrm{e}^{C_E t} \mathcal{E}(0, q_0)$

3.2 Existence result

Abbreviation: **stable sets** $\mathcal{S}(t)$

$$\mathcal{S}(t) \overset{\text{def}}{=} \{\, q \in \mathcal{Q} \,|\, \mathcal{E}(t,q)<\infty,\ \forall \widetilde{q}{:}\mathcal{E}(t,q) \leq \mathcal{E}(t,\widetilde{q})+\mathcal{D}(q,\widetilde{q}) \,\}$$

A sequence $((t_k,q_k))_{k\in\mathbb{N}}$ is called a **stable sequence**, if
$\exists\, C>0\ \forall\, k\in\mathbb{N}$: $\mathcal{E}(t_k,q_k) \leq C$ and $q_k \in \mathcal{S}(t_k)$.

Theorem (Existence of energetic solutions)

- *$\mathcal{Q}=\mathcal{Y}\times\mathcal{Z}$ is a good topological space.*
- *$(\mathcal{Q},\mathcal{E},\mathcal{D})$ satisfies $(\mathcal{E}1)$, $(\mathcal{E}2)$, $(\mathcal{D}1)$, $(\mathcal{D}2)$.*
- *The compatibility conditions (CC1) & (CC2) hold:*
  If $((t_k,q_k))_k$ is a stable sequence and $(t_k,q_k)\to(t_*,q_*)$, then
  (CC1) $\quad \partial_t\mathcal{E}(t_*,q_k) \to \partial_t\mathcal{E}(t_*,q_*)$;
  (CC2) $\quad q_* \in \mathcal{S}(t_*)$.

Then, for any $q_0\in\mathcal{S}(0)$ there exists a an energetic solution $q:[0,T]\to\mathcal{Q}$ for the RIS $(\mathcal{Q},\mathcal{E},\mathcal{D})$ with $q(0)=q_0$.

In fact, we have more (and that is what is used in the proof):

$(\Pi^k)_{k\in\mathbb{N}}$ arbitrary sequence of partitions with fineness$(\Pi^k) \to 0$.

Then, there exists a subsequence $(k_l)_{l\in\mathbb{N}}$ such that with $q_l = \underline{q}_{\Pi^{k_l}}$ for all $t \in [0,T]$ we have

- $z_l(t) \to z(t)$,
- $\mathcal{E}(t, q_l(t)) \to \mathcal{E}(t, q(t))$,
- $\mathrm{Diss}_{\mathcal{D}}(q_l, [0,t]) \to \mathrm{Diss}_{\mathcal{D}}(q, [0,t])$,
- $\partial_t\mathcal{E}(\cdot, q_l(\cdot)) \to \partial_t\mathcal{E}(\cdot, q(\cdot))$ in $\mathrm{L}^1([0,T])$ (or pointw. a.e.)

No convergence $y_l(t) \to y(t)$ can be guaranteed.

Instead, we choose a measurable selection $y : [0,T] \to \mathcal{Y}$ for

$$y(t) \in \mathrm{Argmin}_{\mathcal{Y}}\, \mathcal{E}(t, \cdot, z(t)) \text{ and}$$
$$\partial_t\mathcal{E}(t, y(t), z(t)) = \max\{\, \partial_t\mathcal{E}(t,\widehat{y},z(t)) \mid \widehat{y} \in \mathrm{Argmin}\,\mathcal{E}(t,\cdot,z(t)) \,\}$$

Last condition allows to remove the usage of the axiom of choice.

The proof is an abstract variant of

Dal Maso & Francfort & Toader 2005, see M. & Francfort 2006.

It works in 6 steps:

Step 1: A priori estimates.
Step 2: Selection of subsequences.
Step 3: Stability of the limit function.
Step 4: Upper energy estimate.
Step 5: Lower energy estimate.
Step 6: Improved convergence.

Step 1: A priori estimates.

- Solve $(\mathrm{IMP})^{\Pi_k}$ and construct interpolants $\underline{q}_{\Pi_k} : [0,T] \to \mathcal{Q}$.
- Establish energetic a priori estimates as above.

WIAS

Step 1: A priori estimates.	*Step 4: Upper energy estimate.*
Step 2: Selection of subsequences.	*Step 5: Lower energy estimate.*
Step 3: Stability of the limit function.	*Step 6: Improved convergence.*

Step 2: Selection of subsequences.

Choose subsequence $(k_l)_{l\in\mathbb{N}}$ such that $q_l = \underline{q}_{\Pi_{k_l}}$ satisfies:

$z_l(t) \to z(t)$ for all t; (Helly's selection principle in $(\mathcal{Z},\mathcal{D})$)

Abstract Helly's selection principle

- $(\mathcal{D},\mathcal{Z})$ satisfies $(\mathcal{D}1)$, $(\mathcal{D}2)$;
- $\exists\, C>0\ \forall\, k\in\mathbb{N}$: $\mathrm{Diss}(z_k,[0,T])\le C$;
- $\exists\, \mathcal{K}\subset\mathcal{Z}$ compact: $z_k(t)\in\mathcal{K}$.

Then,

$z_{k_l}(t)\to z(t)$ and $\mathrm{Diss}(z,[0,T])\le \liminf_l \mathrm{Diss}(z_{k_l},[0,T])$.

Step 1: A priori estimates.
Step 2: Selection of subsequences.
Step 3: Stability of the limit function.
Step 4: Upper energy estimate.
Step 5: Lower energy estimate.
Step 6: Improved convergence.

Step 2: Selection of subsequences.

Choose subsequence $(k_l)_{l\in\mathbb{N}}$ such that $q_l = \underline{q}_{\Pi_{k_l}}$ satisfies:

$z_l(t) \to z(t)$ for all t; (Helly's selection principle in $(\mathcal{Z}, \mathcal{D})$)

$\mathrm{Diss}(z_l, [0, t]) \to \delta(t)$ for all t;

Find measurable selection $y : [0, T] \to \mathcal{Y}$ as above (very technical).

Step 3: Stability of the limit function.
Use (CC2) = closedness of the stable sets.

3.3 Sketch of proof

Step 1: A priori estimates.
Step 2: Selection of subsequences.
Step 3: Stability of the limit function.
Step 4: Upper energy estimate.
Step 5: Lower energy estimate.
Step 6: Improved convergence.

Step 4: Upper energy estimate.

We know $e_k + \delta_k \leq e_{k-1} + \int_{t_{k-1}}^{t_k} \partial_s \mathcal{E}(s, q_l(s))\,\mathrm{d}s$

and $q_l(s) \in \mathcal{S}(t_l(s))$ with $(t_l(s), q_l(s)) \to (s, \widehat{y}_s, z(s))$

Using (CC1) (cond. contin. of power) gives

$\partial_s \mathcal{E}(s, q_l(s)) \to \partial_s \mathcal{E}(s, \widehat{y}_s, z(s)) \leq p(s) \overset{\text{def}}{=} \partial_s \mathcal{E}(s, y(s), z(s))$,

since by construction

$\mathcal{E}(s, y(s), z(s)) = \max\{\, \partial_t \mathcal{E}(t,\widehat{y},z(t)) \mid \widehat{y} \in \operatorname{Argmin} \mathcal{E}(t, \cdot, z(t)) \,\}$.

With $e(t) = \mathcal{E}(t, q(t))$ and $\delta(t) = \mathrm{Diss}(q, [0, t])$ we obtain (UE):
$e(t) + \delta(t) \leq e(0) + \int_0^t p(s)\,\mathrm{d}s$.

Step 1: A priori estimates.
Step 2: Selection of subsequences.
Step 3: Stability of the limit function.
Step 4: Upper energy estimate.
Step 5: Lower energy estimate.
Step 6: Improved convergence.

Step 5: Lower energy estimate.

General fact: Stability of q for all t implies (LE):

$e(t) + \delta(t) \geq e(0) + \int_0^t p(s)\,\mathrm{d}s.$

Hence, $q : [0,T] \to \mathcal{Q}$ is an energetic solution for $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$.

Step 6: Improved convergence.

Analyzing the inequalities in more detail we obtain

$$\begin{aligned} e(t) + \delta(t) &\leq_{[\mathrm{lsc}]} \liminf_l \mathcal{E}(t, q_l(t)) + \liminf_l \mathrm{Diss}(q_l, [0,t]) \\ &\leq \liminf_l \mathcal{E}(t, q_l(t)) + \liminf_l \mathrm{Diss}(q_l, [0,t]) \\ &\leq_{[\text{Step 4}]} e(0) + \int_0^t \lim p_l(s)\,\mathrm{d}s \leq e(0) + \int_0^t p(s)\,\mathrm{d}s. \end{aligned}$$

and conclude the desired convergences. ∎

3.4 Application for shape-memory alloys

Model of Souza et al. with improvements by Auricchio et al.

Ongoing research with
F. Auricchio, U. Stefanelli, A. Petrov, L. Paoli.

$u \in \mathcal{U} \overset{\text{def}}{=} \mathrm{H}^1_{\Gamma_{\text{Dir}}}(\Omega;\mathbb{R}^d)$ displacement (Ω bounded, Lipschitz)

$z \in \mathcal{Z} \overset{\text{def}}{=} \mathrm{H}^1(\Omega;\mathbb{S}_d)$ mesoscopic transformation strain ($\mathbb{S}_d = \mathbb{R}^{d\times d}_{\text{sym}}$)

$\mathcal{Q} = \mathcal{U} \times \mathcal{Z}$ state space with weak topology of H^1

$$\mathcal{E}(t,u,z) = \int_\Omega \tfrac{1}{2}(\overset{\text{s}}{\nabla}\, u - z) : \boldsymbol{C} : (\overset{\text{s}}{\nabla} - z) + H(z) + \kappa|\nabla z|^r \,\mathrm{d}x - \langle \ell(t), u\rangle$$

$H : \mathbb{S}_d \to [0,\infty]$ is the hardening function (coercive, lsc)

e.g. $H_{\text{SoAu}}(z) = c_1\sqrt{\delta^2 + |z|^2} + \frac{c_2}{2}|z|^2 + \chi_{|z|\le c_3}(z)$

$\mathcal{R}(z,\dot z) = \rho\|z\|_{\mathrm{L}^1} \quad \rightsquigarrow \quad \mathcal{D}(q_0,q_1) = \rho\|z_1 - z_0\|_{\mathrm{L}^1}$

$\mathcal{Q} = \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega;\mathbb{R}^d)_{\mathrm{weak}} \times \mathrm{H}^1(\Omega;\mathbb{S}_d)_{\mathrm{weak}}$ $\qquad$ $\mathcal{D}(q_0,q_1) = \rho\|z_1 - z_0\|_{\mathrm{L}^1}$

$\mathcal{E}(t,u,z) = \int_\Omega \frac{1}{2}(\overset{\mathrm{s}}{\nabla}\, u - z):\boldsymbol{C}:(\overset{\mathrm{s}}{\nabla}\, - z) + H(z) + \kappa|\nabla z|^r \,\mathrm{d}x - \langle \ell(t), u\rangle$

Clearly, $(\mathcal{E}1)$, $(\mathcal{E}2)$, $(\mathcal{D}1)$, $(\mathcal{D}2)$ hold by standard arguments.

Let us check the compatibility conditions (CC1) & (CC2):
If $((t_k,q_k))_k$ is a stable sequence and $(t_k,q_k) \to (t_*,q_*)$, then
(CC1) $\quad \partial_t\mathcal{E}(t_*,q_k) \to \partial_t\mathcal{E}(t_*,q_*)$; $\qquad$ (CC2) $\quad q_* \in \mathcal{S}(t_*)$.

(CC1) is trivial, since $\partial_t\mathcal{E}(t,u,z) = -\langle \dot{\ell}, u\rangle$

For (CC2) note that $q_k \in \mathcal{S}(t_k)$ gives

$$\underbrace{\mathcal{E}(t_k,q_k)}_{\text{lsc}} \leq \underbrace{\mathcal{E}(t_k,\widetilde{q})}_{\to\mathcal{E}(t_*,\widetilde{q})} + \rho\, \underbrace{\|\widetilde{z} - z_k\|_{\mathrm{L}^1}}_{\text{cont. in } \mathrm{H}^1_{\mathrm{weak}}}$$

Passing to the limit $(t_k,q_k) \rightharpoonup (t_*,q_*)$ gives the desired stability:

$\mathcal{E}(t_*,q_*) \leq \mathcal{E}(t_*,q_*) + \rho\|z_* - \widetilde{z}\|_{\mathrm{L}^1}$. ■

Mechanics introduces **strong geometric nonlinearities**

$\varphi : \Omega \to \mathbb{R}^d$ deformation

$\boldsymbol{F} = \nabla\varphi \in \mathrm{GL}^+(\mathbb{R}^d) \stackrel{\text{def}}{=} \{\, F \in \mathbb{R}^{d\times d} \mid \det F > 0 \,\}$

$\boldsymbol{P} = \boldsymbol{F}_{\text{plast}} \in \mathrm{SL}(\mathbb{R}^d) \stackrel{\text{def}}{=} \{\, F \in \mathbb{R}^{d\times d} \mid \det F = 1 \,\}$

Multiplicative decomposition (Lee'69)

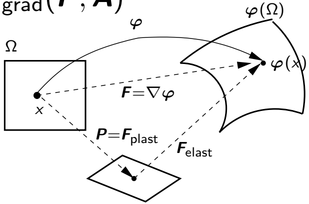
$\nabla\varphi = \boldsymbol{F} = \boldsymbol{F}_{\text{el}}\boldsymbol{F}_{\text{plast}} = \boldsymbol{F}_{\text{el}}\boldsymbol{P} \quad \rightsquigarrow \quad \boxed{\boldsymbol{F}_{\text{el}} = \boldsymbol{F}\boldsymbol{P}^{-1}}$

$W(\boldsymbol{F},\boldsymbol{P},\boldsymbol{A}) = W_{\text{el}}(\underbrace{\boldsymbol{F}\boldsymbol{P}^{-1}}_{=\boldsymbol{F}_{\text{el}}}) + W_{\text{hard}}(\boldsymbol{P}) + W_{\text{grad}}(\boldsymbol{P},\boldsymbol{A})$

$\mathcal{E}(t,\varphi,\boldsymbol{P}) = \int_\Omega W(\nabla\varphi,\boldsymbol{P},\nabla\boldsymbol{P})\,\mathrm{d}x - \langle \ell(t),\varphi\rangle$

$R(\boldsymbol{P},\dot{\boldsymbol{P}}) = \widehat{\boldsymbol{R}}(\dot{\boldsymbol{P}}\boldsymbol{P}^{-1})$ **plastic invariance!**

$\mathcal{R}(\boldsymbol{P},\dot{\boldsymbol{P}}) = \int_\Omega R(\boldsymbol{P},\dot{\boldsymbol{P}})\,\mathrm{d}x$



Plastic dissipation distance $\mathcal{D}$

$\mathcal{D}(\boldsymbol{P}_0, \boldsymbol{P}_1) = \int_\Omega D(x, \boldsymbol{P}_0(x), \boldsymbol{P}_1(x))\,\mathrm{d}x$

where $D(x,\cdot,\cdot) : \mathrm{SL}(\mathbb{R}^d)^2 \to [0,\infty]$ is defined via

$$D(x,P_0,P_1) = \inf\Big\{ \int_0^1 R(x,P(s),\dot{P}(s))\,\mathrm{d}s \;\Big|\; P(0)=P_0, \\ P(1)=P_1,\ P\in \mathrm{C}^1([0,1];\mathrm{SL}(\mathbb{R}^d)), \Big\}$$

Plastic invariance gives $D(x,P_0,P_1) = D(x,I,P_1P_0^{-1})$

Note that $D(x,I,\exp(\boldsymbol{\xi})) \le \widehat{R}(\boldsymbol{\xi}) \sim |\boldsymbol{\xi}|$

Hence, D has at most logarithmic growth

No coercivity in L^q spaces, but positivity still holds!

Admissible deformations $\varphi : \Omega \to \mathbb{R}^d$

$\varphi(t,x) = \varphi_{\mathrm{Dir}}(x)$ for $(t,x) \in [0,T] \times \Gamma_{\mathrm{Dir}}$

$$\mathcal{Y} \stackrel{\text{def}}{=} \left\{ \varphi \in \mathrm{W}^{1,q_{\mathcal{Y}}}(\Omega;\mathbb{R}^d) \,\middle|\, \varphi|_{\Gamma_{\mathrm{Dir}}} = \varphi_{\mathrm{Dir}},\ \text{(GI) holds} \right\}$$

Global invertibility (GI) $\begin{cases} \det \nabla\varphi(x) \geq 0 \text{ a.e. in } \Omega, \\ \int_\Omega \det \nabla\varphi(x)\,\mathrm{d}x \leq \mathrm{vol}(\varphi(\Omega)). \end{cases}$

Ciarlet&Necas'87: $\mathcal{Y}$ is weakly closed in $\mathrm{W}^{1,q_{\mathcal{Y}}}(\Omega;\mathbb{R}^d)$, if $q_{\mathcal{Y}} > d$.

Internal states:

$\mathcal{Z} \stackrel{\text{def}}{=} \{ \boldsymbol{P} \in \big(\mathrm{W}^{1,r} \cap \mathrm{L}^{q_{\boldsymbol{P}}}\big)(\Omega;\mathbb{R}^{d\times d}) \mid \boldsymbol{P}(x) \in \mathrm{SL}(\mathbb{R}^d) \text{ a.e. in } \Omega \}$

WIAS

$$\mathcal{E}(t,\varphi,\boldsymbol{P}) = \int_\Omega W(\nabla\varphi\boldsymbol{P}^{-1},\boldsymbol{P},\nabla\boldsymbol{P})\,\mathrm{d}x - \langle \ell(t),\varphi\rangle$$
$$\mathcal{D}(\boldsymbol{P}_0,\boldsymbol{P}_1) = \int_\Omega D(\boldsymbol{P}_0,\boldsymbol{P}_1)\,\mathrm{d}x$$

- $W : \Omega \times \mathbb{R}^{d\times d} \times \mathbb{R}^{d\times d} \times \mathbb{R}^{d\times d\times d} \to \mathbb{R}_\infty$ is a normal integrand
- $W(x,\cdot,\boldsymbol{P},\boldsymbol{A}) : \mathbb{R}^{d\times d} \to \mathbb{R}_\infty$ is polyconvex
- $W(x,\boldsymbol{F}_{\mathrm{el}},\boldsymbol{P},\cdot) : \mathbb{R}^{d\times d\times d} \to \mathbb{R}_\infty$ is convex
- $W(x,\boldsymbol{F}_{\mathrm{el}},\boldsymbol{P},\boldsymbol{A}) \geq c\big(|\boldsymbol{F}_{\mathrm{el}}|^{q_{\boldsymbol{F}}} + |\boldsymbol{P}|^{q_{\boldsymbol{P}}} + |\boldsymbol{A}|^r\big) - C$

Proposition. Under the above assumptions with $\Gamma_{\mathrm{Dir}} \neq \emptyset$,

$$\frac{1}{q_{\boldsymbol{P}}} + \frac{1}{q_{\boldsymbol{F}}} = \frac{1}{q_{\mathcal{Y}}} < \frac{1}{d} \quad \text{and} \quad r > 1$$

we have that

- $\mathcal{D}$ is weakly continuous on $\mathcal{Z}\times\mathcal{Z}$ and
- $\mathcal{E}(t,\cdot)$ is coercive and weakly lower semi-continuous on $\mathcal{Q}$.

Main Existence Result [Mainik & M.'08].

Under the assumptions (only the major ones)

- W is a normal integrand and is lower semicontinuous;
- W polyconvex in $\boldsymbol{F}_{\mathrm{el}}$ and convex in $\boldsymbol{A} = \nabla \boldsymbol{P}$
- $\varphi_{\mathrm{Dir}} \in \mathrm{W}^{1,q_{\mathcal{Y}}}(\Omega;\mathbb{R}^d)$
- $W(x,\boldsymbol{F}_{\mathrm{el}},\boldsymbol{P},\boldsymbol{A}) \geq c\big(|\boldsymbol{F}_{\mathrm{el}}|^{q_{\boldsymbol{F}}} + |\boldsymbol{P}|^{q_{\boldsymbol{P}}} + |\boldsymbol{A}|^r\big) - C$
- $\dfrac{1}{q_{\boldsymbol{P}}} + \dfrac{1}{q_{\boldsymbol{F}}} = \dfrac{1}{q_{\mathcal{Y}}} < \dfrac{1}{d}$, and $r > 1$,
- dissipation distance D as above

for each stable initial state $\boldsymbol{q}_0 \in \mathcal{Q}$ there exists at least one energetic solution $\boldsymbol{q} : [0,T] \to \mathcal{Q}$ of $(\mathcal{Q},\mathcal{E},\mathcal{D})$ with $\boldsymbol{q}(0) = \boldsymbol{q}_0$.

$\mathcal{Q}$ is a topological space and "$\to$" denotes convergence (e.g. $\mathcal{Q} = \mathrm{W}^{1,p}(\Omega)_{\mathrm{weak}}$)

Definition (Γ-convergence, De Giorgi'75)

$\mathcal{I}_\infty : \mathcal{Q} \to \mathbb{R}_\infty$ is called the **Γ-limit** of the sequence $(\mathcal{I}_k)_{k\in\mathbb{N}}$, (written $\mathcal{I}_k \xrightarrow{\Gamma} \mathcal{I}_\infty$ or $\mathcal{I}_\infty = \Gamma\text{-}\lim_{k\to\infty} \mathcal{I}_k$) if the following holds:

(i) **liminf estimate**:

$$q_k \to q \quad \Longrightarrow \quad \mathcal{I}_\infty(q) \leq \liminf_{k\to\infty} \mathcal{I}_k(q_k)$$

(ii) **limsup estimate** (existence of recovery sequences):

$$\forall\, q \in \mathcal{Q}\ \exists\, (\widehat{q}_k)_{k\in\mathbb{N}} : \quad \widehat{q}_k \to q \quad \text{and} \quad \mathcal{I}_\infty(q) \geq \limsup_{k\to\infty} \mathcal{I}_k(\widehat{q}_k).$$

4.1 Γ-convergence and RIS

Given: energetic solutions $q_k : [0,T] \to \mathcal{Q}$ to $(\mathcal{Q}, \mathcal{E}_k, \mathcal{D}_k)$
$\mathcal{E}_\infty(t,\cdot) = \Gamma\text{-}\lim_{k\to\infty} \mathcal{E}_k(t,\cdot)$ and $\mathcal{D}_\infty = \Gamma\text{-}\lim_{k\to\infty} \mathcal{D}_k$

Question: Under **what additional conditions** are cluster points of the sequence $(q_k)_{k\in\mathbb{N}}$ solutions of the limit system $(\mathcal{Q}, \mathcal{E}_\infty, \mathcal{D}_\infty)$?

RIS	$(\mathcal{Q}, \mathcal{E}_k, \mathcal{D}_k)$	$\overset{\Gamma}{\longrightarrow}$	$(\mathcal{Q}, \mathcal{E}_\infty, \mathcal{D}_\infty)$
initial data	q_k^0	$\xrightarrow[k\to\infty]{}$	q_∞^0
time evolution	$\downarrow$		$\downarrow$
$t > 0$	$q_k(t)$	$\xrightarrow{k\to\infty}$	$q_\infty(t)$

Typical application of (static) Γ-convergence are

- homogenization, two-scale convergence
- singular limits (Cahn-Hilliard $\rightsquigarrow$ sharp interface)
- Young-measure relaxation, penalizations, ...
- finite-dimensional **numerical approximation**

Interchanging Γ-convergence and time evolution can be studied when evolution is driven by functionals:

- in Hamiltonian systems (wave equation, quantum mechanics) or
- in gradient flows
  (..., Sandier&Serfaty, Ortner, Kurzke, ...)

4.2 A simple ODE example

State space $\mathcal{Q} = \mathbb{R}^2$

Stored-energy functional $\quad \mathcal{E}_k(t,q) = \frac{1}{2}q_1^2 + \frac{1}{2}\big(kq_2 - q_1\big)^2 - tq_1$

Dissipation distance $\quad \mathcal{D}_k(q,\widetilde{q}) = \mathcal{R}_k(\widetilde{q} - q)$ with

$\mathcal{R}_k(\dot{q}) = |\dot{q}_1| + k^\beta |\dot{q}_2|$

Limit stored energy

$$\mathcal{E}_k(t,\cdot) \xrightarrow{\Gamma} \mathcal{E}_\infty(t,\cdot) : q \mapsto \begin{cases} \frac{1}{2}q_1^2 - tq_1 & \text{if } q_2 = 0, \\ \infty & \text{else.} \end{cases}$$

Limit dissipation distance

$$\mathcal{D}_k \xrightarrow{\Gamma} \mathcal{D}_\infty : (q,\widetilde{q}) \mapsto \begin{cases} |q_1 - \widetilde{q}_1| & \text{if } \widetilde{q}_2 = q_2, \\ \infty & \text{else.} \end{cases}$$

$q : [0,T] \to \mathbb{R}^2$ with $q(0) = 0$ energetic solution for $(\mathcal{Q}, \mathcal{E}_\infty, \mathcal{D}_\infty)$

- $q_2 \equiv 0$ $\qquad$ • $0 \in \mathrm{Sign}(\dot{q}_1) + q_1 - t$
- $\Longrightarrow \quad q(t) = (\max\{0, t-1\}, 0)^\top$

WIAS

$$0 \in \partial\mathcal{R}_k(\dot q) + \mathrm{D}\mathcal{E}_k(t,q) \subset \mathbb{R}^2_*, \quad q(0)=0$$

$$\boxed{0 \in \mathrm{Sign}(\dot q_1) + 2q_1 - kq_2, \quad 0 \in k^\beta \mathrm{Sign}(\dot q_2) - kq_1 + k^2 q_2}$$

The explicit solution reads

$$q_k(t) = \begin{cases} (0,0)^\top & \text{for } t \in [0,1], \\ \big((t-1)/2, 0\big)^\top & \text{for } t \in [1, T(k)], \\ \big(t-1-k^{\beta-1}, (t-T(k))/k\big)^\top & \text{for } t \geq T(k), \end{cases}$$

where $T(k) = 1 + 2k^{\beta-1}$.

The limit gives $q(t) = \lim_{k\to\infty} q_k(t) =$

$$= \begin{cases} (\max\{0, t-1\}, 0)^\top & \text{for } \beta \in [0,1), & \text{CORRECT} \\ \big(\max\{0, (t-1)/2, t-2\}, 0\big)^\top & \text{for } \beta = 1, & \text{WRONG} \\ (\max\{0, (t-1)/2\}, 0)^\top & \text{for } \beta > 1. & \text{WRONG} \end{cases}$$

$\mathcal{E}_k(t,q) = \frac{1}{2}q_1^2 + \frac{1}{2}\big(kq_2 - q_1\big)^2 - tq_1, \qquad \mathcal{D}_k(0,q) = |\dot q_1| + k^\beta|\dot q_2|$

$$\mathcal{E}_k(t,\cdot) \xrightarrow{\Gamma} \mathcal{E}_\infty(t,\cdot) : q \mapsto \begin{cases} \frac{1}{2}q_1^2 - tq_1 & \text{if } q_2 = 0, \\ \infty & \text{else.} \end{cases}$$

pointwise limit: $\mathcal{E}_{\text{pw}}(t,q) = \begin{cases} (\frac{1}{2}+\frac{1}{2})q_1^2 - tq_1 & \text{if } q_2 = 0, \\ \infty & \text{else.} \end{cases}$

Energetic recovery sequence for $q = (q_1, 0)$ is $\widehat{q}_k = (q_1, q_1/k)$:
$\mathcal{E}_k(t,\widehat{q}_k) = \mathcal{E}(t,q)$.

The dissipation distance gives
$\mathcal{D}_k(0,\widehat{q}_k) = |q_1|(1+k^{\beta-1}) \to \mathcal{D}_\infty(0,q)$ only for $\beta < 1$.

Needed: "JOINT recovery sequences"

WIAS

WIAS

Main assumptions:

- $\mathcal{Q} = \mathcal{Y} \times \mathcal{Z}$ "good" topological space
- $\mathcal{E} = \Gamma\text{-}\lim_{k\to\infty} \mathcal{E}_k$ and $\mathcal{D} = \Gamma\text{-}\lim_{k\to\infty} \mathcal{D}_k$
- $(\mathcal{D}1,2)$: $\mathcal{D}_k, \mathcal{D}_\infty$ lsc, extended quasi-distance on $\mathcal{Z}$
- $(\mathcal{E}1)$: Uniform compactness: all $\mathcal{E}_k(t,\cdot)$ lsc and $\forall\, E\ \forall\, t:\ \cup_{k=1}^{\infty}\{\, q \in \mathcal{Q} \mid \mathcal{E}_k(t,q) \leq E\,\}$ is relatively compact
- $(\mathcal{E}2)$: Uniform control of power of external forces: $\exists\, C_E:\ \mathcal{E}_k(t_*,q_*) < \infty \implies |\partial_t \mathcal{E}_k(t,q_*)| \leq C_E\, \mathcal{E}_k(t,q_*)$
- $(t_k,q_k)_k$ stable sequence with $(t_k,q_k) \to (t_*,q_*)$:

$$\text{(CC1)}\quad \partial_t \mathcal{E}_k(t_*,q_k) \to \partial_t \mathcal{E}_\infty(t_*,q_*)$$

$$\text{(CC2)}\quad q_* \in \mathcal{S}_\infty(t_*).$$

sets of stable states:
$\mathcal{S}_k(t) \overset{\text{def}}{=} \{\, q \in \mathcal{Q} \mid \infty > \mathcal{E}_k(t,q) \leq \mathcal{E}_k(t,\widetilde{q}) + \mathcal{D}_k(q,\widetilde{q})$ for all $\widetilde{q}\,\}$

WIAS

Theorem. (M. & Roubíček & Stefanelli CalcVar'08)
Let the above assumptions hold.

(a) $q_k : [0,T] \to \mathcal{Q}$ be energetic solutions for $(\mathcal{Q}, \mathcal{E}_k, \mathcal{D}_k)$.
If additionally $q_k(0) \to q^0$ and $\mathcal{E}_k(0, q_k(0)) \to \mathcal{E}_\infty(0, q^0)$;
then every (pointwise) cluster point $q : [0,T] \to \mathcal{Q}$ is energetic solution for $(\mathcal{Q}, \mathcal{E}_\infty, \mathcal{D}_\infty)$.

(b) Take sequence of partitions Π_k with fineness$(\Pi_k) \to 0$,
$q_0^k \to q_0 \in \mathcal{S}_\infty(0)$ with $\mathcal{E}_k(0, q_0^k) \to \mathcal{E}_\infty(0, q_0)$ and
solve $(\text{IMP})_k^{\Pi_k}$ to obtain interpolants $\underline{q}_k : [0,T] \to \mathcal{Q}$.
Then, there exists a subseq. $(\underline{q}_{k_l})_l$ converging to an energetic solution $q : [0,T] \to \mathcal{Q}$ for $(\mathcal{Q}, \mathcal{E}_\infty, \mathcal{D}_\infty)$ with $q(0) = q_0$ and
$\mathcal{E}_{k_l}(t, \underline{q}_{k_l}(t)) \to \mathcal{E}_\infty(t, q(t))$, $\text{Diss}_{\mathcal{D}_{k_l}}(\underline{q}_{k_l}, [0,t]) \to \text{Diss}_{\mathcal{D}_\infty}(q, [0,t])$

Proof consists of the 6 steps as in the usual existence result:

Step 1: A priori estimates.
Step 2: Selection of subsequences.
Step 3: Stability of the limit function.
Step 4: Upper energy estimate.
Step 5: Lower energy estimate.
Step 6: Improved convergence.

Real problem: establish **compatibility conditions (CC1)**

(CC2) conditioned upper semicontinuity of the stable sets:

$$(t_k, q_k)_k \text{ stable sequence}, \ (t_k, q_k) \to (t_*, q_*) \implies q_* \in \mathcal{S}_\infty(t_*)$$

Proposition (Joint recovery sequence) [MRS'08]
If $\forall$ stable seq. $(t_k, q_k)_k$ with $(t_k, q_k) \to (t_*, q_*)$ $\forall \widehat{q} \in \mathcal{Q}$
$\exists$ joint recovery seq. $(\widehat{q}_k)_k$ with $\widehat{q}_k \to \widehat{q}$:

$$\limsup_{k\to\infty} \big(\mathcal{E}_k(t_k, \widehat{q}_k) + \mathcal{D}_k(q_k, \widehat{q}_k) - \mathcal{E}_k(t_k, q_k)\big) \le$$

$$\le \mathcal{E}_\infty(t_*, \widehat{q}) + \mathcal{D}_\infty(q, \widehat{q}) - \mathcal{E}_\infty(t_*, q_*),$$

then **(CC2)** holds.

Classical linearized elastoplasticity with hardening:

state space $\mathcal{Q} = \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega;\mathbb{R}^d) \times \mathrm{L}^2(\Omega;\mathbb{S}_d)$
$\mathcal{Q} \ni (u,z) =$ (displacement, plastic strain)

stored-energy functional: quadratic, uniformly coercive

$\mathcal{E}_\varepsilon(t,u,z) = \frac{1}{2}\langle \mathcal{A}_\varepsilon(u,z),(u,z)\rangle - \langle \ell(t),u\rangle$

with $\langle \mathcal{A}_\varepsilon(u,z),(u,z)\rangle = \int_\Omega \binom{\overset{s}{\nabla} u}{z}{:}A(x,\frac{x}{\varepsilon}){:}\binom{\overset{s}{\nabla} u}{z}\,\mathrm{d}x$,

and $A \in \mathrm{C}^0(\overline{\Omega} \times \boldsymbol{Y}, \mathrm{Lin}(\mathbb{R}^{d\times d}_{\mathrm{sym}} \times \mathbb{R}^{d\times d}_{\mathrm{sym}}))$, unif. pos. definite

$\boldsymbol{Y} = \mathbb{R}^d/_\Lambda$ is the periodicity cell associated with lattice Λ

rate-independent dissipation distance: translational invariant

$\mathcal{D}_\varepsilon(z_0,z_1) = \mathcal{R}_\varepsilon(z_1 - z_0)$ with $\mathcal{R}_\varepsilon(z) = \int_\Omega R\big(x,\frac{x}{\varepsilon},z(x)\big)\,\mathrm{d}x$
where $R \in \mathrm{C}^0(\overline{\Omega}\times\boldsymbol{Y}\times\mathbb{R}^m)$ and $R(x,y,\lambda z) = \lambda R(x,y,z)$.

$$0 \in \binom{\{0\}}{\partial_{\dot z}\mathcal{R}_\varepsilon(\dot z)} + \mathcal{A}_\varepsilon \binom{u}{z} - \binom{\ell(t)}{0} \iff \text{energetic formulation (S)\&(E)}$$

Existence, uniqueness of slns. $(u_\varepsilon, z_\varepsilon) \in \mathrm{C}^{\mathrm{Lip}}([0,T],\mathcal{Q})$ standard.

Weak two-scale convergence:

- $z_\varepsilon \overset{2}{\rightharpoonup} Z \in \mathrm{L}^2(\Omega\times \boldsymbol{Y}) \overset{\text{def}}{\iff}$
$$\begin{cases} (i)\ (z_\varepsilon)_\varepsilon \text{ bounded in } \mathrm{L}^2(\Omega), \\ (ii)\ \forall\, \psi \in \mathrm{C}^0(\Omega\times \boldsymbol{Y}): \\ \quad \int_\Omega z_\varepsilon(x)\psi(x,\tfrac{x}{\varepsilon})\,\mathrm{d}x \to \int_{\Omega\times \boldsymbol{Y}} Z(x,y)\psi(x,y)\,\mathrm{d}y\,\mathrm{d}x. \end{cases}$$
- For displacement (gradients) we find
$$u_\varepsilon \rightharpoonup u_0 \text{ in } \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega), \quad \underbrace{\nabla u_\varepsilon}_{\text{fluct.grad.}} \overset{2}{\rightharpoonup} \underbrace{\nabla_x u_0}_{\text{macrosc.}} + \underbrace{\nabla_y U_1}_{\text{microsc.}} \text{ in } \mathrm{L}^2(\Omega\times \boldsymbol{Y})$$

Extended two-scale state space

$Q = (u_0, U_1, Z) \in \boldsymbol{Q} \overset{\text{def}}{=} \mathrm{H}^1_{\Gamma_{\text{Dir}}}(\Omega) \times \mathrm{L}^2(\Omega; \mathrm{H}^1_{\text{av}}(\boldsymbol{Y}))^d \times \mathrm{L}^2(\Omega \times \boldsymbol{Y})$

Two-scale Γ-limits of the two functionals $\mathcal{E}_\varepsilon$ and $\mathcal{D}_\varepsilon$:

$$\boldsymbol{E}(t, Q) = \int_{\Omega \times \boldsymbol{Y}} \frac{1}{2} \begin{pmatrix} \overset{\text{s}}{\nabla}_x u_0 + \overset{\text{s}}{\nabla}_y U_1 \\ Z \end{pmatrix} : A(x, y) : \begin{pmatrix} \overset{\text{s}}{\nabla}_x u_0 + \overset{\text{s}}{\nabla}_y U_1 \\ Z \end{pmatrix} \mathrm{d}y \, \mathrm{d}x - \langle \ell(t), u_0 \rangle$$

$$\boldsymbol{D}(Z_0, Z_1) = \int_{\Omega \times \boldsymbol{Y}} R\big(x, y, Z_1(x, y) - Z_0(x, y)\big) \, \mathrm{d}y \, \mathrm{d}x$$

Theorem. [M./Timofte SIMA'07]
If $(u_\varepsilon^0, z_\varepsilon^0) \overset{2}{\rightharpoonup} (u_0^0, U_1^0, Z^0)$ and $\mathcal{E}_\varepsilon(0, u_\varepsilon^0, z_\varepsilon^0) \to \boldsymbol{E}(0, u_0^0, U_1^0, Z^0)$, then for all $t \in [0, T]$ the solutions $(u_\varepsilon, z_\varepsilon)$ two-scale converge to the unique energetic solution (u_0, U_1, Z) for $(\boldsymbol{Q}, \boldsymbol{E}, \boldsymbol{D})$.

Crucial: **joint recovery sequence** to show **(CC2)**

$\forall\, q_\varepsilon \in \mathcal{S}_\varepsilon(t),\ q_\varepsilon \overset{2}{\rightharpoonup} Q\ \forall\, \widehat{Q}\ \exists\, \widehat{q}_\varepsilon,\ \widehat{q}_\varepsilon \overset{2}{\rightharpoonup} \widehat{Q}:$

$$\limsup_{\varepsilon\to 0} \mathcal{E}_\varepsilon(t,\widehat{q}_\varepsilon)+\mathcal{D}_\varepsilon(q_\varepsilon,\widehat{q}_\varepsilon)-\mathcal{E}_\varepsilon(t,q_\varepsilon) \leq \boldsymbol{E}(t,\widehat{Q})+\boldsymbol{D}(Q,\widehat{Q})-\boldsymbol{E}(t,Q)$$

Choose $\widehat{q}_\varepsilon = \mathcal{F}_\varepsilon(\widehat{Q}-Q) + q_\varepsilon$ and use quadratic structure and translation invariance

well-chosen **folding operator** $\quad \mathcal{F}_\varepsilon : \boldsymbol{Q} \to \mathcal{Q};\ (\mathcal{F}_\varepsilon Z)(x) \approx Z(x,\frac{x}{\varepsilon})$

$$\mathcal{E}_\varepsilon(\widehat{q}_\varepsilon) - \mathcal{E}_\varepsilon(q_\varepsilon) = \Big\langle \mathcal{A}_\varepsilon \underbrace{(q_\varepsilon+\widehat{q}_\varepsilon)}_{\overset{2}{\rightharpoonup}\, Q+\widehat{Q}}\,;\ \underbrace{(\widehat{q}_\varepsilon-q_\varepsilon)}_{\mathcal{F}_\varepsilon(\widehat{Q}-Q)\overset{2}{\to}\widehat{Q}-Q} \Big\rangle \to \boldsymbol{E}(\widehat{Q})-\boldsymbol{E}(Q)$$

$$\mathcal{D}_\varepsilon(q_\varepsilon,\widehat{q}_\varepsilon) = \mathcal{D}_\varepsilon(0,\mathcal{F}_\varepsilon(\widehat{Q}-Q)) \to \boldsymbol{D}(0,\widehat{Q}-Q) = \boldsymbol{D}(Q,\widehat{Q}).$$

Time-incremental minimization for partition Π

$$\textbf{(IMP)}^{\Pi} \qquad q_j \in \text{Argmin}_{\widetilde{q}\in\mathcal{Q}} \left(\mathcal{E}(t_j,\widetilde{q}) - \mathcal{E}(t_{j-1},q_{j-1}) + \mathcal{D}(q_{j-1},\widetilde{q})\right)$$

Additionally choose discrete subspaces

$$\mathcal{Q}_h = \mathcal{Y}_h \times \mathcal{Z}_h \subset \mathcal{Y} \times \mathcal{Z} = \mathcal{Q}, \qquad \mathcal{E}_h(t,q) = \begin{cases} \mathcal{E}(t,q) & \text{in } \mathcal{Q}_h; \\ \infty & \text{else.} \end{cases}$$

> $\mathcal{E}_h(t,\cdot) \xrightarrow{\Gamma} \mathcal{E}(t,\cdot)$ if and only if
> (i) $\mathcal{E}(t,\cdot)$ is lower semi-continuous and
> (ii) for all q with $\mathcal{E}(t,q) < \infty$ there exist $q_h \in \mathcal{Q}_h$, $h > 0$,
> with $q_h \rightharpoonup q$ and $\mathcal{E}(t,q_h) \to \mathcal{E}(t,q)$ for $h \to 0$.

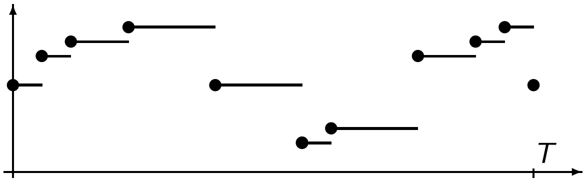
Typical case: $\mathcal{Q} = \mathrm{H}^1_0(\Omega) \times \mathrm{W}^{1,r}(\Omega)$

- $\mathcal{Q}_h$ piecewise affine functions on triangulations $\mathcal{T}_h$ of Ω.
- Q_h dense in strong topol.
- $\mathcal{E}(t,\cdot) : \mathcal{Q} \to \mathbb{R}$ (strongly) contin.

Space-time discretized problem

$$\textbf{(IMP)}^h_\Pi \quad q_j^{h,\Pi} \in \operatorname*{Argmin}_{q\in\mathcal{Q}_h} \big(\mathcal{E}(t_j^\Pi, q) + \mathcal{D}(q_{j-1}^\Pi, q)\big)$$

Temporally piecewise interpolant $\underline{q}^{h,\Pi} : [0,T] \to \mathcal{Q}_h \subset \mathcal{Q}$ with $\underline{q}^{h,\Pi}(t) = q_j^{h,\Pi}$ for $t \in [t_j, t_{j+1})$ and $\underline{q}^{h,\Pi}(T) = q_{n_\Pi}^{h,\Pi}$.



Theorem (**Convergence of space-time approximations**)

- $\mathcal{Q}=\mathcal{Y}\times\mathcal{Z}$ *reflexive Banach spaces*
- $\mathcal{E}(t,\cdot):\mathcal{Q}\to\mathbb{R}_\infty$, $\mathcal{D}:\mathcal{Z}\times\mathcal{Z}\to[0,\infty]$ *coercive, wlsc*
- $\partial_t\mathcal{E}(\cdot,q)\in \mathrm{C}^1([0,T])$ *and* $|\partial_t\mathcal{E}(t,q)|\leq c_1(\mathcal{E}(t,q){+}c_0)$
- **joint-recovery condition** *holds for* $(\mathcal{E},\mathcal{D},(\mathcal{Q}_h)_{h>0})$

For stable $q^0\in\mathcal{Q}$ *choose* $(q_h^0)_{h>0}$ *with* $Q_h\ni q_h^0\rightharpoonup q_0$ *and* $\mathcal{E}(0,q_h)\to\mathcal{E}(0,q)$, *and define* $\underline{q}^{h,\Pi}:[0,T]\to\mathcal{Q}_h$ *as above.*

Then, there exists a subseq. $(h_l,\Pi_l)_{l\in\mathbb{N}}$ *with* $h_l,\Phi(\Pi_l)\to 0$ *and an energetic solution* $q:[0,T]\to\mathcal{Q}$ *with* $q(0)=q^0$ *such that for all* t

- $\mathcal{E}(t,\underline{q}^{h_l,\Pi_l}(t))\to\mathcal{E}(t,q(t))$,
- $\mathrm{Diss}_\mathcal{D}(\underline{q}^{h_l,\Pi_l},[0,t])\to\mathrm{Diss}_\mathcal{D}(q,[0,t])$,
- $\underline{z}^{h_l,\Pi_l}(t)\rightharpoonup z(t)$,
- $\partial_t\mathcal{E}(\cdot,\overline{q}^{h_l,\Pi_l}(\cdot))\to\partial_t\mathcal{E}(\cdot,q(\cdot))$.

- No uniqueness assumption needed
- No assumptions on the smoothness of solutions is made
  ($\rightsquigarrow$ no convergence rates to be expected)

Result in short.

(1) The numerical approximations are relatively compact.

(2) All cluster points of numerical approximations
(as $h, \Phi(\Pi) \to 0$) are true solutions.

$\rightsquigarrow$ no spurious or ghost solutions.

(1) $\widehat{=}$ weakest form of stability of a numerical algorithm

(2) $\widehat{=}$ weakest form of consistency of a numerical algorithm

Last two lectures (w.l.o.g.): $\mathcal{Q} = \{0\} \times \mathcal{Z} \triangleq \mathcal{Q}$

$\widehat{\mathcal{E}}(t,z) \stackrel{\text{def}}{=} \min_{y\in\mathcal{Y}} \mathcal{E}(t,y,z)$

$(\mathcal{Q},\mathcal{E},\mathcal{R})$ Banach-space / manifold with Finsler [eq] metric

(DI) $\quad 0 \in \partial_v \mathcal{R}(z,\dot{z}) + \mathrm{D}\mathcal{E}(t,z)$

$(\mathcal{Q},\mathcal{E},\mathcal{D})$ Space with [eq] distance

Problem: Find a form of (DI) that works in distance spaces!

WIAS

For $\Phi : V \to \mathbb{R}_\infty$ convex, define $\Phi^* = \mathcal{L}\Phi$ via

$\Phi^* : V^* \to \mathbb{R}_\infty; \eta \mapsto \sup_{v \in V} \langle \eta, v\rangle - \Phi(v)$

To allow for viscous regularizations we allow more general dissipation potentials:

$\psi : [0,\infty[\to [0,\infty]$

(e.g.: $\psi_{\text{rate ind.}}(\nu) = \nu$, $\psi(\nu) = \frac{1}{p}\nu^p$, or $\psi(\nu) = \nu + \frac{\varepsilon}{2}\nu^2$)

$\mathcal{R}_\psi(q,v) = \psi(\mathcal{R}(q,v))$, where as usual $\mathcal{R}(q,\lambda v) = \lambda^1 \mathcal{R}(q,v)$

(DI) $\quad 0 \in \partial_v \mathcal{R}_\psi(q,\dot{q}) + \mathrm{D}\mathcal{E}(t,q)$

Define $\mathcal{R}_\psi^*(q,\cdot) = \mathcal{L}\mathcal{R}_\psi(q,\cdot)$

Fundamental property of the Legendre transform

$$\begin{aligned} \eta \in \partial\mathcal{R}_\psi(q,v) \quad &\Longleftrightarrow \quad v \in \partial\mathcal{R}_\psi^*(q,\eta) \\ &\Longleftrightarrow \quad \langle \eta, v\rangle = \mathcal{R}_\psi(q,v) + \mathcal{R}_\psi^*(q,\eta) \end{aligned}$$

Three different formulations

$0 \in \partial_v \mathcal{R}_\psi(q,\dot q) + \mathrm{D}\mathcal{E}(t,q)$ force balance (DI)

$\dot q \in \partial_v \mathcal{R}^*_\psi(q, -\mathrm{D}\mathcal{E}(t,q))$ flow rule (rate eqn.)

$\langle -\mathrm{D}\mathcal{E}(t,q), \dot q\rangle = \mathcal{R}_\psi(q,\dot q) + \mathcal{R}^*_\psi(q,-\mathrm{D}\mathcal{E}(t,q))$ energy balance

Using chain rule $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t,q(t)) - \partial_t \mathcal{E}(t,q(t)) = \langle \mathrm{D}\mathcal{E}(t,q), \dot q\rangle$ gives

(DI) $\iff$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t,q(t)) - \partial_t \mathcal{E}(t,q(t)) = -\mathcal{R}_\psi(q,\dot q) - \mathcal{R}^*_\psi(q,-\mathrm{D}\mathcal{E}(t,q))$$

$\iff$

$$\mathcal{E}(t,q(t)) + \int_0^t \mathcal{R}_\psi(q,\dot q) + \mathcal{R}^*_\psi(q,-\mathrm{D}\mathcal{E}(s,q))\,\mathrm{d}s = \mathcal{E}(0,q(0)) + \int_0^t \partial_s \mathcal{E}(s,q(s))\,\mathrm{d}s$$

Three different formulations

$0 \in \partial_v \mathcal{R}_\psi(q,\dot q) + \mathrm{D}\mathcal{E}(t,q)$	force balance (DI)
$\dot q \in \partial_v \mathcal{R}^*_\psi(q, -\mathrm{D}\mathcal{E}(t,q))$	flow rule (rate eqn.)
$\langle -\mathrm{D}\mathcal{E}(t,q), \dot q\rangle = \mathcal{R}_\psi(q,\dot q) + \mathcal{R}^*_\psi(q,-\mathrm{D}\mathcal{E}(t,q))$	energy balance

Using chain rule $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t,q(t)) - \partial_t\mathcal{E}(t,q(t)) = \langle \mathrm{D}\mathcal{E}(t,q), \dot q\rangle$ gives

(DI) $\iff$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t,q(t)) - \partial_t\mathcal{E}(t,q(t)) = -\underbrace{\mathcal{R}_\psi(q,\dot q)}_{\psi(\mathrm{norm}(\dot q))} - \underbrace{\mathcal{R}^*_\psi(q,-\mathrm{D}\mathcal{E}(t,q))}_{\psi^*(\text{dual norm}(...))}$$

Derivatives $\dot q$ and $\mathrm{D}\mathcal{E}$ only show up in norms (direction not needed)

$[\mathcal{R}_\psi(q,v) = \psi(\mathcal{R}(q,v))$ and $\mathcal{R}^*_\psi(q,\eta) = \psi^*(\mathcal{R}^*(q,\eta))]$

5.2 Metric velocity and slope

For the following see AMBROSIO&GIGLI&SAVARÉ 05

$(\mathcal{Q},\mathcal{D})$ complete distance space

$q:[0,T]\to\mathcal{Q}$ is **absolutely continuous** ($q\in\mathrm{AC}([0,T];\mathcal{Q})$), if $\exists\ m\in\mathrm{L}^1([0,T])$ such that $\mathcal{D}(q(r),q(t))\leq\int_r^t m(s)\,\mathrm{d}s$

Theorem (Metric velocity)

If $q\in\mathrm{AC}([0,T],\mathcal{Q})$, then for a.a. $t\in[0,T]$ the **metric velocity** $|\dot q|_{\mathcal{D}}(t)\overset{def}{=}\lim_{h\to0}\frac{1}{h}\mathcal{D}(q(t),q(t{+}h))$ *exists .*
Moreover, $|\dot q|_{\mathcal{D}}(t)\leq m(t)$ *a.e. and* $\mathrm{Diss}_{\mathcal{D}}(q,[r,t])=\int_r^t|\dot q|_{\mathcal{D}}(s)\,\mathrm{d}s.$

$\mathcal{D}$ generated by $\mathcal{R}$, then
$|\dot q|_{\mathcal{D}}(t)=\mathcal{R}(q(t),\dot q(t))$ a.e. for $q\in\mathrm{W}^{1,1}([0,T],\mathcal{Q})$.

Moreover, $\mathcal{R}_\psi(q,\dot q)=\psi(|\dot q|_{\mathcal{D}}(t))$.

Definition (Metric slope)

$$|\partial\mathcal{E}(t,.)|_{\mathcal{D}}(q) \stackrel{\text{def}}{=} \limsup_{\widetilde{q}\to q} \frac{\big(\mathcal{E}(t,q)-\mathcal{E}(t,\widetilde{q})\big)_+}{\mathcal{D}(q,\widetilde{q})}$$

If $\mathcal{E}(t,.)$ is Gateaux differentiable and $\mathcal{D}$ generated from $\mathcal{R}$, then $|\partial\mathcal{E}(t,.)|_{\mathcal{D}}(q) = \mathcal{R}^*(q,-\mathrm{D}\mathcal{E}(t,q))$.

Throughout, assume **chain-rule inequality** for $(\mathcal{Q},\mathcal{E},\mathcal{D})$

$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t,q(t)) - \partial_t\mathcal{E}(t,q(t)) \geq -|\dot{q}|_{\mathcal{D}}(t)\,|\partial\mathcal{E}(t)|_{\mathcal{D}}(q(t))$

This follows in the setting $(\mathcal{Q},\mathcal{E},\mathcal{R})$ from the classical chain rule

$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t,q(t)) - \partial_t\mathcal{E}(t,q(t)) = \langle \mathrm{D}\mathcal{E}(t,q),\dot{q}\rangle \geq -\mathcal{R}(q,\dot{q})\mathcal{R}^*(q,-\mathrm{D}\mathcal{E}(t,q))$.

Now the third form of (DI) can be written as a ψ–**gradient flow** or **metric evolution** in the sense of De Giorgi:

$$\text{(ME)} \quad \tfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t,q) - \partial_t\mathcal{E}(t,q) \leq -\psi(|\dot q|_{\mathcal{D}}) - \psi^*(|\partial\mathcal{E}(t)|_{\mathcal{D}}(q)) \ \text{ a.e.}$$

(ME), chain rule ineq., $\psi(\nu)+\psi^*(\xi) \geq \nu\xi$ give **energy balance**

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t,q) + \psi(|\dot q|_{\mathcal{D}}) + \psi^*(|\partial\mathcal{E}(t,.)|_{\mathcal{D}}(q)) = \partial_t\mathcal{E}(t,q)$$

as well as the **chain-rule identity** (along solutions of (ME))

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t,q) - \partial_t\mathcal{E}(t,q) = -|\dot q|_{\mathcal{D}}\, |\partial\mathcal{E}(t,.)|_{\mathcal{D}}(q)$$

Throughout, we used heavily $q \in \mathrm{AC}([0,T];\mathcal{Q})$.

How do we model solutions with jumps??

We derive a solutions via the **vanishing-viscosity approach**.
Hence, we obtain all **approximable solutions**.

Use $\psi_\varepsilon(\nu) = \underbrace{\nu}_{\text{rate.ind}} + \underbrace{\frac{\varepsilon}{2}\nu^2}_{\text{small visc.}} \geq 0 \;\; \rightsquigarrow \;\; \psi_\varepsilon(\xi) = \frac{1}{2\varepsilon}\big((\xi-1)_+\big)^2 \geq 0.$

(ME) takes the form of an upper energy estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t,q)+\psi_\varepsilon(|\dot q|_{\mathcal D})+\psi_\varepsilon^*(|\partial\mathcal{E}(t,.)|_{\mathcal D}(q)) \leq \partial_t\mathcal{E}(t,q)$$

(Chain rule supplies the lower energy estimate)

Condition ($\mathcal{E}2$) implies via Gronwall

$$\mathcal{E}(t,q(t)) \leq \mathrm{e}^{C_E t}e(0), \text{ where } e(0)=\mathcal{E}(q(0)), \text{ and}$$
$$\int_0^T \psi_\varepsilon(|\dot q|_{\mathcal D})+\psi_\varepsilon^*(|\partial\mathcal{E}(t,.)|_{\mathcal D}(q))\,\mathrm{d}t \leq \mathrm{e}^{C_E t}e(0)$$

Standard: for $\varepsilon>0$ existence of $q_\varepsilon \in AC([0,T],\mathcal{Q})$ with

$$\|\,|\dot q|_{\mathcal D}\|_{\mathrm{L}^1} \leq C \quad \text{and} \quad \|\,|\dot q|_{\mathcal D}\|_{\mathrm{L}^2} \leq C/\sqrt{\varepsilon}$$

How to control the limit q_ε for $\varepsilon \to 0$.

In general, jumps will develop!

Idea from EFENDIEV & M. JCA'06,
worked out in M. & ROSSI & SAVARÉ '08:

Consider the graphs

$$G_\varepsilon = \{\, (t, q_\varepsilon(t)) \mid t \in [0,T] \,\} \subset \mathcal{Q}_T \overset{\text{def}}{=} [0,T] \times \mathcal{Q}$$

and study graph convergence.
Then the **jump path** will be controlled.

Reparametrize $t = \mathbf{t}_\varepsilon(s)$ and $q = \mathbf{q}_\varepsilon(s) = q_\varepsilon(\mathbf{t}_\varepsilon(s))$, $s \in [0, S]$:

$$G_\varepsilon = \{\, (\mathbf{t}(s), \mathbf{q}_\varepsilon(s)) \mid s \in [0,S] \,\} \subset \mathcal{Q}_T$$

To obtain pointwise convergence we choose a good parametrization.

Fix parametrization via $\mathbf{t}_\varepsilon' + |\mathbf{q}_\varepsilon'|_\mathcal{D} = m_\varepsilon$ given in $\mathrm{L}^1([0,S])$!

Using the chain rule $|\mathbf{q}_\varepsilon'|_\mathcal{D}(s) = \mathbf{t}'(s)|\dot{q}_\varepsilon|_\mathcal{D}(\mathbf{t}(s))$ we have

$$\int_0^S m_\varepsilon \,\mathrm{d}s = \int_0^S \mathbf{t}_\varepsilon' + |\mathbf{q}_\varepsilon'|_\mathcal{D} \,\mathrm{d}s \overset{\text{chain}}{=} \int_0^S \mathbf{t}_\varepsilon'(1+|\dot{q}_\varepsilon'|_\mathcal{D})\,\mathrm{d}s$$
$$= \int_0^T 1+|\dot{q}_\varepsilon'|_\mathcal{D}\,\mathrm{d}t = T + \mathrm{Diss}(q_\varepsilon,[0,T]) \to M$$

The ψ-gradient flow equation

$$\text{(ME)} \qquad \tfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t,q) - \partial_t\mathcal{E}(t,q) \le -\psi_\varepsilon(|\dot{q}|_\mathcal{D}) - \psi_\varepsilon^*(|\partial\mathcal{E}(t,.)|_\mathcal{D}(q))$$

will be multiplied by $\mathbf{t}'$ and written in terms of $(\mathbf{t},\mathbf{q})$:

$$\tfrac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}(\mathbf{t},\mathbf{q}) - \partial_\mathbf{t}\mathcal{E}(\mathbf{t},\mathbf{q})\mathbf{t}' \le -\mathbf{t}'\psi_\varepsilon\big(\tfrac{1}{\mathbf{t}'}|\mathbf{q}'|_\mathcal{D}\big) - \mathbf{t}'\psi_\varepsilon^*(|\partial\mathcal{E}(\mathbf{t},.)|_\mathcal{D}(\mathbf{q}))$$
$$\mathbf{t}'(s) + |\mathbf{q}'|_\mathcal{D}(s) = m(s)$$

WIAS

$$\frac{\mathsf{d}}{\mathsf{ds}}\mathcal{E}(\mathbf{t},\mathbf{q}) - \partial_{\mathbf{t}}\mathcal{E}(\mathbf{t},\mathbf{q})\mathbf{t}' \leq -\mathbf{t}'\psi_\varepsilon\big(\tfrac{1}{\mathbf{t}'}|\mathbf{q}'|_{\mathcal{D}}\big) - \mathbf{t}'\psi_\varepsilon^*(|\partial\mathcal{E}(\mathbf{t},.)|_{\mathcal{D}}(\mathbf{q}))$$

We introduce the function

$$M_\varepsilon(\alpha,\nu,\xi) = \begin{cases} \alpha\psi_\varepsilon(\nu/\alpha) + \alpha\psi_\varepsilon^*(\xi) & \text{for } \alpha > 0, \\ \infty & \text{otherwise.} \end{cases}$$

Explicitly, $M_\varepsilon(\alpha,\nu,\xi) = \nu + \frac{\varepsilon}{2\alpha}\nu^2 + \frac{\alpha}{2\varepsilon}\big((\xi-1)_+\big)^2$ for $\alpha > 0$.
Legendre relation implies $M_\varepsilon(\alpha,\nu,\xi) \geq \nu\xi$.

Thus, the rescaled version of (ME) is equivalent to

$$\forall\, r < s: \qquad \mathcal{E}(\mathbf{t}(s),\mathbf{q}(s)) + \int_r^s M_\varepsilon(\mathbf{t}',|\mathbf{q}'|_{\mathcal{D}},|\partial\mathcal{E}(\mathbf{t},.)|_{\mathcal{D}})\,\mathrm{d}\tau \\ \leq \mathcal{E}(\mathbf{t}(r),\mathbf{q}(r)) + \int_r^s \partial_{\mathbf{t}}\mathcal{E}(\mathbf{t},\mathbf{q})\mathbf{t}'\,\mathrm{d}\tau$$

Today: only formal limit $\varepsilon \to 0$ (justification tomorrow)

$M_\varepsilon(\alpha,\nu,\xi) = \nu + \frac{\varepsilon}{2\alpha}\nu^2 + \frac{\alpha}{2\varepsilon}\big((\xi-1)_+\big)^2$ for $\alpha>0$ and $M_\varepsilon(0,\nu,\xi)=\infty$

$(*)_\varepsilon$ $\mathcal{E}(\mathbf{t}(s),\mathbf{q}(s)) + \int_r^s M_\varepsilon(\mathbf{t}', |\mathbf{q}'|_{\mathcal{D}}, |\partial\mathcal{E}(\mathbf{t},.)|_{\mathcal{D}})\mathrm{d}\tau \leq \mathcal{E}(\mathbf{t}(r),\mathbf{q}(r)) + \int_r^s \partial_{\mathbf{t}}\mathcal{E}(\mathbf{t},\mathbf{q})\mathbf{t}'\mathrm{d}\tau$

$$M_\varepsilon \xrightarrow{\Gamma} M_0 : (\alpha,\nu,\xi) \mapsto \begin{cases} \nu + \nu(\xi-1)_+ & \text{for } \alpha = 0, \\ \nu + \chi_{[0,1]}(\xi) & \text{for } \alpha > 0. \end{cases}$$

Formal limit $\varepsilon \to 0$:

- $(\mathbf{t}_\varepsilon, \mathbf{q}_\varepsilon) \in \mathrm{AC}([0,T];\mathcal{Q}_T)$ satisfy $(*)_\varepsilon$
- $(\mathbf{t}_\varepsilon(s), \mathbf{q}_\varepsilon(s)) \to (\mathbf{t}(s),\mathbf{q}(s))$ (with equibounded velocities)

Then, $(\mathbf{t},\mathbf{q})$ solves $(*)_0$:

$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}(\mathbf{t},\mathbf{q}) - \partial_{\mathbf{t}}\mathcal{E}(\mathbf{t},\mathbf{q})\mathbf{t}' \leq -M_0\big(\mathbf{t}', |\mathbf{q}'|_{\mathcal{D}}, |\partial\mathcal{E}(\mathbf{t},.)|_{\mathcal{D}}(\mathbf{q})\big)$ a.e.

Definition (Parametrized metric solutions)

Assume that $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$ satisfies the chain-rule inequality.

$(\mathbf{t}, \mathbf{q}) \in \mathrm{AC}([s_0, s_1]; \mathcal{Q}_T)$ is called **parametrized metric solution**, if a.e. in $[s_0, s_1]$ we have

- $\mathbf{t}'(s) \geq 0$ and $\mathbf{t}'(s) + |\mathbf{q}'|(s) > 0$
- $\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}(\mathbf{t}, \mathbf{q}) - \partial_\mathbf{t}\mathcal{E}(\mathbf{t}, \mathbf{q})\mathbf{t}' \leq -M_0\big(\mathbf{t}', |\mathbf{q}'|_\mathcal{D}, |\partial\mathcal{E}(\mathbf{t}, .)|_\mathcal{D}(\mathbf{q})\big)$.

Using the chain-rule inequality we must have equalities

$$-|\mathbf{q}'|_\mathcal{D}|\partial\mathcal{E}(\mathbf{t}, .)|_\mathcal{D}(\mathbf{q}) = \frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}(\mathbf{t}, \mathbf{q}) - \partial_\mathbf{t}\mathcal{E}(\mathbf{t}, \mathbf{q})\mathbf{t}'$$
$$= -M_0\big(\mathbf{t}', |\mathbf{q}'|_\mathcal{D}, |\partial\mathcal{E}(\mathbf{t}, .)|_\mathcal{D}(\mathbf{q})\big)$$

With $\mathbf{\Xi} \overset{\text{def}}{=} \{\, (\alpha, \nu, \xi) \mid M_0(\alpha, \nu, \xi) = \nu\xi \,\}$ this implies

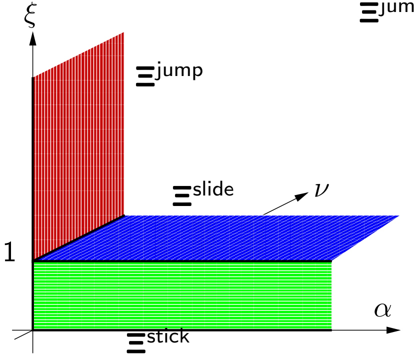
$$(\mathbf{t}'(s), |\mathbf{q}'|_\mathcal{D}(s), |\partial\mathcal{E}(\mathbf{t}, .)|_\mathcal{D}(\mathbf{q}(s))) \in \mathbf{\Xi} \text{ a.e. in } [s_0, s_1]$$

$M_0(\alpha,\nu,\xi) = \nu + \chi_{[0,1]}(\xi)$ for $\alpha > 0$ and $M_0(0,\nu,\xi) = \nu + \nu(\xi-1)_+$

$\boldsymbol{\Xi} \overset{\text{def}}{=} \{\, (\alpha,\nu,\xi) \mid M_0(\alpha,\nu,\xi) = \nu\xi \,\}$

$\boldsymbol{\Xi} = \boldsymbol{\Xi}^{\text{stick}} \cup \boldsymbol{\Xi}^{\text{slide}} \cup \boldsymbol{\Xi}^{\text{jump}}$

$$\boldsymbol{\Xi}^{\text{stick}} = \{\, (\alpha,0,\xi) \mid \alpha \geq 0,\ \xi \leq 1 \,\}$$
$$\boldsymbol{\Xi}^{\text{slide}} = \{\, (\alpha,\nu,1) \mid \alpha,\nu \geq 0 \,\}$$
$$\boldsymbol{\Xi}^{\text{jump}} = \{\, (0,\nu,\xi) \mid \nu \geq 0,\ \xi \geq 1 \,\}$$



Three distinct regimes
sticking
sliding
jumping

Alternative definition

$(\mathbf{t},\mathbf{q}) \in \mathrm{AC}([s_0,s_1];\mathcal{Q}_T)$ is a *parametrized metric solution*, if and only if a.e. in $[s_0,s_1]$:

- $\mathbf{t}'(s) \geq 0$ and $\mathbf{t}'(s) + |\mathbf{q}'|(s) > 0$;
- $\mathbf{t}'(s) > 0 \quad \Longrightarrow \quad |\partial\mathcal{E}(\mathbf{t},.)|_{\mathcal{D}}(\mathbf{q}(s)) \leq 1;$
- $|\mathbf{q}'|_{\mathcal{D}}(s) > 0 \quad \Longrightarrow \quad |\partial\mathcal{E}(\mathbf{t},.)|_{\mathcal{D}}(\mathbf{q}(s)) \geq 1;$
- $\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}(\mathbf{t},\mathbf{q}) - \partial_{\mathbf{t}}\mathcal{E}(\mathbf{t},\mathbf{q})\mathbf{t}' = -|\mathbf{q}'|_{\mathcal{D}}|\partial\mathcal{E}(\mathbf{t},.)|_{\mathcal{D}}(\mathbf{q}).$

Last condition: if $\mathbf{q}$ moves, then like a rescaled “gradient flow”.

Energy balance

$$\mathcal{E}(\mathbf{t}(s),\mathbf{q}(s)) + \int_r^s |\mathbf{q}'|_{\mathcal{D}}(\tau) + g(\tau)\,\mathrm{d}\tau = \mathcal{E}(\mathbf{t}(r),\mathbf{q}(r)) + \int_r^s \partial_{\mathbf{t}}\mathcal{E}(\mathbf{t},\mathbf{q})\mathbf{t}'\,\mathrm{d}\tau$$

with $g = |\mathbf{q}'|_{\mathcal{D}}\Big(|\partial\mathcal{E}(\mathbf{t},.)|_{\mathcal{D}}(\mathbf{q}) - 1\Big)_+$

g = additional dissipation power during a jump.
It arises as limit of rescaled viscosity contributions.

Vanishing-viscosity approach for $\varepsilon \to 0$.

$\psi_\varepsilon(\nu) = \nu + \frac{\varepsilon}{2}\nu^2 \geq 0$ and $\psi^*_\varepsilon(\xi) = \frac{1}{2\varepsilon}\big((\xi{-}1)_+\big)^2$

$$(\mathrm{ME})_\varepsilon \quad \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t,q_\varepsilon) - \partial_t\mathcal{E}(t,q_\varepsilon) \leq -\psi_\varepsilon(|\dot q_\varepsilon|_{\mathcal{D}}) - \psi^*_\varepsilon(|\partial\mathcal{E}(t)|_{\mathcal{D}}(q_\varepsilon))$$

Theorem [MRS'08]. Let $(\mathcal{Q},\mathcal{E},\mathcal{D})$ be as above with $\mathcal{E}, \partial_t\mathcal{E}{:}\mathcal{Q} \to \mathbb{R}$ are continuous and $|\partial\mathcal{E}(.,.)|_{\mathcal{D}}{:}\mathcal{Q} \to \mathbb{R}$ is lsc.

For solutions q_ε of $(\mathrm{ME})_\varepsilon$, define parametrizations $(\mathbf{t}_\varepsilon, \mathbf{q}_\varepsilon) \in \mathrm{AC}([0,S],\mathcal{Q}_T)$ with $\mathbf{t}'_\varepsilon + |\mathbf{q}'_\varepsilon|_{\mathcal{D}} = m_\varepsilon \to m \in \mathrm{L}^1$.

Then, there exists a subsequence $(\varepsilon_l)_l$ and a parametrized metric flow $(\mathbf{t},\mathbf{q})$ such that

$$(\mathbf{t}_\varepsilon(s), \mathbf{q}_\varepsilon(s)) \to (\mathbf{t}(s), \mathbf{q}(s)) \text{ for all } s \in [0,S].$$

Sketch of proof: Parametrized version of $(\mathrm{ME})_\varepsilon$:

$$\mathcal{E}(\mathbf{t}_\varepsilon(s),\mathbf{q}_\varepsilon(s)) + \int_r^s M_\varepsilon(\mathbf{t}_\varepsilon', |\mathbf{q}_\varepsilon'|_{\mathcal{D}}, |\partial\mathcal{E}(\mathbf{t}_\varepsilon,.)|_{\mathcal{D}}(\mathbf{q}_\varepsilon))\,\mathrm{d}\tau \\ \leq \mathcal{E}(\mathbf{t}_\varepsilon(r),\mathbf{q}_\varepsilon(r)) + \int_r^s \partial_{\mathbf{t}}\mathcal{E}(\mathbf{t}_\varepsilon,\mathbf{q}_\varepsilon)\mathbf{t}_\varepsilon'\,\mathrm{d}\tau$$

Extraction of a convergenct subsequence (not relabeled) by Helly's selection principle

$$\mathbf{t}_\varepsilon \to \mathbf{t},\ \mathbf{q}_\varepsilon \to \mathbf{q} \text{ in } \mathrm{C}^0([0,S],\mathcal{Q}).$$

$$\mathbf{t}_\varepsilon' \rightharpoonup \mathbf{t}', \quad \nu_\varepsilon \stackrel{\text{def}}{=} |\mathbf{q}_\varepsilon'|_{\mathcal{D}} \rightharpoonup \nu_*$$

$$\xi_*(s) = \liminf_{\varepsilon\to 0} |\partial\mathcal{E}(\mathbf{t}_\varepsilon(s),.)|_{\mathcal{D}}(\mathbf{q}_\varepsilon(s))$$

We find $\mathbf{t}' + \nu_* = m$, $|\mathbf{q}'|_{\mathcal{D}} \leq \nu_*$, and $|\partial\mathcal{E}(\mathbf{t},.)|_{\mathcal{D}}(\mathbf{q}_\varepsilon) \leq \xi_*$.

Using continuity of $\mathcal{E}$ and $\partial_t\mathcal{E}$ the limit $\varepsilon \to 0$ gives

$$\mathcal{E}(\mathbf{t}(s),\mathbf{q}(s)) + \liminf_{\varepsilon\to 0} \int_r^s M_\varepsilon(\mathbf{t}_\varepsilon', |\mathbf{q}_\varepsilon'|_{\mathcal{D}}, |\partial\mathcal{E}(\mathbf{t}_\varepsilon,.)|_{\mathcal{D}}(\mathbf{q}_\varepsilon))\,\mathrm{d}\tau \\ \leq \mathcal{E}(\mathbf{t}(r),\mathbf{q}(r)) + \int_r^s \partial_{\mathbf{t}}\mathcal{E}(\mathbf{t},\mathbf{q})\mathbf{t}'\,\mathrm{d}\tau$$

It remains to show $\int_r^s M_0(\mathbf{t}', |\mathbf{q}'|_{\mathcal{D}}, |\partial\mathcal{E}(\mathbf{t}, .)|_{\mathcal{D}}(\mathbf{q}))\,\mathrm{d}\tau$

$$\leq \mu_\varepsilon \overset{\text{def}}{=} \liminf_{\varepsilon\to 0} \int_r^s M_\varepsilon(\mathbf{t}_\varepsilon', |\mathbf{q}_\varepsilon'|_{\mathcal{D}}, |\partial\mathcal{E}(\mathbf{t}_\varepsilon, .)|_{\mathcal{D}}(\mathbf{q}_\varepsilon))\,\mathrm{d}\tau$$

We have the following properties:

$(\varepsilon, \alpha, \nu, \xi) \mapsto M_\varepsilon(\alpha, \nu, \xi)$ is lsc.

$M_\varepsilon(\cdot, \cdot, \xi) : [0, \infty[\to \mathbb{R}_\infty$ is convex.

$M_\varepsilon(\alpha, \cdot, \cdot)$ is monotone in each entry.

$$\mu_\varepsilon \quad \overset{\text{Joffe}}{\geq} \quad \int_r^s M_0(\mathbf{t}', \nu_*, \xi_*)\,\mathrm{d}\tau$$

$$\int_r^s M_0(\mathbf{t}', |\mathbf{q}'|_{\mathcal{D}}, |\partial\mathcal{E}(\mathbf{t}, .)|_{\mathcal{D}}(\mathbf{q}))\,\mathrm{d}\tau$$

Thus, the limit is a parametrized metric flow. ■

6.2 BV solutions

We want to return to solutions $q : [0,T] \to \mathcal{Q}$.

Given $(\mathbf{t},\mathbf{q}) \in \mathrm{AC}([0,S];\mathcal{Q}_T)$ with $\mathbf{t}(0) = 0$ and $\mathbf{t}(S) = T$,
define $q(t) = \mathbf{q}(\widehat{s}(t))$ with $\widehat{s}(t) = \min\{\, s \mid \mathbf{t}(s) \geq t \,\}$.

Obviously, we then have
$\{\, (\mathbf{t}(s),\mathbf{q}(s)) \mid s \in [0,S] \,\} = \mathrm{Graph}(q) \cup \bigcup_k \{k\text{th jump curve}\}$

Questions:

- Is it possible to characterize these solutions directly?
- Can we show directly the convergence $q_\varepsilon(t) \to q(t)$?

We first establish a chain-rule inequality for $q \in \mathrm{BV}([0,T];\mathcal{Q})$.
It should be consistent with the case $q \in \mathrm{AC}([0,T];\mathcal{Q})$, namely

AC: $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t,q(t)) - \partial_t\mathcal{E}(t,q(t)) \geq -\underbrace{|\dot{q}|_{\mathcal{D}}(t)\,|\partial\mathcal{E}(t)|_{\mathcal{D}}(q(t))}_{\mathcal{R}_{\mathrm{slope}}(t,q,\dot{q})}$

The **slope distance** $\boldsymbol{S}$ associated with $\mathcal{R}_{\mathrm{slope}}$ is defined via

$\boldsymbol{S}(t_0,q_0,t_1,q_1) \stackrel{\mathrm{def}}{=}$
$\inf\big\{\int_0^1 |\mathbf{q}'(s)|_{\mathcal{D}}|\partial\mathcal{E}(\mathbf{t}(s),.)|_{\mathcal{D}}(\mathbf{q}(s))\,\mathrm{d}s \,\big|\, (\mathbf{t},\mathbf{q}) \in \mathbb{A}(t_0,q_0,t_1,q_1)\big\}$

with the set of admissible pathes $\mathbb{A}(t_0,q_0,t_1,q_1) \stackrel{\mathrm{def}}{=}$
$\{\,(\mathbf{t},\mathbf{q}) \in \mathrm{AC}([0,1];\mathcal{Q}_T) \mid (\mathbf{t}(j),\mathbf{q}(j)) = (t_j,q_j), j=0,1,\ \mathbf{t}' \geq 0\,\}$

Slope variation along a BV curve

$$\boldsymbol{\Sigma}(q,[r,t]) = \mathrm{Var}_{\boldsymbol{S}}(q,[r,t]) \stackrel{\mathrm{def}}{=} \sup_{\text{all part.}} \sum_1^N \boldsymbol{S}(t_{j-1},q(t_{j-1}),t_j,q(t_j))$$

Chain-rule inequality for BV solutions:
Consider $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$ with $\mathcal{E}, \partial\mathcal{E}$, and $|\partial\mathcal{E}|_{\mathcal{D}}$ being continuous, then for all $q \in \mathrm{BV}([0,T];\mathcal{Q})$ and all $[r,t] \subset [0,T]$ we have

$$\mathcal{E}(t,q(t)) - \mathcal{E}(r,q(r)) - \int_r^t \partial_s\mathcal{E}(s,q(s))\,\mathrm{d}s \geq -\mathbf{\Sigma}(q,[r,t])$$

For BV functions q we have left and right limits

$$q(t^{\pm}) \overset{\text{def}}{=} \lim_{\tau\to 0+} q(t{\pm}\tau).$$

Set J_q of jump times is countable:
$\mathrm{J}_q \overset{\text{def}}{=} \{\, t \in [0,T] \mid \#\{q(t^-), q(t), q(t^+)\} > 1 \,\}$ (countable)

Set of sticking times $\mathrm{S}_q \overset{\text{def}}{=} \{\, t \mid \exists \delta > 0\colon\, q|_{[t-\delta,t+\delta]} \text{ is const.} \,\}$

Definition of BV solutions

$q \in \mathrm{BV}([0,T];\mathcal{Q})$ is called BV solution for $(\mathcal{Q},\mathcal{E},\mathcal{D})$, if the following holds:

(1) $\mathcal{E}(t,q(t)) - \mathcal{E}(r,q(r)) - \int_r^t \partial_s\mathcal{E}(s,q(s))\,\mathrm{d}s \leq -\mathbf{\Sigma}(q,[r,t])$;

(2) $\forall\, t \in [0,T]\backslash \mathrm{J}_q$: $|\partial\mathcal{E}(t,.)|_{\mathcal{D}}(q(t)) \leq 1$;

(3) $\forall\, t \in [0,T]\backslash \mathrm{S}_q$: $|\partial\mathcal{E}(t,.)|_{\mathcal{D}}(q(t)) \geq 1$;

(4) $\forall\, t \in \mathrm{J}_q\ \exists\, (\mathbf{t},\mathbf{q}^t) \in \mathbb{A}(t,q(t^-),t,q(t^+))$:

(a) $\exists\, \theta^t \in [0,1]$: $q(t) = \mathbf{q}^t(\theta^t)$,

(b) $\forall\, \theta \in [0,1]$: $|\partial\mathcal{E}(t,.)|_{\mathcal{D}}(\mathbf{q}(\theta)) \geq 1$,

(c) $\mathcal{E}(t,q(t^+)) - \mathcal{E}(t,q(t^-)) = -\int_0^1 |\mathbf{q}'|(\theta)|\partial\mathcal{E}(t,.)|_{\mathcal{D}}(\mathbf{q}(\theta))\mathrm{d}\theta$

BV solutions are closely related to parametrized solutions:
To obtain $(\mathbf{t},\mathbf{q})$ from q, one fills in the jumps from (4)

Theorem
If $q_\varepsilon \in \mathrm{AC}([0,T];\mathcal{Q})$ are solutions of the ψ_ε-gradient flow $(\mathrm{ME})_\varepsilon$ and $q_\varepsilon(0) \to q^0$.

Then, there exists a subsequence $(\varepsilon_l)_l$ and a BV solution $q : [0,T] \to \mathcal{Q}$ for $(\mathcal{Q},\mathcal{E},\mathcal{D})$, such that $q_{\varepsilon_l}(t) \to q(t)$.

Consequence: All *approximable solutions* are BV solutions.

The opposite is not true.

Proof:

- Construct parametrized solutions $(\mathbf{t}_\varepsilon, \mathbf{q}_\varepsilon)$ of $(*)_\varepsilon$ first.
- Apply previous theorem to obtain a param. metric flow $(\mathbf{t}, \mathbf{q})$.
- $q_\varepsilon(t) = \mathbf{q}_\varepsilon(\widehat{s}_\varepsilon(t)) \to \mathbf{q}(\widehat{s}_0(t)) = q(t)$ ■

6.3 Stability of the solution set

Sequence of RIS $(\mathcal{Q}, \mathcal{E}_k, \mathcal{D})$ with $\mathcal{E}_k \to \mathcal{E}_\infty$

Stability $=$

upper semi-continuity of solution set with respect to data

(cf. Γ-convergence result for energetic solutions)

Proposition Let $(\mathcal{Q}, \mathcal{D})$ by a complete distance space; $\mathcal{E}_k, \mathcal{E}$ and their slopes are continuous, $(\mathcal{E}1, 2)$ hold uniformly, $\mathcal{E}_k \to \mathcal{E}_\infty$ pointwise, and $q_k^0 \to q_\infty^0$. Then

$$\operatorname*{Limsup}_{k\to\infty} \mathrm{ParMetrSol}\big(\mathcal{Q}, \mathcal{E}_k, \mathcal{D}, q_k^0\big) \subset \mathrm{ParMetrSol}\big(\mathcal{Q}, \mathcal{E}_\infty, \mathcal{D}, q_\infty^0\big)$$

and

$$\operatorname*{Limsup}_{k\to\infty} \mathrm{BVSol}\big(\mathcal{Q}, \mathcal{E}_k, \mathcal{D}, q_k^0\big) \subset \mathrm{BVSol}\big(\mathcal{Q}, \mathcal{E}_\infty, \mathcal{D}, q_\infty^0\big)\ .$$

As shown above we have

$$\mathrm{ApprSol}\big(\mathcal{Q},\mathcal{E},\mathcal{D},q^0\big) \subset \mathrm{BVSol}\big(\mathcal{Q},\mathcal{E},\mathcal{D},q^0\big)$$

But we do not have stability for **approximable solutions**.

$$\operatorname*{Limsup}_{k\to\infty} \mathrm{ApprSol}\big(\mathcal{Q},\mathcal{E}_k,\mathcal{D},q^0_k\big) \not\subset \mathrm{ApprSol}\big(\mathcal{Q},\mathcal{E}_\infty,\mathcal{D},q^0_\infty\big)$$

By definition $\mathrm{ApprSol}\big(\mathcal{Q},\mathcal{E},\mathcal{D},q^0\big) = \operatorname*{Limsup}_{\varepsilon\to 0} \mathrm{Sol}((\mathrm{ME})_\varepsilon, q^0)$.

Thus, $\mathrm{Limsup}_{k\to\infty}\,\mathrm{Limsup}_{\varepsilon\to 0} \neq \mathrm{Limsup}_{\varepsilon\to 0}\,\mathrm{Limsup}_{k\to\infty}$.

Example from above:

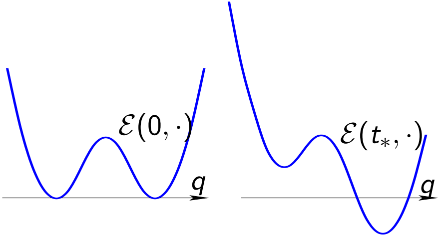
$\mathcal{Q} = \mathbb{R}$

$\mathcal{E}_k(t,q) = \Phi(q) - \ell_k(t)q$ with $\Phi(q) = \begin{cases} \frac{1}{2}(q+4)^2 & \text{for } q \le -2, \\ 4 - \frac{1}{2}q^2 & \text{for } |q| \le 2, \\ \frac{1}{2}(q-4)^2 & \text{for } q \ge 2; \end{cases}$

$\mathcal{R}(q,v) = |v|$

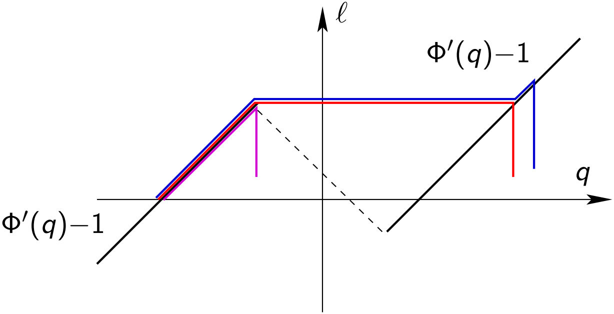
$\Rightarrow \quad \mathcal{D}(q_0,q_1) = |q_1 - q_0|$.

Initial state $q(0) = -4$.



$\ell_k(t) = \min\{t, 6-t\} + \frac{1}{k}$

$\ell = 3 =$ critical loading for the jump and $\ell_k(3) = \max \ell_k = 3 + \frac{1}{k}$



$k \in \mathbb{N}$ $\quad \mathrm{BVSol}_k = \mathrm{ApprSol}_k$

$k = \infty$ $\quad \mathrm{ApprSol}_\infty \not\subset \mathrm{BVSol}_\infty = \mathrm{ApprSol}_\infty \cup \operatorname*{Limsup}_{k\to\infty} \mathrm{ApprSol}_k$

6.4 Direct incremenal approximation

BV solutions and parametric metric solutions:

- behave mostly rate independent
- have some viscous remainders,
  namely jumps via rescaled gradient flows

Task: Find direct approximation scheme to these solutions.

Parametrized metric setting:

Find approximations for $(\mathbf{t},\mathbf{q}) \in \mathrm{AC}([0,S];\mathcal{Q}_T)$.

Without loss of generality assume exact arclength parametrization
$\mathbf{t}'(s) + |\mathbf{q}'|_{\mathcal{D}}(s) = 1$ for a.a. $s \in [0,S]$

Scheme from EFENDIEV&M.'06:

For an **arclength stepsize** $\sigma > 0$ we want to find approximations $(\mathbf{t}_j,\mathbf{q}_j) \approx (\mathbf{t}(j\sigma),\mathbf{q}(j\sigma))$.

$$(\mathrm{IP})_\sigma \qquad \begin{cases} \mathbf{q}_j \in \mathrm{Argmin}\{\, \mathcal{E}(\mathbf{t}_{j-1},q) + \mathcal{D}(\mathbf{q}_{j-1},q) \,|\, \mathcal{D}(\mathbf{q}_{j-1},q) \le \sigma \,\} \\ \mathbf{t}_j = \mathbf{t}_{j-1} + \sigma - \mathcal{D}(\mathbf{q}_{j-1},\mathbf{q}_j). \end{cases}$$

Only local minimization (with nonsmooth constraint)

$$(\mathrm{IP})_\sigma \qquad \begin{cases} \mathbf{q}_j \in \mathrm{Argmin}\{\, \mathcal{E}(\mathbf{t}_{j-1}, q) + \mathcal{D}(\mathbf{q}_{j-1}, q) \mid \mathcal{D}(\mathbf{q}_j, q) \leq \sigma \,\} \\ \mathbf{t}_j = \sigma - \mathcal{D}(\mathbf{q}_{j-1}, \mathbf{q}_j). \end{cases}$$

If $(\mathcal{Q}, \mathcal{D})$ admits geodesic curves, then define the **geodesic interpolant** $(\widetilde{\mathbf{t}}_\sigma, \widetilde{\mathbf{q}}_\sigma) \in \mathrm{AC}([0, S]; \mathcal{Q})$.

Under the assumption from above subsequences converge to a parametrized metric solution still having arclength parametrization.

For calculation **BV solutions** directly consider

a partition Π of $[0,T]$ and a viscosity $\varepsilon > 0$:

$$(\text{IMP})^{\varepsilon}_{\Pi} \quad q_j \in \underset{q\in\mathcal{Q}}{\text{Argmin}} \Big(\mathcal{E}(t_j, q) + \mathcal{D}(q_{j-1}, q) + \frac{\varepsilon\,\mathcal{D}(q_{j-1}, q)^2}{t_j - t_{j-1}} \Big)$$

(Almost proved) Theorem: (Cetraro'08)
Assume that $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$ satisfies the assumption from above.
Take sequence of partions $(\Pi_k)_{k\in\mathbb{N}}$ and of viscosities $(\varepsilon_k)_{k\in\mathbb{N}}$ with

$$\varepsilon_k \to 0 \quad \text{and} \quad \frac{\text{fineness}(\Pi_k)}{\varepsilon_k} \to 0.$$

Then, the De Giorgi interpolants $(\widetilde{q}_k)_{k\in\mathbb{N}}$ associated with $(\text{IMP})^{\varepsilon_k}_{\Pi_k}$ have a pointwise convergent subsequence, whose limit q is a BV solution for $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$.

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