A new identity for the surface impedance matrix and its application to the determination of surface-wave speeds

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Abstract

It is well-known in surface wave theory that the secular equation for the surface-wave speed v can be written as $\det M = 0$ in terms of the surface impedance matrix M. It is shown in this paper that M satisfies the simple identity $(M - \mathrm{i}R)T^{-1}(M + \mathrm{i}R^T) - Q + \rho v^2 I = 0$ in the usual notation in the Stroh formalism. This identity provides an efficient method for calculating the wave speed of surface waves in unstressed or prestressed elastic half-spaces. The method is explained and illustrated by examples. It is also shown that the buckling/wrinkling prestress for a prestressed elastic half-space can be calculated using the same procedure but with prestress playing the role of v.

Keywords: Stroh formalism; surface waves; elastic half-space; buckling; surface impedance matrix.

1 Introduction

Free surface waves are travelling waves that can propagate along the surface of an elastic half-space. They satisfy the traction-free boundary condition and decay to zero exponentially away from the surface. Plane body waves can always propagate in any infinite, well-behaved elastic body, as guaranteed by the satisfaction of the the strong ellipticity condition. But given an elastic half-space, the existence of a surface wave solution is not guaranteed since in general one needs a linear combination of three inhomogeneous body wave solutions (that have the right decaying behaviour) to construct a surface wave solution, and it is not always possible for such a construction to satisfy the traction-free boundary condition.

The central question addressed in the linear theory of surface waves is the existence and uniqueness: given a half-space, can it support a surface wave? and if it does, is the surface wave solution unique? Over the past three decades or so, this question has been fully resolved, with the aid of the powerful Stroh formalism (Stroh 1958, 1962), even for a generally anisotropic elastic half-space. We refer to the review article by Chadwick and Smith (1977) and the book by Ting (1996) for a detailed description of the theory and for a comprehensive collection of relevant references. It is now known that whenever a subsonic surface wave exists, it is unique. Furthermore, given an elastic half-space, there are simple procedures to determine whether a surface wave can propagate or not.

Once the existence has been established by the general theory, there remains the question of finding the surface-wave speed from a *secular equation.* A number of approaches have been suggested under the Stroh formalism; see, for instance, Barnett and Lothe (1985) and Chadwick and Wilson (1992). Although all these approaches are straightforward, their use requires familiarity with the Stroh formalism and a considerable amount of numerical work. Recently, Mozhaev (1994) proposed a novel method based on first integrals of displacement components, and obtained an explicit secular equation which, in the most general case, involves the evaluation of the determinant of an 18×18 matrix. More recently, motivated by Mozhaev's (1994) approach and ideas in the Stroh formalism, Destrade (2001) developed another efficient method based on first integrals of traction components. Although both Mozhaev's (1994) and Destrade's (2001) methods yield explicit secular equations that can be solved numerically for the wave speed, these secular equations also admit spurious roots that have to be carefully eliminated.

It is well-known that the secular equation for the surface-wave speed v can be written as

$$\det M(v) = 0, \tag{1.1}$$

where $M(v)$ is the *surface impedance matrix* (Ingebrigtsen and Tonning 1969). This matrix has the attractive properties that it is Hermitian so that $\det M(v)$ is real and that $\det M(v)$ is a monotone decreasing function of v in a sufficiently large interval starting from $v = 0$ so that it is very amenable to numerical calculations. To simplify notation, we shall write $M(v)$ simply as M hereafter. In this paper, we show that M can be determined from the simple algebraic matrix equation

$$(M - \mathrm{i}R)T^{-1}(M + \mathrm{i}R^T) - Q + \rho v^2 I = 0, \tag{1.2}$$

where ρ is the material density, I is the identity matrix, the superscripts "T" and "-1" denote matrix transpose and inverse, respectively, and in terms of the elastic stiffnesses C_{ijks} the components of the three matrices T, R, Q are defined by

$$T_{ik} = C_{i2k2}, \quad R_{ik} = C_{i1k2}, \quad Q_{ik} = C_{i1k1}. \tag{1.3}$$

Throughout this work we assume that surface waves propagate along the the x_1-direction and the half space occupies the region $x_2 > 0$ relative to a rectangular coordinate system with coordinates (x_i).

Relation (1.2) does not seem to have been noticed previously by researchers working with the Stroh formalism. The discovery of this relation is motivated by recent results of Mielke and Sprenger (1998) on a topic which is indirectly related to the surface wave problem. In fact (1.2) is called an "algebraic Riccati equation" in the mathematical literature, especially in Control Theory where an extended literature exists, cf. Lancaster and Rodman (1995).

Once (1.2) is known, one would naturally try to solve (1.2) for M and substitute the solution into (1.1) to obtain an explicit secular equation. We note, however, that solutions of (1.2) for M are not unique. For instance, if M is a solution, then so is $-M^T$. In the following section, we summarize some results from the general surface wave theory under the Stroh formalism; these results show that whenever a surface wave with speed v_R exists, M must necessarily be positive definite for $0 \leq v < v_R$ and at $v = v_R$ it must be positive semi-definite and must have 0 as an unrepeated eigenvalue. It is further shown in Section 3 that the solution of (1.2) that has the above properties is unique. These results suggest the following simple method for determining the surface wave speed: If equation (1.2) can be solved exactly, we simply find the unique solution of M that is positive definite at $v = 0$ and substitute it into (1.3) to obtain an explicit secular equation for v. If it is not possible to solve (1.2) exactly, we may numerically solve (1.1) and (1.2) simultaneously, making use of the selection criterion that the correct M must be positive semi-definite and must have 0 as an unrepeated eigenvalue.

In Section 3 we also derive (1.2) using two different approaches. One approach is based on the Stroh formalism whereas the other is entirely free from the Stroh formalism. The above method for computing the surface wave speed is then explained and illustrated by examples in Section 4. We explain how in the numerical solution of (1.1) and (1.2) an initial guess may be chosen that should usually converge to the correct solution. In Section 5, we extend our analysis to surface waves in *prestressed* elastic half-spaces. With the prestress acting as an extra parameter, there exists the possibility that a standing surface wave may be supported by the half-space. When this happens, the prestressed half-space is said to be marginally stable, and the corresponding condition on the prestress is called the *buckling/wrinkling condition*. The latter condition is also the condition at which the complementing condition is marginally violated (see, e.g., Thompson 1969, Simpson and Spector 1987 and 1989, Renardy and Rogers 1996). For variational problems this condition is called the Agmon's condition in the linear setting and quasiconvexity at the boundary in the nonlinear setting (Ball and Marsden 1984, Mielke and Sprenger 1998). Since the buckling condition can be obtained from the secular equation by setting $v = 0$, it can also be derived by the present method. This is discussed in Section 6. In all these sections, the half-space is assumed to be compressible. In the final section, we discuss how an incompressible elastic half-space could be treated as the limit of a nearly incompressible elastic half-space.

2 Elements of the surface wave theory under the Stroh formalism

We first consider a homogeneous, un-stressed, generally anisotropic elastic half-space defined by

$$0 < x_2 < \infty, \quad -\infty < x_1, x_3 < \infty$$

relative to a rectangular coordinate system with coordinates (x_i). Free surface waves are governed by the equation of motion

$$C_{ijks} u_{k,sj} = \rho \ddot{u}_i, \quad 0 < x_2 < \infty, \tag{2.1}$$

the traction-free boundary condition

$$C_{i2ks}u_{k,s} = 0 \quad \text{on} \;\; x_2 = 0, \tag{2.2}$$

and the decay condition

$$u_k \to 0 \quad \text{as } x_2 \to \infty, \tag{2.3}$$

where (u_k) is the displacement, ρ the material density, a comma denotes differentiation with respect to spatial coordinates and a dot denotes material time derivative. The C_{ijks} are elastic stiffnesses and are assumed to satisfy the symmetry relations

$$C_{ijks} = C_{ksij} = C_{jiks}, \tag{2.4}$$

and the strong convexity condition

$$C_{ijks}\xi_{ij}\xi_{ks} > 0 \;\; \forall \text{ non-zero real symmetric tensors } \boldsymbol{\xi}. \tag{2.5}$$

The strong ellipticity condition is given by

$$C_{ijks}\eta_i\eta_k\gamma_j\gamma_s > 0 \;\; \forall \text{ non-zero real vectors } \boldsymbol{\eta} \text{ and } \boldsymbol{\gamma}, \tag{2.6}$$

and is implied by the strong convexity condition (2.5).

Without loss of generality, we may assume that the surface wave is propagating along the x_1-direction and that

$$\boldsymbol{u} = \boldsymbol{z}(mx_2)\mathrm{e}^{\mathrm{i}m(x_1 - vt)} + \text{C.C.}, \tag{2.7}$$

where $\mathbf{u} = (u_k)$, $\mathrm{i} = \sqrt{-1}$, m is a positive wave number, v is the propagation speed and C.C. denotes the complex conjugate of the preceding term. The elastic half-space is said to support a surface wave if we can find a real positive value v and a non-trivial vector function $\boldsymbol{z}(y)$ such that (2.1)–(2.3) are satisfied.

On substituting (2.7) into (2.1) and (2.2), we obtain

$$T\mathbf{z}''(y) + \mathrm{i}(R + R^T)\mathbf{z}'(y) - (Q - \rho v^2 I)\mathbf{z}(y) = \mathbf{0}, \quad 0 < y < \infty, \tag{2.8}$$

$$T\mathbf{z}' + \mathrm{i}R^T\mathbf{z} = \mathbf{0}, \quad \text{on } y = 0, \tag{2.9}$$

where a prime signifies differentiation with respect to y $(= mx_2)$ and the matrices T, R, Q are defined by (1.3). We note that satisfaction of the strong ellipticity condition (2.6) ensures that T and Q are both positive definite and hence they are invertible.

To solve (2.8), we look for a solution of the form

$$\mathbf{z} = \mathbf{a}\mathrm{e}^{ipy}, \quad \mathrm{Im}\,(p) > 0, \tag{2.10}$$

where $\mathrm{Im}\,(p)$ denotes the imaginary part of p and the inequality is imposed to ensure satisfaction of the decay condition (2.3). The number p and vector $\mathbf{a}$ are determined by the following eigenvalue problem resulting from the substitution of (2.10) into (2.8):

$$\Big(p^2T + p(R + R^T) + Q - \rho v^2 I\Big)\,\mathbf{a} = \mathbf{0}. \tag{2.11}$$

Thus, the eigenvalues p are determined by

$$\det\left(p^2T + p(R + R^T) + Q - \rho v^2 I\right) = 0. \tag{2.12}$$

We note that corresponding to (2.10), $\mathbf{u}$ given by (2.7) becomes

$$\mathbf{u} = \mathbf{a}\,\mathrm{e}^{\mathrm{i}m(x_1+px_2-vt)}. \tag{2.13}$$

If p were real, $p = \tan\phi$ say, (2.13) would then represent a plane body wave with wave number $m/\cos\phi$ and speed $v\cos\phi$, propagating in the direction $\mathbf{n} = (n_k) = (\cos\phi, \sin\phi, 0)$. Equation (2.11), which could be re-written as

$$\left(\sin^2\phi\, T + \sin\phi\cos\phi\,(R + R^T) + \cos^2\phi\, Q - \rho v^2\cos^2\phi\, I\right)\mathbf{a} = \mathbf{0}, \tag{2.14}$$

or equivalently,

$$(C_{ijks}n_s n_j - \rho v^2\cos^2\phi\,\delta_{ik})\,a_k = 0, \tag{2.15}$$

would then become the propagation condition for the body wave.

Define by v_b the speed of body waves and by $\hat{v}$ the minimum of $v_b/\cos\phi$ over all possible values of ϕ and all possible plane body waves propagating in the direction $(\cos\phi, \sin\phi, 0)$. This means that any value v determined by the eigenvalue problem (2.14) must necessarily be greater than or equal to $\hat{v}$. The condition $0 \le v < \hat{v}$ then ensures that none of the roots of (2.12) can be real, for if one such real root existed, $p = \tan\phi$ say, the above argument would imply that (2.14) has a solution for v that is smaller than $\hat{v}$, which contradicts the statement in the last sentence. Surface waves with wave speeds satisfying $0 \le v < \hat{v}$ are said to be *subsonic*, and the number $\hat{v}$ is called the *limiting speed* (Chadwick and Smith 1977, p. 335).

We now assume that $0 \le v < \hat{v}$. We denote by p_1, p_2, p_3 the three complex roots of (2.12) that have positive imaginary parts and by $\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \mathbf{a}^{(3)}$ a set of associated eigenvectors. By considering the complex conjugate of (2.11), we deduce that $\overline{p}_1, \overline{p}_2, \overline{p}_3$, where a bar signifies complex conjugation, are also roots of (2.12) and that a set of associated eigenvectors is $\overline{\mathbf{a}}^{(1)}, \overline{\mathbf{a}}^{(2)}, \overline{\mathbf{a}}^{(3)}$. It can be shown (see, e.g., Chadwick and Smith 1977, p. 350) that when p_1, p_2, p_3 are all distinct, $\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \mathbf{a}^{(3)}$ must necessarily be linearly independent. For a simple exposition we assume that when there exist repeated roots, such a set of linearly independent eigenvectors can still be found. We remark, however, all the main results presented in this paper can be shown to be independent of this assumption.

A general solution that satisfies the decay condition is then given by

$$\mathbf{z} = \sum_{k=1}^{3} q_k \mathbf{a}^{(k)} \mathrm{e}^{\mathrm{i}p_k y}, \tag{2.16}$$

where q_1, q_2, q_3 are disposable constants. This solution yields a surface wave solution only if the traction-free boundary condition (2.9) is satisfied, that is if

$$\sum_{k=1}^{3} q_k \mathbf{b}^{(k)} = B\mathbf{q} = \mathbf{0}, \tag{2.17}$$

where

$$\mathbf{b}^{(k)} = p_k T \mathbf{a}^{(k)} + R^T \mathbf{a}^{(k)}, \quad B = (\mathbf{b}^{(1)}, \mathbf{b}^{(2)}, \mathbf{b}^{(3)}), \quad \mathbf{q} = (q_1, q_2, q_3)^T. \tag{2.18}$$

In the above definition, B is the 3×3 matrix formed by putting the three column vectors $\mathbf{b}^{(k)}$ side by side. It then follows that a surface wave exists only if the *secular equation*

$$\det B = 0 \tag{2.19}$$

has a positive real root for v. The central question addressed in surface wave theory is the existence and uniqueness of such a root.

There are a number of situations for which the answer to the above question is straightforward. For instance, for a half-space that is made of an isotropic material or an orthotropic material whose axes of symmetry coincide with the three coordinate axes, the equation for u_3 decouples from those for u_1 and u_2. We may then restrict the range of all subscripts to 1 and 2, and as a result the three matrices T, R, Q have the simple form

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}, \quad R = \begin{pmatrix} 0 & R_1 \\ R_2 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix}. \tag{2.20}$$

Equation (2.12) then reduces to a quadratic equation for p^2. Since the secular equation (2.19) can be manipulated into a form that depends on p_1 and p_2 through $p_1^2+p_2^2$ and p_1p_2 whose expressions can be expressed explicitly in terms of the coefficients in the quadratic equation for p^2, it can be reduced to an *explicit* form: a form that depends only on the elastic stiffnesses and the speed v. The existence and uniqueness question can then be settled either by elementary analysis or by a straightforward numerical calculation.

For more general anisotropic materials, elimination of p_1, p_2, p_3 from the secular equation (2.19) is in general not possible. As a result, the secular equation cannot be reduced to an explicit form. This is the source of difficulty in characterizing surface waves in generally anisotropic elastic half-spaces. We note, however, that although an explicit secular equation is not possible by the above method of elimination, it can still be obtained by a different route, as has been demonstrated by Mozhaev (1994) and Destrade (2001).

In the surface wave theory that has been developed over the past three decades under the framework of the Stroh formalism, the existence and uniqueness problem for surface waves in generally anisotropic materials has been beautifully resolved. We now summarize its main results that are relevant to the present study. For a comprehensive review of various aspects of the Stroh formalism, we refer the reader to the reviw article by Chadwick and Smith (1977) and the book by Ting (1996). For a concise account of the surface wave theory based on the surface impedance matrix, we refer the reader to the paper by Barnett and Lothe (1985).

We first consider the problem of solving (2.8) subject to the usual decay condition and, instead of the traction-free boundary condition (2.9), the condition that $\mathbf{z}(0)$ is prescribed. Since the latter condition corresponds to (cf. (2.7))

$$\mathbf{u}(x_1, 0, t) = \mathbf{z}(0) \mathrm{e}^{im(x_1 - vt)} + C.C.,$$

the half-space under consideration is subjected to a forcing that is travelling with speed v. The surface traction required to produce such a surface displacement is given by $C_{i2ks}u_{k,s}$.

The averaged work done by the surface traction over the area $0 < x_1 < 2\pi/m$, $0 < x_3 < 1$ and over one period of oscillation from $t = 0$ to $t = 2\pi/(mv)$ is given by

$$\mathcal{L} = \frac{mv}{2\pi} \cdot \frac{m}{2\pi} \int_0^{2\pi/(mv)} \int_0^{2\pi/m} C_{i2ks} u_{k,s} u_i dx_1 dt. \tag{2.21}$$

Since the integrand takes the same value at $x_1 = 0$ and $x_1 = 2\pi/m$ and decays to zero exponentially as $x_2 \to \infty$, the above integral can be replaced by

$$\mathcal{L} = \frac{m^2 v}{4\pi^2} \int_0^{2\pi/(mv)} \oint_{\partial D} C_{ijks} u_{k,s} u_i n_j dx_1 dt, \tag{2.22}$$

where ∂D is the boundary of the domain $0 < x_1 < 2\pi/m$, $0 < x_2 < \infty$, and (n_j) is the unit normal to ∂D. Applying the divergence theorem and making use of the equation of motion (2.1), we obtain

$$\mathcal{L} = \frac{m^2 v}{4\pi^2} \int_0^{\infty} \int_0^{2\pi/m} \int_0^{2\pi/(mv)} L \, dt dx_1 dx_2, \tag{2.23}$$

where

$$\begin{aligned} L &= C_{ijks} u_{i,j} u_{k,s} + \rho \ddot{u}_i u_i, \\ &= Q_{ik} u_{k,1} u_{i,1} + R_{ki} u_{k,1} u_{i,2} + R_{ik} u_{k,2} u_{i,1} + T_{ik} u_{i,2} u_{k,2} + \rho v^2 u_{i,11} u_i, \\ &= 2m^2 \left\{ Q_{ik} z_k \overline{z}_i + \mathrm{i} R_{ki} (z_k \overline{z}_i' - \overline{z}_k z_i') + T_{ik} z_i' \overline{z}_k' - \rho v^2 z_i \overline{z}_i \right\} \\ &\quad + w(kx_2) \mathrm{e}^{2\mathrm{i}m(x_1 - vt)} + \overline{w}(kx_2) \mathrm{e}^{-2\mathrm{i}m(x_1 - vt)}, \end{aligned} \tag{2.24}$$

a bar signifies complex conjugation and the expression for $w(kx_2)$ is not required in our analysis. In obtaining the last equation we have also made use of the representation (2.7). It then follows that

$$\mathcal{L} = 2m^2 \int_0^{\infty} \left(Q\mathbf{z} \cdot \overline{\mathbf{z}} + \mathrm{i} R^T \mathbf{z} \cdot \overline{\mathbf{z}}' - \mathrm{i} R \mathbf{z}' \cdot \overline{\mathbf{z}} + T\mathbf{z}' \cdot \overline{\mathbf{z}}' - \rho v^2 \mathbf{z} \cdot \overline{\mathbf{z}} \right) dy, \tag{2.25}$$

where a dot denotes the usual dot product of two vectors. By evaluating the integral in (2.25) by parts and making use of (2.8), we may reduce the above expression to

$$\frac{1}{2m^2} \mathcal{L} = (T\mathbf{z}' + \mathrm{i} R^T \mathbf{z}) \cdot \overline{\mathbf{z}} \Big|_0^{\infty} = -(T\mathbf{z}'(0) + \mathrm{i} R^T \mathbf{z}(0)) \cdot \overline{\mathbf{z}}(0). \tag{2.26}$$

It is seen that the right hand side of (2.26) is simply the dot product of the reduced traction vector $-(T\mathbf{z}'(0) + \mathrm{i} R^T \mathbf{z}(0))$ with $\overline{\mathbf{z}}(0)$ (note that the actual traction vector and reduced traction vector are minus the left hand sides of (2.2) and (2.9), respectively). This reduced traction vector can be evaluated with the use of (2.16) which is a general solution for $\mathbf{z}$ with the correct decay behaviour. First, we have

$$\mathbf{z}(0) = \sum_{k=1}^{3} q_k \mathbf{a}^{(k)} = A\mathbf{q}, \tag{2.27}$$

and

$$T\mathbf{z}'(0) = \sum_{k=1}^{3} \mathrm{i} q_k p_k T \mathbf{a}^{(k)} = \sum_{k=1}^{3} \mathrm{i} q_k (\mathbf{b}^{(k)} - R^T \mathbf{a}^{(k)}) = \mathrm{i}(B - R^T A)\mathbf{q}, \tag{2.28}$$

where we have made use of (2.18) and the matrix A is defined by

$$A = (\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \mathbf{a}^{(2)}). \tag{2.29}$$

It then follows that

$$-(T\mathbf{z}'(0) + \mathrm{i}R^T\mathbf{z}(0)) = -\mathrm{i}(B - R^T A)A^{-1}\mathbf{z}(0) - \mathrm{i}R^T\mathbf{z}(0) = -\mathrm{i}BA^{-1}\mathbf{z}(0). \tag{2.30}$$

This motivates the introduction of M, defined by

$$M = -\mathrm{i}BA^{-1}, \tag{2.31}$$

as the *surface impedance* matrix. It can be verified by straightforward substitution that M has the representation

$$M = H^{-1} - \mathrm{i}S^T H^{-1}, \tag{2.32}$$

where the two matrices H and S, defined by

$$H = 2\mathrm{i}AA^T, \quad S = \mathrm{i}(2AB^T - I), \tag{2.33}$$

play an important role in the Stroh formalism (here we follow the notation of Ting 1996; our matrices M, H, S respectively correspond to $Z, -Q, S$ in Barnett and Lothe 1985 and to Z, S_2, S_1 in Chadwick and Smith 1977). It is known that when $\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \mathbf{a}^{(3)}$ are appropriately normalized, then for $0 \leq v \leq \hat{v}$ the matrix H^{-1} is real and symmetric, whereas $S^T H^{-1}$ is real and skew-symmetric. It is also known that H^{-1} is positive definite for $0 \leq v < \hat{v}$, and at $v = \hat{v}$ at least one of its eigenvalues must vanish. Thus, for $0 \leq v \leq \hat{v}$, M is Hermitian,

$$\mathbf{w} \cdot M\mathbf{w} = \mathbf{w} \cdot H^{-1}\mathbf{w} \geq 0 \quad \text{for all real vectors } \mathbf{w}, \tag{2.34}$$

and

$$\operatorname{tr} M = \operatorname{tr} H^{-1} \geq 0. \tag{2.35}$$

See Barnett and Lothe (1985, p. 139). These properties are crucial in establishing existence and uniqueness of surface waves.

The above analysis is valid for any $\mathbf{z}(0)$. If $\mathbf{z}(0)$ and M are such that

$$M\mathbf{z}(0) = -(T\mathbf{z}'(0) + \mathrm{i}R^T\mathbf{z}(0)) = \mathbf{0},$$

then the traction-free boundary condition (2.9) is satisfied. Conversely, had we additionally imposed the boundary condition (2.9), we would have arrived at $M\mathbf{z}(0) = \mathbf{0}$. Thus, a (free) surface wave solution exists only if (1.1) is satisfied. Equation (1.1) provides a secular equation alternative to (2.19). In view of the relation (2.31) and the fact that $\det A \neq 0$, these two alternative secular equations are clearly equivalent.

Return now to the forcing problem. In terms of M, equation (2.26) becomes

$$\frac{1}{2m^2}\mathcal{L} = \overline{\mathbf{z}}(0) \cdot M\mathbf{z}(0). \tag{2.36}$$

From the fact that the right hand side of (2.36) should be real for arbitrary $\mathbf{z}(0)$ (since the left hand is), we may again deduce that M must necessarily be Hermitian. When $v = 0$, the forcing is static and the work done by the surface traction is converted entirely into strain energy which, under the assumption (2.5), is positive. Equation (2.36) then implies that the surface impedance matrix M is positive definite when $v = 0$ (since $\mathbf{z}(0)$ can be arbitrarily chosen). Thus, all the three real eigenvalues of M are positive when $v = 0$.

We observe that although the matrices A and B in the definition (2.31) are not uniquely defined, by (2.36) the matrix M must necessarily be uniquely defined.

Following Barnett and Lothe (1985, p. 145), we equate the two expressions for $d\mathcal{L}/dv$, one from (2.25) and the other from (2.36), to obtain

$$\overline{\mathbf{z}}(0) \cdot \frac{dM}{dv}\mathbf{z}(0) = -2\rho v \int_0^\infty \mathbf{z} \cdot \overline{\mathbf{z}} dy, \tag{2.37}$$

where in obtaining the expression on the right hand side use has been made of the fact that the first variation of the Lagrangian $\mathcal{L}$ vanishes when $\mathbf{z}$ satisfies (2.8) and (2.9).

Equation (2.37) shows that dM/dv is negative definite, and hence that in $0 \le v < \hat{v}$ the eigenvalues of M are monotone decreasing functions of v. Thus, a (subsonic) surface wave exists only if an eigenvalue of M, originally positive at $v = 0$, decreases to zero at $v = v_R < \hat{v}$. It may further be concluded that whenever such a v_R exists, it is unique, for if it is not unique, then two of the eigenvalues of M must be negative at $v = \hat{v}$ and any real vector $\mathbf{w}$ lying in the eigenspace of these two negative eigenvalues will violate (2.34) (see Barnett and Lothe 1985, p. 145). This argument also implies that 0 cannot be a repeated eigenvalue of M at $v = v_R$. Finally, from the facts that $\det M$ equals the product of the three eigenvalues and that each eigenvalue is a monotone decreasing function of v, we deduce that $\det M$ is a monotone decreasing function of v at least for $0 \le v \le v_R$. This last property implies that the secular equation (1.1) is very amenable to numerical calculations.

We denote by $\lambda_1, \lambda_2, \lambda_3$ the three eigenvalues of M at $v = \hat{v}$, and assume that they are ordered such that $\lambda_1 \ge \lambda_2 \ge \lambda_3$. The above discussion shows that a necessary and sufficient condition for the existence of a unique subsonic surface wave is that $\lambda_3 < 0$. To derive a more convenient form of this condition (in terms of M), we observe that if $\lambda_3 < 0$ then the other two eigenvalues must be such that either $\lambda_1 > 0$, $\lambda_2 > 0$ or $\lambda_1 > 0$, $\lambda_2 = 0$ (we note that (2.35) precludes the possibility $\lambda_1 = \lambda_2 = 0$). Since $\lambda_3 < 0, \lambda_1 > 0, \lambda_2 > 0$ can be characterized by $\det M < 0$ and $\lambda_3 < 0, \lambda_1 > 0, \lambda_2 = 0$ by $2(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3) = (\operatorname{tr} M)^2 - \operatorname{tr} M^2 < 0$, we have the following important result due to Barnett and Lothe (1985).

Theorem 1. *A necessary and sufficient condition for the existence of a unique subsonic surface wave is that at $v = \hat{v}$ either $\det M < 0$ or $(\operatorname{tr} M)^2 - \operatorname{tr} M^2 < 0$.*

3 Derivation of (1.2) and uniqueness of its solution

In this section we derive (1.2) using two different approaches. First, from (2.11) we obtain

$$\left(p_k^2 T + p_k(R + R^T) + Q - \rho v^2 I\right) \mathbf{a}^{(k)} = \mathbf{0}, \quad k = 1, 2, 3, \tag{3.1}$$

which can be rewritten as

$$p_k(p_k T + R^T)\mathbf{a}^{(k)} + (p_k R + Q - \rho v^2 I)\mathbf{a}^{(k)} = \mathbf{0}, \quad k = 1, 2, 3. \tag{3.2}$$

With the use of (2.18), (3.2) becomes

$$p_k \mathbf{b}^{(k)} + p_k R\mathbf{a}^{(k)} + (Q - \rho v^2 I)\mathbf{a}^{(k)} = \mathbf{0}, \quad k = 1, 2, 3. \tag{3.3}$$

It then follows that

$$BD + RAD + (Q - \rho v^2 I)A = 0, \tag{3.4}$$

where

$$D = \text{diag}\,(p_1, p_2, p_3). \tag{3.5}$$

From (2.18) we obtain

$$B = TAD + R^T A, \tag{3.6}$$

and hence

$$D = A^{-1}T^{-1}B - A^{-1}T^{-1}R^T A. \tag{3.7}$$

On substituting (3.7) into (3.4), multiplying the resulting equation from the right by A^{-1} and making use of the definition (2.31), we obtain the algebraic Ricatti equation (1.2):

$$(M - \mathrm{i}R)T^{-1}(M + \mathrm{i}R^T) - Q + \rho v^2 I = 0.$$

Alternatively, this equation can be derived by viewing (2.8) as an initial value problem and looking for a solution of the form

$$\mathbf{z} = \mathrm{e}^{-yE}\mathbf{z}(0), \tag{3.8}$$

where E is a 3×3 matrix to be determined. This representation is justified since (2.16) can in fact be manipulated into this form. On substituting (3.8) into (2.8) and (2.9), we obtain

$$TE^2 - \mathrm{i}(R + R^T)E - Q + \rho v^2 I = 0, \quad (-TE + \mathrm{i}R^T)\,\mathbf{z}(0) = 0. \tag{3.9}$$

Equation $(3.9)_2$ motivates the introduction of M through $-TE + \mathrm{i}R^T = -M$, or equivalently,

$$E = T^{-1}(M + \mathrm{i}R^T). \tag{3.10}$$

This definition of M is clearly consistent with (2.30) and (2.31). Substituting (3.10) into $(3.9)_1$, we again obtain (1.2). Mathematically, we may interpret the transformation from E to M as being necessary in converting $(3.9)_1$ (for E) into a Hermitian equation (for M).

The solution (3.8) is a decaying solution only if all the eigenvalues of E have positive real parts. We shall prove shortly that this will be the case if and only if M is as constructed in Section 2. This implies that when we solve (1.2) to find M and use (1.1) to find the surface-wave speed, the solution of (1.2) that we are seeking must have the following properties: (i) it must be Hermitian; (ii) it must be positive definite for $0 \leq v < v_R$; and (iii) at $v = v_R$ it must be positive semi-definite having 0 as an unrepeated eigenvalue. In the rest of this section, we show that (1.2) has a unique solution having these properties. More specifically, we shall show that (i) for $0 \leq v < v_R$ equation (1.2) has a unique solution that is positive definite; (ii) at $v = v_R$ if the eigenvalues of (2.11) are all distinct, then (1.2) has a unique solution that is positive semi-definite; (iii) at $v = v_R$ if (2.11) have repeated eigenvalues, then (1.2) may have more than one solution that is positive semi-definite, but all the spurious solutions of (1.2) must necessarily have 0 as a repeated eigenvalue.

We first establish the following preliminary results.

Proposition 1. *Let M be an arbitrary solution of the Riccati equation (1.2) and E be the corresponding matrix calculated from (3.10) so that E satisfies (3.9)$_1$. If λ is an eigenvalue of E and $\mathbf{d}$ an associated eigenvector, then $p = \mathrm{i}\lambda, \mathbf{a} = \mathbf{d}$ and $p = -\mathrm{i}\overline{\lambda}, \mathbf{a} = \overline{\mathbf{d}}$ are both solutions of the eigenvalue problem (2.11).*

Proof. We first note that in terms of M the eigenvalue problem (2.11) can be factorized as

$$\left\{(M - \mathrm{i}R)T^{-1} - \mathrm{i}pI\right\} T \left\{T^{-1}(M + \mathrm{i}R^T) + \mathrm{i}pI\right\} \mathbf{a} = \mathbf{0}. \tag{3.11}$$

This can be verified by simply expanding the left hand side of (3.11) and making use of the fact that M is a solution of (1.2). In terms of E, equation (3.11) may be written as

$$(\overline{E}^T - \mathrm{i}pI)T(E + \mathrm{i}pI)\,\mathbf{a} = \overline{(E + \mathrm{i}\overline{p}I)}^T T(E + \mathrm{i}pI)\,\mathbf{a} = \mathbf{0}. \tag{3.12}$$

It then follows that if $E\mathbf{d} = \lambda\mathbf{d}$, then $p = \mathrm{i}\lambda$, $\mathbf{a} = \mathbf{d}$ satisfy the above equation and hence the eigenvalue problem (2.11). Since the eigenvalues and eigenvectors of (2.11) always appear as complex conjugate pairs, $p = -\mathrm{i}\overline{\lambda}, \mathbf{a} = \overline{\mathbf{d}}$ is also a solution. QED

Proposition 2. *Consider the matrix problem*

$$TE^2 - \mathrm{i}(R + R^T)E - Q + \rho v^2 I = 0, \qquad \text{Re spec } E > 0, \tag{3.13}$$

where "Re spec E" means the "real parts of the spectra of E".

(i) The problem (3.13) has a unique solution, given by

$$E = -\mathrm{i}ADA^{-1}, \tag{3.14}$$

where A and D are as defined in Section 2;

(ii) If p is an eigenvalue of (2.11) with associated eigenvector $\mathbf{a}$ and $\mathrm{Im}\,(p) > 0$, then $\lambda = -\mathrm{i}p, \mathbf{d} = \mathbf{a}$ is an eigensolution of $E\mathbf{d} = \lambda\mathbf{d}$.

Proof. (i) If (3.13) has two different solutions, then by Proposition 1 the eigenvalue problem (2.11) must have more than three eigenvalues with positive imaginary parts (counting multiplicity). This contradicts the fact that (2.11) can only have three eigenvalues with positive imaginary parts.

On substituting (2.31) and (3.6) into (3.10), we obtain (3.14). From (3.5) the eigenvalues of this solution of E are $-\mathrm{i}p_1, -\mathrm{i}p_2, -\mathrm{i}p_3$ and they all have positive real parts. Thus, (3.14) is a solution of (3.13) and is the unique solution.
(ii) Let the E in Proposition 1 be the solution of (3.13). Since $\mathrm{Re}\,(-\mathrm{i}\overline{p}) < 0$, we have $\det(E + \mathrm{i}\overline{p}I) \neq 0$. It then follows from (3.12) that $(E + \mathrm{i}pI)\mathbf{a} = \mathbf{0}$, i.e. $E\mathbf{a} = -\mathrm{i}p\mathbf{a}$. QED

Item (i) in the above Proposition and its proof imply that the eigenvalues of E in (3.8) all have positive real parts if and only if M is as constructed in Section 2.

In the rest of this section, we use M exclusively to denote the solution given by (2.31) and E to denote the corresponding E calculated according to (3.10) or equivalently (3.14).

Proposition 3. *Let M_0 be any other solution of (1.2). Then $\overline{\boldsymbol{\eta}} \cdot M_0\boldsymbol{\eta} \leq \overline{\boldsymbol{\eta}} \cdot M\boldsymbol{\eta}$ for all complex vectors $\boldsymbol{\eta}$.*

Proof. This proposition is a special case of a more general result known in Control Theory, see, e.g., Knobloch *et al.* (1993, Lemma A2.2). To prove this inequality, we consider the function $V(y)$ defined by

$$V(y) = (M - M_0)\boldsymbol{\eta}(y) \cdot \overline{\boldsymbol{\eta}}(y), \quad \text{where } \boldsymbol{\eta}(y) = \mathrm{e}^{-yE}\boldsymbol{\eta}_0, \quad \boldsymbol{\eta}_0 \text{ arbitrary.}$$

It follows that

$$\frac{\mathrm{d}}{\mathrm{d}y}V(y) = [M_0E + \overline{E}^T M_0 - ME - \overline{E}^T M]\boldsymbol{\eta} \cdot \overline{\boldsymbol{\eta}}.$$

It is easy to verify with the use of (3.10) that

$$M_0E + \overline{E}^T M_0 - ME - \overline{E}^T M = -(M - M_0)T^{-1}(M - M_0),$$

where we have also made use of the fact that both M and M_0 are solutions of (1.2). Since the matrix on the right hand side is negative semi-definite because T^{-1} is positive definite, we have

$$\frac{\mathrm{d}}{\mathrm{d}y}V(y) = -(M - M_0)T^{-1}(M - M_0)\boldsymbol{\eta} \cdot \overline{\boldsymbol{\eta}} \leq 0.$$

Since $\boldsymbol{\eta}(y) \to 0$ as $y \to \infty$, we obtain $(M - M_0)\boldsymbol{\eta}_0 \cdot \overline{\boldsymbol{\eta}}_0 = V(0) \geq V(\infty) = 0$. The result in the Proposition then follows from the fact that $\boldsymbol{\eta}_0$ can be chosen arbitrarily. QED

We are now in a position to prove the following theorem.

Theorem 2. *For $0 \leq v \leq v_R$ if M_0 is another positive semi-definite solution of (1.2) that is not given by (2.31), then 0 must be a repeated eigenvalue of M_0.*

Proof. By Propositions 2(i), the matrix E_0, given by $E_0 = T^{-1}(M_0 + \mathrm{i}R^T)$, must have at least one eigenvalue with a negative real part. We denote this eigenvalue by q and its associated eigenvector by $\mathbf{d}$. By Proposition 1, $p = -\mathrm{i}\overline{q}$ is an eigenvalue of (2.11) with associated eigenvector $\overline{\mathbf{d}}$ and $\mathrm{Re}\,(p) > 0$. It then follows from Proposition 2(ii) that $-\overline{q}$ is an eigenvalue of E with associated eigenvector $\overline{\mathbf{d}}$. Thus,

$$T^{-1}(M + \mathrm{i}R^T)\overline{\mathbf{d}} = -\overline{q}\overline{\mathbf{d}}, \quad T^{-1}(M_0 + \mathrm{i}R^T)\mathbf{d} = q\mathbf{d}. \tag{3.15}$$

Taking the complex conjugate of the second equation and adding it to the first one, we find $(M+\overline{M}_0)\overline{\mathbf{d}}=\mathbf{0}$. Since M and M_0 are both positive semidefinite (then so is $\overline{M}_0$), we conclude $M\overline{\mathbf{d}}=\overline{M}_0\overline{\mathbf{d}}=\mathbf{0}$. Furthermore, from Proposition 3, $0\le M_0\overline{\mathbf{d}}\cdot\mathbf{d}\le M\overline{\mathbf{d}}\cdot\mathbf{d}=0$ which implies $M_0\overline{\mathbf{d}}=\mathbf{0}$. Thus, $\mathbf{d}$ and $\overline{\mathbf{d}}$ are both eigenvectors of M_0 corresponding to the eigenvalue 0. The theorem is proved if it can be shown that $\mathbf{d}$ and $\overline{\mathbf{d}}$ are linearly independent. Suppose for contradition that $\mathbf{d}$ is a multiple of $\overline{\mathbf{d}}$. Then $\mathbf{d}$ must necessarily be a multiple of a real vector, and so without loss of generality we may take $\mathbf{d}$ to be real. Equation $(3.15)_2$ would imply $\mathrm{Re}(q)\mathbf{d}\cdot T\mathbf{d}=0$, a contradiction since $\mathrm{Re}(q)<0$ and $\mathbf{d}\cdot T\mathbf{d}>0$. QED.

The following corollary follows immediately from Theorem 2:

Corollary 1. *For $0\le v<v_R$, the Riccati equation (1.2) has a unique positive definite solution for M, given by (2.31).*

We remark, however, that for any v, including $v=v_R$, equation (1.2) may have more than one solution that is positive semi-definite. For instance, when $v=0$, $T=I$, and R,Q are given by

$$R=\begin{pmatrix}0 & 2 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},\quad Q=\begin{pmatrix}4 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 9\end{pmatrix}, \tag{3.16}$$

equation (1.2) has two positive semi-definite solutions, given by

$$\begin{pmatrix}4\sqrt{2}/3 & 4\mathrm{i}/3 & 0\\ -4\mathrm{i}/3 & 2\sqrt{2}/3 & 0\\ 0 & 0 & 3\end{pmatrix}\quad\text{and}\quad\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 3\end{pmatrix}, \tag{3.17}$$

respectively. The following theorem shows that in general at $v=v_R$ equation (1.2) has a unique positive semi-definite solution.

Theorem 3. *At $v=v_R$, if the eigenvalues of (2.11) are all distinct, then the Riccati equation (1.2) has no other positive semi-definite solutions than (2.31).*

Proof. Suppose M_0 is another solution of (1.2) that is positive semi-definite. We define matrix E_0 through $E_0=T^{-1}(M_0+\mathrm{i}R^T)$, and denote its eigenvalues and associated eigenvectors by

$$q_1,\ q_2,\ q_3,\quad\text{and}\quad \mathbf{d}_1,\ \mathbf{d}_2,\ \mathbf{d}_3,$$

respectively. By Proposition 1, the six distinct eigenvalues of (2.11) are

$$\mathrm{i}q_1,\ \mathrm{i}q_2,\ \mathrm{i}q_3,\ -\mathrm{i}\overline{q}_1,\ -\mathrm{i}\overline{q}_2,\ -\mathrm{i}\overline{q}_3. \tag{3.18}$$

By Propositions 2(i), E_0 must have at least one eigenvalue with a negative real part. Suppose without loss of generality that this eigenvalue is q_3. Following the same argument as that used in the proof of Theorem 2, we may deduce that $M_0\mathbf{d}_3=M_0\overline{\mathbf{d}}_3=\mathbf{0}$. It then follows that

$$E_0\overline{\mathbf{d}}_3=T^{-1}(M_0+\mathrm{i}R^T)\overline{\mathbf{d}}_3=T^{-1}(\mathrm{i}R^T)\overline{\mathbf{d}}_3=T^{-1}(-\overline{M}_0+\mathrm{i}R^T)\overline{\mathbf{d}}_3=-\overline{E}_0\overline{\mathbf{d}}_3=-\overline{q}_3\overline{\mathbf{d}}_3,$$

which shows that $-\overline{q}_3$, which has a positive real part, is an eigenvalue of E_0 with associated eigenvector $\overline{\mathbf{d}}_3$. Thus, we must have either $-\overline{q}_3=q_1$ or $-\overline{q}_3=q_2$, both implying that two of the eigenvalues in (3.18) are equal, which contradicts our assumption that the six eigenvalues of (2.11) are all distinct. QED.

4 Determination of surface-wave speeds

In the most general case when u_3 is coupled with u_1 and u_2, the surface impedance matrix takes the form

$$M = \begin{pmatrix} M_1 & M_3 + \mathrm{i}M_4 & M_5 + \mathrm{i}M_6 \\ M_3 - \mathrm{i}M_4 & M_2 & M_7 + \mathrm{i}M_8 \\ M_5 - \mathrm{i}M_6 & M_7 - \mathrm{i}M_8 & M_9 \end{pmatrix}, \quad M_i \in \mathrm{I\!R}. \tag{4.1}$$

On substituting (4.1) into (1.2) and equating both the real and imaginary parts of each component on the left to zero, we obtain nine real algebraic equations. These equations can easily be obtained with the aid of a symbolic manipulation package. For instance, with the use of Mathematica (Wolfram 1991), the real and imaginary parts of the (1, 2)-component of the left hand side of (1.2) can be extracted using the commands

$$\textbf{ComplexExpand}[\textbf{Re}[W[[1,2]]]], \quad \textbf{ComplexExpand}[\textbf{Im}[W[[1,2]]]],$$

where W denotes the left hand side of (1.2) and **ComplexExpand** expands a complex expression by assuming that all the non-numerical symbols are real. For each numerical value of $v < v_R$, these nine equations can also be solved with the aid of Mathematica to find $M_1, \ldots, M_9$ numerically. In using the command **FindRoot** to find the desired M that is positive definite, a reasonable initial guess is essential in preventing convergence to a spurious solution (i.e. a solution that is not positive definite). The choice of initial guess will be discussed in subsequent examples. To determine the surface-wave speed, we may simply increase v from zero in small steps and for each v we first determine M and then evaluate $\det M$. The value of v at which $\det M = 0$ yields the surface-wave speed v_R.

Alternatively, we may solve the above nine equations and (1.1) simultaneously for the ten unknowns ρv^2 and M_i ($i = 1, 2, \ldots, 9$), bearing in mind the selection criterion that the correction solution of M must be positive semi-definite and must have 0 as an unrepeated eigenvalue. This alternative procedure is more efficient. Only when this fails to find a solution, does it become necessary to use the above searching method (usually just to verify that there is indeed no solution). We now illustrate this procedure through some examples.

We first consider the simple case when the plane $x_3 = 0$ is a symmetry plane of the anisotropic half-space. In this case the variation of u_3 is de-coupled from that of u_1 and u_2. We may set $u_3 = 0$ and restrict the range of all subscripts to 1 and 2. Equation (4.1) may now be replaced by

$$M = \begin{pmatrix} M_1 & M_3 + \mathrm{i}M_4 \\ M_3 - \mathrm{i}M_4 & M_2 \end{pmatrix}, \quad M_i \in \mathrm{I\!R}, \tag{4.2}$$

and the three matrices T, R and Q can all be assumed to be 2×2. When these three matrices are of the form given by (2.20), equation (1.2) can be solved analytically and we obtain

$$M_1 = \sqrt{T_1(Q_1 - \rho v^2) - \frac{T_1}{T_2}\left(\frac{R_1 + R_2}{1+\gamma}\right)^2},$$

$$M_2 = \gamma \frac{T_2}{T_1} M_1, \quad M_3 = 0, \quad M_4 = \frac{\gamma R_1 - R_2}{1+\gamma}, \tag{4.3}$$

where

$$\gamma = \sqrt{\frac{T_1(Q_2 - \rho v^2)}{T_2(Q_1 - \rho v^2)}}, \tag{4.4}$$

and in obtaining (4.3) we have made use of the fact that M is positive semi-definite so that $M_1 \geq 0, M_2 \geq 0$. On substituting (4.3) into $\det M = M_1 M_2 - M_3^2 - M_4^2 = 0$, we obtain the explicit secular equation

$$\sqrt{T_1 T_2 (Q_1 - \rho v^2)(Q_2 - \rho v^2)} - \frac{\gamma R_1^2 + R_2^2}{1+\gamma} = 0. \tag{4.5}$$

By the general surface wave theory summarized in Section 2, the left hand side of (4.5) is a monotone decreasing function of v in a sufficiently large interval starting from $v = 0$, and a single root v_R can easily be located if it exists.

We next consider the case when the three matrices T, R and Q are not of the simple form given by (2.20), but instead are of the general form given by

$$T = \begin{pmatrix} c_{66} & c_{26} \\ c_{26} & c_{22} \end{pmatrix}, \quad R = \begin{pmatrix} c_{16} & c_{12} \\ c_{66} & c_{26} \end{pmatrix}, \quad Q = \begin{pmatrix} c_{11} & c_{16} \\ c_{16} & c_{66} \end{pmatrix}, \tag{4.6}$$

where we have used the standard notation $C_{ijkl} = c_{s(i,j)s(k,l)}$ with

$$s(m,n) = \begin{cases} m & \text{if } m = n \\ 9 - m - n & \text{if } m \neq n. \end{cases}$$

In this case it does not seem possible to solve (1.2) analytically. But the four algebraic equations for $\rho v^2, M_1, M_2, M_3$ are easily solved with the use of the command **FindRoot** in Matheamtica. As an initial guess, we may use the exact solution when c_{26} and c_{16} are set to zero. This exact solution is given by (4.3)–(4.5) with

$$T_1 = c_{66}, \;\; T_2 = c_{22}, \;\; R_1 = c_{12}, \;\; R_2 = c_{66}, \;\; Q_1 = c_{11}, \;\; Q_2 = c_{66}.$$

We have performed this calculation for all the 12 monoclinic crystals considered in Destrade (2001) and our calculations confirm the results in his Table 1. We may also consider the general case when the [100] crystallographic axis of each monoclinic crystal makes an arbitrary angle θ with the x_1-axis while the [001] axis remains coincident with the x_3-axis (thus the plane $x_3 = 0$ is always a symmetry plane and Destrade's (2001) calculations correspond to $\theta = 0$). We increase θ gradually from 0 and at each step we may use either the result from the previous step or the exact solution (4.3)–(4.5) as an initial guess. We have carried out such calculations and have been able to reproduce the curves for the surface-wave speed in Chadwick and Wilson's (1992) Figures 5–7.

For the more general case when $x_3 = 0$ is not a symmetry plane, u_3 becomes coupled with u_1 and u_2 and we have to solve the full system of ten equations for $\rho v^2, M_1, \ldots, M_9$. In this case it is more important to have a good initial guess when using the command

FindRoot in Mathematica. A good strategy is to start the calculation from a configuration of the elastic half-space in which $x_3 = 0$ is a symmetry plane and then rotate the crystallographic axes in small steps to obtain the target configuration. For some crystals we may find a configuration in which all the three coordinate axes are symmetry axes. In this case the three matrices T, R, Q take the simple form

$$T = \begin{pmatrix} T_1 & 0 & 0 \\ 0 & T_2 & 0 \\ 0 & 0 & T_3 \end{pmatrix}, \quad R = \begin{pmatrix} 0 & R_1 & 0 \\ R_2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} Q_1 & 0 & 0 \\ 0 & Q_2 & 0 \\ 0 & 0 & Q_3 \end{pmatrix}, \tag{4.7}$$

and the exact solution of (1.2) is given by (4.3) and

$$M_5 = M_6 = M_7 = M_8 = 0, \quad M_9 = \sqrt{T_3 Q_3}. \tag{4.8}$$

Then $\det M$ is simply M_9 times the left hand side of (4.5) and so $\det M = 0$ reduces to (4.5). As an example, we consider surface waves propagating on the $(1\bar{1}0)$ plane of cubic crystal nickel. When the x_1-axis coincides with the [001] direction, the three coordinate axes are symmetry axes and equation (1.2) has the above exact solution. The surface-wave speed can be determined from the secular equation (4.5). Once this solution is obtained, we may rotate the crystal, in small steps, from when the x_1-axis coincides with the [001] direction to, for instance, when the x_1-axis coincides with the [110] direction. At each step, the result from the previous step is used as an initial guess. Using this procedure we have been able to reproduce the result in Farnell's (1970) Figure 11.

If one chooses the initial guess according to the methods explained in this section, it would be unlikely for the solution to converge to a solution that is not positive semi-definite (since the initial guess is positive semi-definite and is sufficiently close to the correct solution). One might think that convergence to a spurious positive semi-definite solution would still be possible. However, Theorem 3 assures us that a spurious positive semi-definite solution does not even exist in most cases. In fact, in all the calculations which we have conducted, including all those presented in this section, our initial guesses have always converged to the correct solutions.

5 Surface waves in prestressed elastic half-spaces

The extension of the Stroh formalism to surface waves in a prestressed elastic body has been established by Chadwick and Jarvis (1979). In this context, the coordinates (x_i) used in the previous sections describe the position of material particles in a finitely-deformed configuration. By a surface wave we mean a travelling wave along the x_1-axis that has the same properties as before except that the traction-free boundary condition is replaced by the condition that the surface traction corresponding to the finite deformation is a dead-load and hence it is the incremental surface traction that mush vanish.

Surface waves are now governed by the incremental equation of motion

$$\mathcal{A}_{jilk} u_{k,lj} = \rho \ddot{u}_i, \quad 0 < x_2 < \infty, \tag{5.1}$$

and the incremental boundary condition

$$\mathcal{A}_{2ilk} u_{k,l} = 0, \quad \text{on } x_2 = 0, \tag{5.2}$$

together with the usual decay condition (2.3), where (u_k) is the incremental displacement field and $\mathcal{A}_{jilk}$ are usually referred to as the instantaneous elastic moduli. The expressions of $\mathcal{A}_{jilk}$ relative to the principal axes of stretch are well-documented and can be found in many papers on incremental deformations (see, e.g., Fu and Ogden 2001, p. 249). We do not require these moduli to satisfy the strong convexity condition but instead we assume that they satisfy the strong ellipticity condition and the symmetry relations $\mathcal{A}_{jilk} = \mathcal{A}_{lkji}$. In general $\mathcal{A}_{jilk} \neq \mathcal{A}_{ijlk}$, but none of the results presented in Sections 2 and 3 depend on $C_{ijks} = C_{jiks}$. Thus, with the identification $C_{ijks} = \mathcal{A}_{jisk}$ almost all of the results from Sections 2 and 3 also apply to surface waves in a prestressed elastic half-space. The only exception is that as we do not require $\mathcal{A}_{jilk}$ to satisfy the rather restrictive strong convexity condition, we cannot claim immediately that the surface impedance matrix M is positive definite when $v = 0$.

For easy reference, we assume that the size of the finite deformation is characterized by a controlling parameter, ω say, with $\omega = 0$ corresponding to no deformation. The surface impedance matrix M now depends on ω as well as v and the secular equation takes the form

$$\det M(v, \omega) = 0. \tag{5.3}$$

We assume that when $\omega = 0$, the elastic half-space under condition satisfies the strong convexity condition so that in that case M is positive definite at $v = 0$. We now assume that ω is increased gradually from zero and we consider the corresponding variation of the three eigenvalues of M at $v = 0$. M will remain positive definite until ω reaches a critical value, ω_{cr} say, at which one of the eigenvalues of M vanish. From our analysis in Section 2, at this value of ω, the elastic half-space supports a standing wave solution and the half-space is said to be marginally stable. If $\omega > \omega_{cr}$, the prestressed half-space is said to be unstable with respect to surface-wave type perturbations. The critical value ω_{cr} thus separates the region of stability from that of instability. The relation $\omega = \omega_{cr}$ is referred to as the buckling condition which will be discussed further in the next section. We remark that we may also hit marginal stability when ω is decreased from zero and the region of stability may be bounded by two critical values/curves.

We now assume that the prestressed half-space is stable with respect to surface-wave type perturbations. The above argument shows that under this condition M must necessarily be positive definite at $v = 0$. Then all the results concerning existence and uniqueness of surface waves presented in Sections 2 and 3 can be applied, and the identity (1.2) can be used together with (1.1) to locate the unique surface-wave speed if it exists.

For surface waves propagating in a pre-stressed half-space in which the principal axes of stretch are aligned with the three coordinate axes, the explicit secular equation is given by (4.5) together with

$$T_1 = \mathcal{A}_{2121}, \;\; T_2 = \mathcal{A}_{2222}, \;\; R_1 = \mathcal{A}_{1122}, \;\; R_2 = \mathcal{A}_{2112}, \;\; Q_1 = \mathcal{A}_{1111}, \;\; Q_2 = \mathcal{A}_{1212}. \tag{5.4}$$

This secular equation agrees with that derived by Dowaikh and Ogden (1991).

6 Buckling condition

The discussion in the previous section indicates that for a prestressed elastic half-space, the buckling condition can be obtained by setting $v = 0$ in the secular equation (5.3), and so it is given by $\det M(0, \omega) = 0$. The determination of the critical value of ω follows the same procedure as that of the surface-wave speed described in previous sections, with ω playing the same role as v.

There are situations where an explicit expression for the buckling condition can be obtained. For instance, when the principal axes of stretch coincide with the three coordinates axes, the buckling condition can be obtained by setting $v = 0$ in (4.5) and making the substitutions (5.4). The result is

$$\mathcal{A}_{2112}^2\sqrt{\mathcal{A}_{2222}\mathcal{A}_{1111}} + \mathcal{A}_{1122}^2\sqrt{\mathcal{A}_{2121}\mathcal{A}_{1212}} - \mathcal{A}_{2121}\mathcal{A}_{1212}\sqrt{\mathcal{A}_{2222}\mathcal{A}_{1111}} - \mathcal{A}_{2222}\mathcal{A}_{1111}\sqrt{\mathcal{A}_{2121}\mathcal{A}_{1212}} = 0, \tag{6.1}$$

which is precisely the buckling condition obtained in Dowaikh and Ogden (1991) (see also Steigmann and Ogden 1997, Mielke and Sprenger 1998, Cai and Fu 2000).

We observe that the buckling condition is also the condition at which quasiconvexity at the boundary is marginally violated, see Mielke and Sprenger (1998). The following theorem follows from Mielke and Sprenger's (1998) Theorem 3.4.

Theorem 4. *The buckling condition for a pre-stressed elastic half-space is independent of the orientation of the surface as long as the surface normal remains in the* (x_1, x_2)*-plane.*

Proof. Suppose that we wish to derive the buckling condition for a pre-stressed elastic half-space Π that is defined by $|x_1| < \infty$, $|x_3| < \infty$, $0 < x_2 < \infty$ in terms of coordinates (x_i). Now consider another pre-stressed elastic half-space Π^* whose interior is identical with that of Π but the outward unit normal to whose surface is $(-\sin\theta, \cos\theta, 0)^T$. We assume that the surface traction on the surface of Π^* must be consistent with the state of pre-stress in its interior. The theorem says that Π and Π^* have the same buckling condition.

We introduce a new coordinate system with coordinates (x_i^*) such that the half-space Π^* is defined by $|x_1^*| < \infty$, $|x_3^*| < \infty$, $0 < x_2^* < \infty$. It is then clear that $x_i^* = \Omega_{ij}x_j$, where Ω_{ij} are the components of

$$\Omega = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We use a superscript "*" to signify any quantity that is referred to the new coordinate system. Thus, for instance, $\mathcal{A}_{jilk}^*$ denotes the instantaneous elastic moduli relative to the coordinate system (x_i^*), and we have

$$\mathcal{A}_{jisk}^* = \Omega_{jm}\Omega_{it}\Omega_{sp}\Omega_{kq}\mathcal{A}_{mtpq}.$$

We note that all the elastic moduli are functions of the prestress. It is straightforward to show that

$$T^* = \Omega T(\theta)\Omega^T, \quad R^* = \Omega R(\theta)\Omega^T, \quad Q^* = \Omega Q(\theta)\Omega^T, \quad \Omega = (\Omega_{ij}), \tag{6.2}$$

where, for instance, T^* is the matrix whose components are defined by $T^*_{ik} = \mathcal{A}^*_{2i2k}$, and

$$\begin{aligned} T(\theta) &= T\cos^2\theta - (R + R^T)\sin\theta\cos\theta + Q\sin^2\theta. \\ R(\theta) &= R\cos^2\theta + (T - Q)\sin\theta\cos\theta - R^T\sin^2\theta, \\ Q(\theta) &= Q\cos^2\theta + (R + R^T)\sin\theta\cos\theta + T\sin^2\theta, \end{aligned} \tag{6.3}$$

For the half-space Π^*, the eigenvalue problem (2.11) with $v = 0$ becomes

$$\left(p^{*2}T^* + p^*(R^* + R^{*T}) + Q^*\right)\mathbf{a}^* = \mathbf{0}. \tag{6.4}$$

On substituting (6.2) and (6.3) into (6.4), we obtain

$$\left\{Q + \frac{\sin\theta + p^*\cos\theta}{\cos\theta - p^*\sin\theta}(R + R^T) + \left(\frac{\sin\theta + p^*\cos\theta}{\cos\theta - p^*\sin\theta}\right)^2 T\right\}\Omega^T\mathbf{a}^* = 0, \tag{6.5}$$

from which we deduce that

$$p = \frac{\sin\theta + p^*\cos\theta}{\cos\theta - p^*\sin\theta}, \quad \mathbf{a} = \Omega^T\mathbf{a}^*. \tag{6.6}$$

From the formula $\mathbf{b}^* = p^*T^*\mathbf{a}^* + R^{*T}\mathbf{a}^*$ (cf. $(2.18)_1$), we obtain $\mathbf{b}^* = \Omega\mathbf{b}$. It then follows that

$$A^* = \Omega A, \quad B^* = \Omega B, \quad M^* = \Omega M \Omega^T. \tag{6.7}$$

Thus, $\det M^* = \det M$, and so the buckling conditions for Π and Π^* are the same. QED.

We remark that the above method of proof is borrowed entirely from the surface wave theory based on the Stroh formalism. The result in Theorem 4 should also have been obvious from that theory although previously it does not seem to have been stated explicitly.

Theorem 4 makes it possible to derive an explicit buckling condition for a pre-stressed elastic half-space even when the principal axes of stretch do not coincide with the coordinate axes relative to which the half-space is defined by $0 < x_2 < \infty$. According to the above theorem, if a rotation about the x_3-axis can bring the new axes to coincide with the principal axes of stretch, then the explicit buckling equation is given by (6.1) with $\mathcal{A}_{jisk}$ replaced by $\mathcal{A}^*_{jisk}$.

7 Conclusion

In this paper, we have derived a new identity (namely (1.2)) for the surface impedance matrix and proposed a method for determining the speed of surface waves in unstressed or prestressed elastic half-spaces. Although the properties that M is Hermitian, positive semi-definite and that $\det M$ is a monotone decreasing function of v are underpinned by the Stroh formalism, the use of this method is entirely free from the latter. Thus, it offers an attractive alternative to the procedures proposed in Barnet and Lothe (1985) and Chadwick and Wilson (1992). A simple but efficient Mathematica program is written and is used to reproduce results reported in some previous investigations.

All the results presented so far are for compressible elastic half-spaces. Chadwick (1997) has shown how to extend the Stroh formalism to incompressible materials by treating an incompressible material as the limit of a nearly incompressible material. An elastic material is said to be nearly incompressible if its strain energy function is of the form

$$\hat{W}(\mathbf{F}) = W(\mathbf{F}) + \frac{1}{2}c(J-1)^2, \tag{7.1}$$

where $\mathbf{F}$ is the deformation gradient, $J = \det \mathbf{F}$, and c is a positive constant with physical dimensions of stress. The strain-energy given by (7.1) reduces to the pseudo-strain-energy function appropriate to a general incompressible material under the limit

$$J \to 1, \quad c \to \infty, \quad c(J-1) \to -p, \tag{7.2}$$

where p is the pressure associated with the incompressibility constraint. Oldroyd (1950) seems to have been the first to use such a limiting process, followed by Spencer (1959, 1962, 1970), Scott (1986) and Rogerson and Scott (1992), and Chadwick (1997).

With the aid of (7.1) and the definitions

$$\hat{\mathcal{A}}_{jilk} = \bar{J}^{-1}\bar{F}_{jp}\bar{F}_{lq}\frac{\partial^2 \hat{W}}{\partial F_{ip}\partial F_{kq}}(\bar{\mathbf{F}}), \quad \mathcal{A}_{jilk} = \bar{J}^{-1}\bar{F}_{jp}\bar{F}_{lq}\frac{\partial^2 W}{\partial F_{ip}\partial F_{kq}}(\bar{\mathbf{F}}), \tag{7.3}$$

for the instantaneous elastic moduli for the nearly incompressible and the incompressible materials, respectively, we obtain

$$\hat{\mathcal{A}}_{jilk} = \mathcal{A}_{jilk} + c(2\bar{J}-1)\delta_{ij}\delta_{kl} - c(\bar{J}-1)\delta_{il}\delta_{jk}, \tag{7.4}$$

where $\bar{\mathbf{F}}$ is the deformation gradient associated with the finite deformation, $\bar{J} = \det \bar{\mathbf{F}}$ and δ_{ij} is the usual Kroneker delta. For an nearly incompressible elastic half-space with elastic moduli given by (7.4), the surface-wave speed or the buckling condition can be obtained following the procedure explained in the previous sections. By taking the limit

$$\bar{J} \to 1, \quad c \to \infty, \quad c(\bar{J}-1) \to -\bar{p}, \tag{7.5}$$

where $\bar{p}$ is the pressure associated with the finite deformation, we may obtain the corresponding results for a prestressed incompressible elastic half-space.

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