# On the unfolding of reversible vector fields with SO(2)-symmetry and a non-semisimple eigenvalue 0 *

Andrei Afendikov† and Alexander Mielke‡

October 29, 1998§

## 1 Introduction

We consider four-dimensional ordinary differential equations depending on a vector-valued parameter $\lambda$ in a neighborhood of the origin. For $\lambda = 0$ the origin is supposed to be an equilibrium whose linearization has a fourfold non-semisimple eigenvalue 0. Moreover, we assume that the vector fields are SO(2)-invariant and reversible. Such systems occur typically from spatial dynamical systems in physics where the evolution variable is obtained from a one-dimensional axial direction. The reversibility is then associated to a reflection symmetry and the SO(2)-invariance might arise for instance from an additional spatial variable in which periodicity is assumed, see [2, 4, 9, 10] for such applications.

Our aim is to describe the generic unfoldings of such a singularity. It turns out that there are two cases. In Case 1 the SO(2) action and the reversor commute, and in Case 2 they do not commute. In both cases the lowest order terms which are derived via quasihomogeneous truncation lead to the steady one-dimensional Ginzburg-Landau equation

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}A + a(\lambda)A + b(\lambda)\frac{\mathrm{d}}{\mathrm{d}x}A + d|A|^2A = 0. \tag{1}$$

In Case 1 the reversibility acts as $(x, A) \mapsto (-x, A)$; and we have $b(\lambda) = 0$ and general coefficients $a(\lambda), d \in \mathbb{C}$. Then, we call (1) the complex Ginzburg-Landau equation (cGL). In Case 2 the reversibility is $(x, A) \mapsto (-x, \overline{A})$ which implies $a(\lambda), \mathrm{i}b(\lambda), d \in \mathbb{R}$. Then, (1) is called the real Ginzburg-Landau equation (rGL).

We demonstrate that (rGL) is completely integrable while (cGL) in general has complicated dynamics. For instance, there are cascades of $n$-homoclinic orbits in (cGL) while in (rGL) only simple homoclinic orbits occur, see Section 4.

*Research partially supported by RFBR 97-01-00541 and Volkswagen-Stiftung under grant I/71016

†Keldysh Institute of Applied Mathematics, Miusskaya sq. 4, Moscow, Russia

‡Institut für Angewandte Mathematik, Universität Hannover, Welfengarten 1, Hannover, Germany

§Published as "Afendikov, A.L. and Mielke, A.: On families of reversible SO(2)–invariant vector fields with a non–semi–simple zero eigenvalue of multiplicity four. *Dokl. Akad. Nauk.* **369** (1999) 153–157.

# 2 A linear unfolding

We identify the space $\mathbb{R}^4$ with $\mathbb{C}^2$ in order to simplify notations. Throughout the action of SO(2) is

$$T_\alpha(y_1, y_2) = (\mathrm{e}^{\mathrm{i}\alpha} y_1, \mathrm{e}^{\mathrm{i}\alpha} y_2), \quad \alpha \in \mathbb{S}^1 \overset{\text{def}}{=} \mathbb{R}/_{2\pi\mathbb{Z}}.$$

The differential equation has the form

$$\dot{y} = f(y, \lambda) \quad \text{with } f(y, \lambda) = L(\lambda)y + \mathcal{O}(|y|^2),$$

and SO(2)-invariance means $T_\alpha f(y, \lambda) = f(T_\alpha y, \lambda)$. The only possible non-semisimple linearization $L(0)$ commuting with the symmetry is $L(0) = L_0 \overset{\text{def}}{=} \left(\begin{smallmatrix} 0 & I \\ 0 & 0 \end{smallmatrix}\right) \in \mathbb{R}^{4\times 4}$. We additionally impose reversibility with the reversor $R$, that is $R^2 = I$ and $Rf(y, \lambda) = -f(Ry, \lambda)$.

**Lemma 2.1**
*Assume $RL_0 = -L_0 R$.*
**Case 1.** *If $R$ commutes with all $T_\alpha$, then we have $R = \pm R_{\mathrm{com}}$ with $R_{\mathrm{com}} \overset{\text{def}}{=} \left(\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right)$.*
**Case 2.** *If $R$ does not commute with some $T_\alpha$, then we have $R = T_\beta R_{\mathrm{nc}} T_{-\beta}$ for a suitable $\beta \in \mathbb{S}^1$, where $R_{\mathrm{nc}} \overset{\text{def}}{=} \left(\begin{smallmatrix} K & 0 \\ 0 & -K \end{smallmatrix}\right)$ with $K = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.*

This result follows easily by letting $R = \begin{pmatrix} R_1 & R_2 \\ R_3 & R_4 \end{pmatrix}$. Then $L_0 R = -RL_0$ gives $R_1 = -R_4$ and $R_2 = R_3 = 0$. Together with $R_1^2 = I$ the two cases arise if either both eigenvalues of $R_1$ are equal (both $+1$ or both $-1$) or if they are different (one $+1$ and one $-1$).

In Case 1 we may assume without loss of generality that $R = R_{\mathrm{com}}$, since by $T_\pi = -I$ we automatically see that $-R$ is also a reversor. In Case 2 we may assume $R = R_{\mathrm{nc}}$ after rotating the coordinate system by $T_\beta$. Before studying the nonlinear problem we study parameter-dependent matrices with the given symmetries.

**Lemma 2.2**
*Let $L(\lambda) \in \mathbb{R}^{4\times 4}$ depend smoothly on $\lambda$ such that $L(0) = L_0$, $RL(\lambda) = -L(\lambda)R$ and $T_\alpha L(\lambda) = L(\lambda) T_\alpha$ for all $\alpha$. Then there exists a smooth and SO(2)-invariant transformation $M(\lambda)$ with $RM(\lambda)R = M(\lambda)$ and $M(0) = I$ such that $\widetilde{L}(\lambda) = M(\lambda)L(\lambda)M(\lambda)^{-1}$ has the form*

$$\widetilde{L}(\lambda) = \begin{cases} \begin{pmatrix} 0 & I \\ \mu(\lambda)I + \nu(\lambda)J & 0 \end{pmatrix} & \textit{for } R = R_{\mathrm{com}} \\ \begin{pmatrix} 0 & I \\ \mu(\lambda)I & \nu(\lambda)J \end{pmatrix} & \textit{for } R = R_{\mathrm{nc}}, \end{cases}$$

*where $J = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $\mu(0) = \nu(0) = 0$.*

This result again follows by writing $L = \begin{pmatrix} L_1 & L_2 \\ L_3 & L_4 \end{pmatrix}$ and setting $M = \begin{pmatrix} I & 0 \\ L_1 & L_2 \end{pmatrix}$. This gives the upper two block matrices of $\widetilde{L}$. The special form of the lower blocks follows from reversibility and SO(2) invariance.

With this lemma we have obtained the general unfolding of the matrix $L_0$ under the given symmetry, see [5]. In both cases we find that the unfolding depends on two real parameters $\mu$ and $\nu$. If $\lambda \in \mathbb{R}^2$, then we say that the family $(L(\lambda))_{\lambda\in\mathbb{R}^2}$ is *linearly nondegenerate in* $\lambda = 0$ if the matrix $D_\lambda(\mu, \nu)|_{\lambda=0}$ is invertible.

# 3 Unfolding the nonlinear problem

We now extract the lowest order terms in a generic unfolding of the linear part $L_0$ under the given SO(2) symmetry and the reversibility with respect to $R_{\rm com}$ or $R_{\rm nc}$. We write

$$\begin{pmatrix} \dot y_1 \\ \dot y_2 \end{pmatrix} = \widetilde{L}(\lambda) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} n_1(y_1, \overline{y}_1, y_2, \overline{y}_2, \lambda) \\ n_2(y_1, \overline{y}_1, y_2, \overline{y}_2, \lambda) \end{pmatrix} \tag{2}$$

with $n_j(\ldots) = \mathcal{O}(|y|^2), j = 1, 2$. By SO(2) symmetry we have

$$n_j(\ldots) = y_1 p_j(|y_1|^2, |y_2|^2, y_1\overline{y}_2, \lambda) + y_2 q_j(|y_1|^2, |y_2|^2, y_1\overline{y}_2, \lambda), \quad j = 1, 2$$

where the functions $p_j$ and $q_j$ vanish for $y = 0$. In Case 1 the reversibility gives that $p_1$ and $q_2$ are odd in $y_1\overline{y}_2$ while $p_2$ and $q_1$ are even. In Case 2 the reversibility is $(y_1, y_2) \mapsto (\overline{y}_1, -\overline{y}_2)$ and we find

$$p_j(r_1, r_2, -\overline{z}, \lambda) = (-1)^{j-1}\overline{p_j(r_1, r_2, z, \lambda)}, \quad q_j(r_1, r_2, -\overline{z}, \lambda) = (-1)^j \overline{q_j(r_1, r_2, z, \lambda)}.$$

**Definition 3.1**
*Let $\lambda \in \mathbb{R}^2$ and $D_y f(0,0) = L_0$, then we say that the SO(2)-invariant and reversible family $\dot y = f(y, \lambda)$ is* **nondegenerate** *if the matrix family $L(\lambda)$ is linearly nondegenerate and if the coefficient $d = \partial_{r_1} p_2(0,0,0,0)$ is nonzero. We call such a family an* **unfolding** *of $\dot y = L_0 y$.*

In this situation we can use without loss of generality the parameter $\lambda = (\mu, \nu) \in \mathbb{R}^2$ with $\mu$ and $\nu$ as defined in Lemma 2.2. Observe that $d \in \mathbb{C}$ for Case 1 but $d \in \mathbb{R}$ for Case 2.

**!a!**The dynamics in unfoldings of eigenvalues 0 is analyzed most efficiently using the Newton-Brjuno polyhedron, see [6, 7]. To this end we study the six-dimensional system obtained by adding the equations $\dot\mu = 0$ and $\dot\nu = 0$. Because of the SO(2)-invariance the polyhedron lies in a five-dimensional subspace. It turns out that in both of our cases there is only one hyperface with positive normal vector that gives a nontrivial truncation of the problem (2). The corresponding truncation is quasihomogeneous (cf. [5]) and can be found by the relevant scaling. **?a?**

**Theorem 3.2**
*In Case 1 we have the scaling $(t, \mu, \nu, y_1, y_2) = (x/\eta, \eta^2\widehat\mu, \eta^2\widehat\nu, \eta\widehat y_1, \eta^2\widehat y_2)$ and the scaled equation*

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix} \widehat y_1 \\ \widehat y_2 \end{pmatrix} = \begin{pmatrix} \widehat y_2 \\ (\widehat\mu + \mathrm{i}\widehat\nu)\widehat y_1 + d|\widehat y_1|^2 \widehat y_1 \end{pmatrix} + \mathcal{O}(\eta^2) \tag{3}$$

*In Case 2 we have the scaling $(t, \mu, \nu, y_1, y_2) = (x/\eta, \eta^2\widehat\mu, \eta\widehat\nu, \eta\widehat y_1, \eta^2\widehat y_2)$ and the scaled equation*

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix} \widehat y_1 \\ \widehat y_2 \end{pmatrix} = \begin{pmatrix} \widehat y_2 \\ \widehat\mu\widehat y_1 + \mathrm{i}\widehat\nu\widehat y_2 + d|\widehat y_1|^2 \widehat y_1 \end{pmatrix} + \mathcal{O}(\eta^2) \tag{4}$$

*(In both cases $\mathcal{O}(\eta^2)$ is uniform in $\widehat y$ on any bounded set in $\mathbb{C}^2$.)*

**!a!**Neglecting $\mathcal{O}(\eta^2)$ and letting $A = \widehat y_1$ and $(\cdot)' = \mathrm{d}/\mathrm{d}x$ we obtain the truncations **?a?**

$$A'' - (\widehat\mu + \mathrm{i}\widehat\nu)A - d|A|^2 A = 0, \quad d \in \mathbb{C} \setminus \{0\}, \tag{cGL}$$

$$A'' - \widehat{\mu}A - \mathrm{i}\widehat{\nu}A' - d|A|^2 A = 0, \quad d \in \mathbb{R} \setminus \{0\}, \tag{rGL}$$

for Case 1 and 2, resp.

Since these equations are obtained via scaling and the limit $\eta \to 0$ it is important to note that now $\widehat{\mu}$, $\widehat{\nu}$ and $A$ are not nessessarily small. Whenever we find any bounded solution of (cGL) or (rGL) for a given parameter value $(\widehat{\mu}, \widehat{\nu})$, this solution can be understood, via the scaling, as a limit of small bounded solutions of (3) or (4), respectively, with $\lambda \approx 0$. Of course, existence of certain solution classes (like (quasi-) periodic, chaotic or homoclinic solutions) in the limit equations does not immediately imply the existence of such solutions in the original problem (2). To this end one has to show that the corresponding solution class is structurally stable under perturbations with the given symmetries.

The dynamics of (rGL) is much simpler than that of (cGL). In fact, (rGL) can be understood as a completely integrable system for $a(x) = \mathrm{e}^{-\mathrm{i}\widehat{\nu}x/2}A(x)$ with Hamiltonian $\mathcal{H} = \frac{1}{2}|a'|^2 - (\frac{\widehat{\mu}}{2} - \frac{\widehat{\nu}^2}{8})|a|^2 - \frac{d}{4}|a|^4$ and the additional first integral $\mathcal{K} = \mathrm{Im}\,(a'\overline{a})$. In particular, for (rGL) the origin is a center for $\widehat{\mu} \le \widehat{\nu}^2/4$ and a saddle-focus else. If $d < 0$ all solutions of (rGL) remain bounded, whereas for $d > 0$ nontrivial bounded solutions only exist for $\widehat{\mu} < \widehat{\nu}^2/4$, and then they lie in the region $|A| \le \sqrt{m/d}$, $|A'| \le (\sqrt{2m} + |\widehat{\nu}|)\sqrt{m/d}/2$, where $m = \widehat{\nu}^2/4 - \widehat{\mu} > 0$.

For (cGL) a similarly simple estimate for the region of bounded solutions cannot be given. A necessary condition for the existence of a bounded nontrivial solution is that $\widehat{\nu}\,\mathrm{Im}\,d \le 0$. Moreover, explicit bounds can be derived by taking the bounded solutions as steady states of the time-dependent complex Ginzburg-Landau equation

$$\partial_t A = (1+\mathrm{i}\rho)\Big[\partial_x^2 A - (\widehat{\mu}+\mathrm{i}\widehat{\nu})A - d|A|^2 A\Big]$$

with $\rho \in \mathbb{R}$. If $\mathrm{Im}\,d \ne 0$ then it is possible to choose $\rho$ such that $\mathrm{Re}[(1+\mathrm{i}\rho)d] > 0$. Hence the general bounds for time-dependent solutions in [11] give explicit bounds for bounded solutions of (cGL).

# 4 Homoclinic bifurcation

As one particular example, we show that in the unfolding (2) there are bifurcations of homoclinic orbits. In both cases we know that if one homoclinic solution exists then the SO(2)-orbit of this solution forms a two-dimensional manifold which in fact coincides with the global stable as well as the global unstable manifold of the origin. If $H : \mathbb{R} \to \mathbb{R}^4$ is one of these homoclinic orbits then together with the reversibility we conclude $H(t) = RT_\beta H(\tau - t)$ for suitable $\beta \in \mathbb{S}^1$ and $\tau \in \mathbb{R}$. In Case 1 we have $RT_\alpha = T_\alpha R$ implying $T_\beta = \pm I$. In Case 2 we have $RT_\alpha = T_{-\alpha}R$ such that $RT_\beta$ always has the eigenvalue $+1$.

**Lemma 4.1**
*In Case 1 each homoclinic orbit $H$ is either reversible ($s = 1$) or antireversible ($s = -1$):*

$$H(t) = sRH(\tau - t) \text{ with } s \in \{-1, 1\}.$$

*The value $s$ is called the parity of $H$, $s = \mathrm{par}(H)$.*
*If in Case 2 a homoclinic orbit exist, then there are exactly two reversible and two antireversible orbits.*

The parity in case 1 is only defined after the choice of $R$. Similarly we could have used $-R$ as the reversor, then all parities would be interchanged.

In both cases the associated limit equations admit explicit homoclinic solutions. For (rGL) with $d<0$ and $\widehat{\mu}>\widehat{\nu}^2/4$ we have the reversible homoclinic solution

$$A_{\mathrm{rGL}}(x)=\rho\widehat{\eta}\mathrm{e}^{\mathrm{i}\widehat{\nu}x/2}/\cosh(\widehat{\eta}x)$$

with $\rho=(-d)^{-1/2}$ and $\widehat{\eta}=(\widehat{\mu}-\widehat{\nu}^2/4)^{1/2}$.

The persistence of these homoclinic orbits follows easily by using $y_1=r\mathrm{e}^{\mathrm{i}\phi}$ such that (4) reduces, after factoring out the SO(2)-symmetry, to a three-dimensional system for $(r,r',\Phi)$ where $\Phi=\phi'$. The reversibility reduces to $R^{\mathrm{red}}_{\mathrm{nc}}:(r,r',\Phi)\mapsto(r,-r',\Phi)$ and a homoclinic orbits occurs as soon as the unstable manifold $M^{\mathrm{red}}_{\mathrm{s}}$, which is one-dimensional after reduction, intersects the plane $\mathrm{Fix}(R^{\mathrm{red}}_{\mathrm{nc}})$. As $A_{\mathrm{rGL}}(x)$ has a transversal intersection we arrive at the following result.

**Theorem 4.2**
*If the unfolding* (2) *in Case 2 satisfies $d<0$ then for all sufficiently small $\lambda=(\mu,\nu)$ with $\mu>\nu^2/4$ the stable and unstable manifold coincide and are obtained by* SO(2) *symmetry from one homoclinic orbit $H_\lambda$ which has the expansion*

$$H_\lambda(t)=\frac{\rho\widetilde{\eta}\mathrm{e}^{\mathrm{i}\nu t/(2\sqrt{\mu})}}{\cosh(\widetilde{\eta}t)}\begin{pmatrix}1\\ \mathrm{i}\nu/2-\widetilde{\eta}\tanh(\widetilde{\eta}t)\end{pmatrix}+\mathcal{O}(e^{-\widetilde{\eta}|t|})\begin{pmatrix}\mathcal{O}(\widetilde{\eta}^{3/2})\\ \mathcal{O}(\widetilde{\eta}^2)\end{pmatrix}$$

*where $\widetilde{\eta}=\sqrt{\mu-\nu^2/4}$.*

Case 1 yields a completely different picture, since the reduced reversibility is $R^{\mathrm{red}}_{\mathrm{com}}:(r,r',\Phi)\mapsto(r,-r',-\Phi)$. Thus, $\mathrm{Fix}(R^{\mathrm{red}}_{\mathrm{com}})$ is one-dimensional and we need one parameter to intersect $M^{\mathrm{red}}_{\mathrm{s}}$ and $\mathrm{Fix}(R^{\mathrm{red}}_{\mathrm{com}})$. The explicit solution of (cGL) is the so-called Hocking-Stewartson pulse ([8])

$$A_{\mathrm{cGL}}(x)=\rho\widetilde{\eta}[\cosh(\widetilde{\eta}x)]^{-(1+\mathrm{i}\omega)}$$

with $\rho$ and $\omega$ defined via $d=-\rho^{-2}(1+\mathrm{i}\omega)(2+\mathrm{i}\omega)$ and $\widehat{\mu}+\mathrm{i}\widehat{\nu}=\widetilde{\eta}^2(1+\mathrm{i}\omega)^2$. This family occurs on a ray in the $(\widehat{\mu},\widehat{\nu})$ plane, as $d\notin[0,\infty)$ fixes $\omega$ uniquely.

This homoclinic orbit is transversal with respect to parameter variations for all $\omega$ which do not lie in an exceptional set

$$\mathcal{E}=\{0,\pm\omega_1,\pm\omega_2,\dots\},$$

see [3]. The set $\mathcal{E}$ is the set of zeros of an integral of Melnikov type which depends analytically on $\omega$. Numerical calculations give $\omega_1\approx 8.032$ and $\omega_2\approx 9.51$.

**Theorem 4.3**
*If the unfolding in Case 2 satisfies $d=-\rho^{-2}(1+\mathrm{i}\omega)(2+\mathrm{i}\omega)$ with $\omega\notin\mathcal{E}$, then there exists a smooth curve $\mathcal{C}^{(1)}:[0,\varepsilon_0]\ni\varepsilon\mapsto c(\varepsilon)\in\mathbb{R}^2$ with $c(0)=0$ and $c'(0)=(1-\omega^2,2\omega)$ such that* (2) *with $\lambda=(\mu,\nu)=c(\varepsilon)$, $\varepsilon\in(0,\varepsilon_0]$, has a homoclinic solution $H^{(1)}_\varepsilon$ with parity $s=+1$ which satisfies the expansion*

$$H^{(1)}_\varepsilon(t)=\frac{\sqrt{\varepsilon}\,\rho}{[\cosh(\sqrt{\varepsilon}\,t)]^{1+\mathrm{i}\omega}}\begin{pmatrix}1\\ -(1+\mathrm{i}\omega)\sqrt{\varepsilon}\tanh(\sqrt{\varepsilon}\,t)\end{pmatrix}+\mathcal{O}(\mathrm{e}^{-\sqrt{\varepsilon}|t|})\begin{pmatrix}\mathcal{O}(\varepsilon^{3/2})\\ \mathcal{O}(\varepsilon^2)\end{pmatrix}.$$

These homoclinic orbits are also called single-pulse solutions. However, they are not the only homoclinic orbits. In every neighborhood of the curve $\varepsilon \mapsto c(\varepsilon)$ there are parameter values such that we have homoclinic orbits which are called multi-pulse solutions. A solution $H^{(n)}$ is called $n$-pulse solution with respect to a single-pulse solution $H^{(1)}$ if it can be written in the form

$$H^{(n)}(t) = \sum_{j=1}^{n} T_{\alpha_j} H^{(1)}(t - t_j) + \text{h.o.t.}$$

with suitable $\alpha_j \in \mathbb{S}^1$ and $t_j \in \mathbb{R}$, where $t_{j+1} - t_j \gg 1$. For a geometrical characterization of $n-$pulses in terms of the factorized Poincaré map we refer to [3].

**Theorem 4.4**
*Consider the case of Theorem 4.3. Then on any curve $\mathcal{C} \subset \mathbb{R}^2$ which intersects the curve $\mathcal{C}^{(1)}$ transversally in a point $c(\varepsilon)$ with $\varepsilon \in (0, \varepsilon_0)$ and for any $n \in \mathbb{N} \setminus \{1\}$ there exist infinitely many points $c_k^{(n)} \in \mathcal{C}$, $k \in \mathbb{N}$, such that $c_k^{(n)} \to c(\varepsilon)$ for $k \to \infty$ and that (2) has an $n$-pulse solution $H_k^{(n)}$ for the parameters $\lambda = (\mu, \nu) = c_k^{(n)}$.*

*If $n$ is odd then $\text{par}(H_k^{(n)}) = 1 = \text{par}(H^{(1)})$. If $n$ is even we may choose the points $c_k^{(n)}$ such that $\text{par}(H_k^{(n)}) = (-1)^k$.*

In fact, the result can be strengthened by considering the obtained multi-pulses as new single-pulses and studying multi-pulses with respect to these homoclinic orbits. Thus, we obtain cascades of homoclinic orbits and the parameter set where homoclinic orbits exist has a fractal structure, see [3].

# References


[1] Afendikov A.L., Mielke A.// *Dynamics, Bifurcation and Symmetry.* ed. P. Chossat, ASI Series C: **Vol. 437,** P. 1-10, Kluwer AP, Dortrecht, Netherlands, 1994.

[2] Afendikov A.L., Mielke A.// *Arch. Rat. Mech. Anal.* 1995. **129**. P. 101-127.

[3] Afendikov A.L., Mielke A. *Bifurcation of homoclinic orbits to a saddle-focus in reversible systems with* SO(2)*-symmetry.* Preprint DANSE 1/1998, Universität Hannover. p. 20. (*J. Diff. Eqns.* (1998) Submitted.)

[4] Afendikov A.L., Mielke A. *Multi-pulse solutions to the Navier-Stokes problem between parallel plates.* Preprint DANSE 58/1998, Universität Hannover. p. 19. (*ZAMP* (1998) Submitted.)

[5] V.I. Arnold: *Geometrical Methods in the Theory of Ordinary Differential Equations.* Springer-Verlag; New York, Heidelberg, Berlin; 1983.

[6] A.D. Brjuno: Power asymptotics of solutions to nonlinear systems. *Izv. Acad. Sci. USSR, Math. Series*, **29** (1965) 329-364.

[7] A.D. Brjuno: *Power Geometry in Algebraic and Differential Equations.* Moscow, Nauka, 1998 (in Russian).

[8] Hocking L.M., Stewartson K.// *Proc. Royal Soc. London* 1972. **A 326**. P. 289-313.

[9] Iooss G., Mielke A.// *J. Nonlinear Science* 1991. **v. 1.** P. 106-146.

[10] Iooss G., Mielke A.// ZAMP. 1992. **v. 43**. P. 125-138.

[11] Mielke A.// *Physica* **D** 1998. **v. 117**. P. 106-116.