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A vanishing viscosity approach to rate-independent modelling in metric spaces

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Abstract

We present a new variational formulation of rate-independent evolutionary problems in metric spaces, which arises in the limit of a vanishing viscosity approximation.

Keywords: Rate-independent evolutions, vanishing viscosity, analysis in metric spaces.

1 Introduction

Rate-independent problems arise in a variety of applicative contexts, among which elastoplasticity, damage, the quasistatic evolution of fractures, shape memory alloys, delamination and ferromagnetism, see [3] and the references therein. In several situations, the evolution of rate-independent systems is described by the doubly nonlinear equation

$$\partial\Psi(u'(t)) + \mathrm{D}_u\mathcal{E}(t, u(t)) \ni 0 \quad \text{in } B' \ \text{ for a.a. } t \in (0, T)\,, \tag{1}$$

where B is a (separable) Banach space, $\Psi : B \to \mathbb{R}^+$ a lower semicontinuous and convex *dissipation* functional (for simplicity, here we omit its dependence on the state variable u), fulfilling $\Psi(\lambda v) = \lambda\Psi(v)$ for all $\lambda \geq 0$ and $v \in B$; the symbol ∂ denotes its subdifferential, and, hereafter, we shall assume the *energy potential* $\mathcal{E} : [0,T] \times B \to (-\infty, +\infty]$ (possibly accounting for an external loading through the dependence on the variable t) to be smooth with respect to t. Indeed, (1) renders a balance between a potential restoring force $-\mathrm{D}_u\mathcal{E}(t,u)$ and a frictional force $f \in \partial\Psi(u')$. Since $\partial\Psi$ is homogeneous of degree 0, (1) is invariant for time rescalings, meaning that the process under consideration is insensitive to changes in the time scale. Such a feature is typical of mechanical systems driven by an external loading set on a time scale much slower than the system internal time scale, which is thus neglected. This corresponds to taking the vanishing viscosity limit of systems with a viscous, rate-dependent dissipation.

When B is a reflexive Banach space and $\mathcal{E}(t,\cdot)$ is uniformly convex and smooth, the Cauchy problem for (1) has a unique solution $u \in \mathrm{W}^{1,\infty}([0,T];B)$, see [3, Sec. 3]. On the other hand, the relevant energy functionals in many applications are neither smooth, nor, in general, convex. In such cases, existence can still be proved by passing to the limit in a time-discretization scheme via energy a priori estimates and compactness/lower semicontinuity arguments, but solutions are in general only BV with respect to time, hence they may jump. Furthermore, the natural ambient space for problems in, e.g., shape-memory alloys, quasistatic crack growth, finite-strain elastoplasticity, need not be reflexive, and may even lack a linear/differential structure. These considerations show that the subdifferential formulation (1), involving both the pointwise derivative of u and the (Gâteaux) differential of $\mathcal{E}(t,\cdot)$ with respect to u, is often not adequate for rate-independent modelling. Hence, a *derivative-free* formulation (leading to the notion of *energetic solution* of the rate-independent system (1)) has been proposed, combining an energy balance with a *global stability* inequality, see [3, Sec. 2.1]. However, for nonconvex energies the latter condition may force solutions to jump "too early" to avoid energy losses.

The purpose of this note is to present a novel formulation of rate-independent evolutions which, on the one hand, is based on a *local*, rather than a global, stability condition, and does not enforce premature or spurious jumps, at the same time providing a description of the jump paths. Furthermore, we aim to develop our analysis in a general setting, and for nonsmooth, nonconvex energies. Indeed, we shall work in a complete metric space and, according to physical intuition, derive our new solution notion by passing to the limit as $\varepsilon \searrow 0$ in the viscous problem

$$\varepsilon J(u'(t)) + \partial\Psi(u'(t)) + \mathrm{D}_u\mathcal{E}(t,u(t)) \ni 0 \quad \text{in } B' \quad \text{for a.a. } t \in (0,T) \tag{2}$$

($J : B \to B'$ being the duality operator). In doing so, we shall follow the approach of [2], which we briefly sketch.

The vanishing viscosity analysis by Efendiev & Mielke. In [2], the case $B = \mathbb{R}^N$, $\Psi(v) = K\|v\|$ for all $v \in \mathbb{R}^N$ and some $K > 0$, and $\mathcal{E} \in \mathrm{C}^1([0,T] \times \mathbb{R}^N)$ was considered. The key idea in [2] is that jumps are in fact viscous transitions of the system between two metastable states, which are very fast with respect to the slow external time scale. In order to capture the viscous transition path at jumps, the authors go over to the trajectory phase space, reparametrize the sequence of solutions $\{u_\varepsilon\}_\varepsilon$ to (2) by their arclength τ_ε, and study the limiting behavior as $\varepsilon \searrow 0$ of the sequence $\{(\widehat{t}_\varepsilon := \tau_\varepsilon^{-1}, \widehat{u}_\varepsilon := u_\varepsilon(\widehat{t}_\varepsilon))\}_\varepsilon$. With some calculations, one sees that, setting

$$\widehat{\Psi}_\varepsilon(v) := \Psi(v) + \varepsilon(-\|v\| - \log(1 - \|v\|)) + I_{[0,1)}(\|v\|) \quad \text{for all } v \in \mathbb{R}^N$$

(where $I_{[0,1)}$ is the indicator function of $[0,1)$), the pair $(\widehat{t}_\varepsilon, \widehat{u}_\varepsilon)$ fulfils for every $\varepsilon > 0$

$$\begin{aligned} &\partial\widehat{\Psi}_\varepsilon(\widehat{u}_\varepsilon'(\tau)) + \mathrm{D}_u\mathcal{E}(\widehat{t}_\varepsilon(\tau), \widehat{u}_\varepsilon(\tau)) \ni 0 && \text{for a.a. } \tau \in (0, \widehat{T}_\varepsilon)\,,\\ &\widehat{t}_\varepsilon'(\tau) + \|\widehat{u}_\varepsilon'(\tau)\| = 1 && \text{for a.a. } \tau \in (0, \widehat{T}_\varepsilon)\,, \end{aligned}$$

with $\widehat{T}_\varepsilon = \tau_\varepsilon(T)$. Hence, in [2] it was proved that, up to a subsequence, $\{(\widehat{t}_\varepsilon, \widehat{u}_\varepsilon)\}_\varepsilon$ converges to a pair $(\widehat{t}, \widehat{u}) \in \mathrm{AC}([0,\widehat{T}]; [0,T] \times X)$ satisfying

$$\begin{aligned} &\partial\widehat{\Psi}(\widehat{u}'(\tau)) + \mathrm{D}_u\mathcal{E}(\widehat{t}(\tau), \widehat{u}(\tau)) \ni 0 && \text{for a.a. } \tau \in (0, \widehat{T})\,,\\ &\widehat{t}'(\tau) + \|\widehat{u}'(\tau)\| = 1 && \text{for a.a. } \tau \in (0, \widehat{T})\,, \end{aligned} \tag{3}$$

and $\widehat{t}(0) = 0$, $\widehat{t}(\widehat{T}) = T$, where $\widehat{T} = \lim_{\varepsilon\searrow 0} \widehat{T}_\varepsilon$ and $\widehat{\Psi}_\varepsilon$ Mosco-converges to the function $\widehat{\Psi}(v) := \Psi(v) + I_{[0,1]}(\|v\|)$ for all $v \in \mathbb{R}^N$. The limiting problem (3) encompasses three regimes, which completely describe the evolution of the rate-independent system:

sticking corresponding to $\|\widehat{u}'\| = 0$ ($\Leftrightarrow \widehat{t}' = 1$);

sliding (or *rate-independent motion*), occurring when $0 < \widehat{t}', \|\widehat{u}'\| < 1$: in this case, (3) may be reparametrized to yield (1);

viscous slip when $\|\widehat{t}'\| = 0$: then, the system switches to a rate-dependent behavior encoded by (3), which yields a gradient flow at the fixed process time $\widehat{t}$.

2 The metric formulation of doubly nonlinear evolution equations

Before revisiting the vanishing viscosity approach of [2] in the framework of

a *complete metric space* (X, d),

we show how doubly nonlinear equations of the type (1) (where for the moment Ψ is no longer 1-homogeneous) may be reformulated in the absence of a linear/differentiable structure on X. We note that (1) is equivalent for a.e. $t \in (0,T)$ to

$$\Psi(u'(t)) + \Psi^*(-\mathrm{D}_u\mathcal{E}(t, u(t))) + \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t, u(t)) - \partial_t\mathcal{E}(t, u(t)) = 0\,, \tag{4}$$

where Ψ^* is the convex conjugate of Ψ and we have combined the convex analysis property

$$\Psi(v) + \Psi^*(\sigma) = \langle\sigma, v\rangle \;\Leftrightarrow\; \sigma \in \partial\Psi(v)$$

with the (formal) chain rule

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t,u(t)) - \partial_t\mathcal{E}(t,u(t)) = \langle \mathrm{D}_u\mathcal{E}(t,u(t)), u'(t)\rangle.$$

Now, choosing $\Psi(v) = \|v\|^p$, $1 < p < \infty$, we see that (4) highlights the role of the modulus of the derivatives u' and $-\mathrm{D}_u\mathcal{E}$, rather than of the derivatives themselves.

Thus, (4) somehow points in the direction of the appropriate formulation of (1) in the metric space (X,d). There, one in fact disposes of surrogates of the modulus of derivatives, in the realm of E. De Giorgi's theory of *Curves of Maximal Slope*, (i.e. of gradient flow equations in metric spaces, see [1] and the references therein). Indeed, for every absolutely continuous curve $u : (0,T) \to X$,

$$\text{the limit } \lim_{h\to 0}\frac{d(u(t+h),u(t))}{|h|} =: |u'|(t) \text{ exists for a.a. } t \in (0,T),$$

defining the *metric derivative* of u (which "replaces" the norm of the pointwise derivative $\|u'\|$). In the same way, given a functional $\mathcal{E} : [0,T]\times X \to (-\infty,+\infty]$, the *local slope* of $\mathcal{E}$ at $u \in \mathrm{dom}(\mathcal{E}(t,\cdot))$

$$|\partial\mathcal{E}|(t,u) := \limsup_{z\to u}\frac{(\mathcal{E}(t,u)-\mathcal{E}(t,z))^+}{d(u,z)} \tag{5}$$

surrogates $\|-\mathrm{D}_u\mathcal{E}(t,u(t))\|$. In this framework, the usual chain rule identity is substituted by a *chain rule inequality*, viz.: For all $u \in \mathrm{AC}([0,T];X)$

$$\begin{gathered}\text{the map } t \mapsto \mathcal{E}(t,u(t)) \text{ is absolutely continuous on } [0,T], \text{ and}\\ \tfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t,u(t)) - \partial_t\mathcal{E}(t,u(t)) \geq -|u'|(t)\cdot|\partial\mathcal{E}|(t,u(t)) \quad \text{for a.a. } t \in (0,T).\end{gathered} \tag{6}$$

With the help of these concepts, in [4] we have proposed this definition: given a (l.s.c., convex) function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ a curve $q \in \mathrm{AC}([0,T];X)$ fulfils the ψ-gradient system associated with a functional $\mathcal{E} : [0,T]\times X \to (-\infty,+\infty]$ if for a.a. $t \in (0,T)$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t,u(t)) - \partial_t\mathcal{E}(t,u(t)) = -\psi(|u'|(t)) - \psi^*\big(|\partial\mathcal{E}|(t,u(t))\big)\,, \tag{7}$$

(which is in fact formally analogous to (4)). In [4] it is shown that, under suitable assumptions, the Cauchy problem for (7) has at least a solution, and that (7) is a reformulation of (1) when X is a Banach space with the Radon-Nikodým property.

3 Parametrized metric solutions of rate-independent systems

Setting $\psi_\varepsilon(x) = x + \frac{\varepsilon}{2}x^2$ for all $x \in \mathbb{R}^+$, the metric formulation of (2) reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t,u_\varepsilon(t)) - \partial_t\mathcal{E}(t,u(t)) = -\psi_\varepsilon(|u_\varepsilon'|(t)) - \psi_\varepsilon^*\big(|\partial\mathcal{E}|(t,u_\varepsilon(t))\big)\,. \tag{8}$$

for a.a. $t \in (0,T)$. In [5, 7] we adapt the techniques of [2] to the metric framework: we rescale u_ε by the arclength $\tau_\varepsilon(t) := t + \int_0^t |u'_\varepsilon|(s)\mathrm{d}s$, $t \in [0,T]$, and pass to the limit as $\varepsilon \searrow 0$ in the (rescaled) metric formulation fulfilled by the pair $(\widehat{t}_\varepsilon, \widehat{u}_\varepsilon)$. Under suitable assumptions on $\mathcal{E}$ and the chain rule (6), combining a priori estimates with compactness/lower semicontinuity arguments, we indeed prove that, up to a subsequence, $\{(\widehat{t}_\varepsilon, \widehat{u}_\varepsilon)\}$ converges to a pair $(\widehat{t}, \widehat{u}) \in \mathrm{AC}([0,\widehat{T}];[0,T]\times X)$ (with $\widehat{T} = \lim_\varepsilon \tau_\varepsilon(T)$), fulfilling $\widehat{t}(0) = 0$, $\widehat{t}(\widehat{T}) = T$ and

$$\left.\begin{array}{l} \frac{\mathrm{d}}{\mathrm{d}\tau}\mathcal{E}(\widehat{t}(\tau),\widehat{u}(\tau)) - \partial_t\mathcal{E}(\widehat{t}(\tau),\widehat{u}(\tau))\widehat{t}'(\tau) = -\widehat{\psi}(|\widehat{u}'|(\tau)) - \widehat{\psi}^*\big(|\partial\mathcal{E}|(\widehat{t}(\tau),\widehat{u}(\tau))\big), \\ \widehat{t}'(\tau) + |\widehat{u}'|(\tau) = 1 \end{array}\right\} \quad (9)$$

for a.a. $\tau \in (0,\widehat{T})$, where, in accordance with (3), $\widehat{\psi}(x) = x + I_{[0,1]}(x)$ for all $x \in \mathbb{R}^+$. Let us point out that, in view of the chain rule (6) and elementary convex analysis, the first of (9) can be decoupled as

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\mathcal{E}(\widehat{t}(\tau),\widehat{u}(\tau)) - \partial_t\mathcal{E}(\widehat{t}(\tau),\widehat{u}(\tau))\widehat{t}'(\tau) = -|\widehat{u}'|(\tau)\cdot|\partial\mathcal{E}|(\widehat{t}(\tau),\widehat{u}(\tau)) \quad \text{for a.a. } \tau \in (0,\widehat{T}), \quad (10)$$

$$|\partial\mathcal{E}|(\widehat{t}(\tau),\widehat{u}(\tau)) \in \partial\widehat{\psi}(|\widehat{u}'|(\tau)) \quad \text{for a.a. } \tau \in (0,\widehat{T}). \quad (11)$$

Taking into account that $\partial\widehat{\psi}(0) = (-\infty,1]$, $\partial\widehat{\psi}(v) = \{1\}$ for $v \in (0,1)$, and $\partial\widehat{\psi}(1) = [1,+\infty)$, also in view of the second of (9) we rephrase (11) as

$$\left.\begin{array}{lll} |\widehat{u}'|(s) = 1 \ (\Leftrightarrow \widehat{t}'(s) = 0) & \Rightarrow & |\partial\mathcal{E}|\,(\widehat{t}(s),\widehat{u}(s)) \geq 1\,, \\ |\widehat{u}'|(s) \in (0,1) \ (\Leftrightarrow \widehat{t}'(s) \in (0,1)) & \Rightarrow & |\partial\mathcal{E}|\,(\widehat{t}(s),\widehat{u}(s)) = 1\,, \\ |\widehat{u}'|(s) = 0 \ (\Leftrightarrow \widehat{t}'(s) = 1) & \Rightarrow & |\partial\mathcal{E}|\,(\widehat{t}(s),\widehat{u}(s)) \leq 1\,, \end{array}\right\} \quad (12)$$

which highlights three regimes: *sticking*, corresponding to $|\widehat{u}'| = 0$, *sliding*, occurring when $\widehat{t}'\cdot|\widehat{u}'| > 0$, and *viscous slip*, at frozen process time (i.e. $\widehat{t}' = 0$). The notion of rate-independent metric evolution we propose retains (10) and (12), replacing the second of (9) with a general "non-degeneracy" condition.

Definition 3.1. *A curve* $(\widehat{t},\widehat{q}) \in \mathrm{AC}([0,\widehat{T}];[0,T]\times X)$ *is a* parametrized metric solution *of the rate-independent system* $(X,d,\mathcal{E})$ *if* $\widehat{t} : [0,\widehat{T}] \to [0,T]$ *is nondecreasing, the energy identity* (10) *and the differential conditions* (12) *hold, and*

$$\widehat{t}'(\tau) + |\widehat{u}'|(\tau) > 0 \qquad \textit{for a.a. } \tau \in (0,\widehat{T}). \quad (13)$$

We refer to [5] for a thorough discussion of this concept in a specific metric setting (viz., a finite-dimensional smooth manifold endowed with a Finsler distance), and comparison with *energetic solutions* and other solution notions for rate-independent problems. Here, we just point out that (10), (12), and (13) (unlike the second of (9)), are invariant with respect to strictly increasing reparametrizations: hence, the notion of parametrized metric solution is truly rate-independent. As we have seen, existence of parametrized metric solutions can be obtained by passing to the limit in the arclength-rescaled viscous approximation (8). In the forthcoming [7], we shall prove this in a general metric setting. Therein, we shall also prove existence of parametrized metric solutions, through approximation by time discretization and solution of incremental (*local*) minimization problems (cf. also the results in [6], in a Banach context). In doing so, we shall exploit the techniques of De Giorgi's theory of *Minimizing Movements* for the approximation of *Curves of Maximal Slope*, see [1, Chap. III].

Finally, let us gain further insight into the evolution (10) and (12).

Sliding : if $\widehat{t}'(\bar\tau), |\widehat{u}'|(\bar\tau) > 0$ at $\bar\tau \in (0, \widehat{T})$, in some neighborhood $(\bar\tau - \delta, \bar\tau + \delta)$ the system evolution is described by

$$\begin{cases} \mathcal{E}(\widehat{t}(\tau_2), \widehat{u}(\tau_2)) - \mathcal{E}(\widehat{t}(\tau_1), \widehat{u}(\tau_1)) = \int_{\tau_1}^{\tau_2} \big(\partial_t \mathcal{E}(\widehat{t}(\tau), \widehat{u}(\tau))\, \widehat{t}'(\tau) - |\widehat{u}'|(\tau)\big)\, \mathrm{d}\tau\,, \\ |\partial\mathcal{E}|\,(\widehat{t}(\tau), \widehat{u}(\tau)) = 1 \end{cases}$$

for a.a. $\tau \in (\bar\tau - \delta, \bar\tau + \delta)$, where the former relation, obtained by integrating (10) for all $\tau_1, \tau_2 \in (\bar\tau - \delta, \bar\tau + \delta)$, is an *energy balance* and the latter one a *local stability* condition, since the slope notion (5) has an intrinsically local character.

Viscous slip : if $\widehat{t}'(\bar\tau) = 0$ at $\bar\tau \in (0, \widehat{T})$, then $\widehat{t}(\tau) = \widehat{t}(\bar\tau)$ for $\tau \in (\bar\tau - \delta, \bar\tau + \delta)$, and

$$\begin{cases} \mathcal{E}(\widehat{t}(\bar\tau), \widehat{u}(\tau_2)) - \mathcal{E}(\widehat{t}(\bar\tau), \widehat{u}(\tau_1)) = -\int_{\tau_1}^{\tau_2} |\partial\mathcal{E}|\,(\widehat{t}(\bar\tau), \widehat{u}(\tau))\, |\widehat{u}'|(\tau)\, \mathrm{d}\tau\,, \\ |\partial\mathcal{E}|\,(\widehat{t}(\tau), \widehat{u}(\tau)) \geq 1, \quad |\widehat{u}'|(\tau) > 0 \end{cases}$$

for a.a. $\tau \in (\bar\tau - \delta, \bar\tau + \delta)$, where the energy identity for all $\tau_1, \tau_2 \in (\bar\tau - \delta, \bar\tau + \delta)$ provides a description of the system evolution along the jump path, in terms of a (*generalized*) gradient flow at the fixed process time $\widehat{t}(\bar\tau)$.

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