films with mid-surface or arbitrary (non-flat) geometry, an infinite hierarchy of models was proposed, by means of asymptotic expansion in [13], and it remains in agreement with all the rigorously obtained results [2, 9, 10, 11].

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Vanishing-viscosity solutions for rate-independent systems

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In these notes we give an overview of the recently developed theory for rate-independent systems. Such systems are used to model hysteresis, dry friction, elastoplasticity, magnetism, and phase transformation, and they are characterized by the fact that the changes of the state are driven solely by changes of the loading.

General energy-driven systems, also called generalized gradient systems, are characterized by a triple $(\boldsymbol{Z},\mathcal{I},\mathcal{R})$ where the Banach space $\boldsymbol{Z}$ is the state space and $\mathcal{I} : [0,T] \times \boldsymbol{Z} \to \mathbb{R}_\infty := \mathbb{R} \cup \{\infty\}$ is the energy functional. The dissipation potential $\mathcal{R} : \boldsymbol{Z} \times \boldsymbol{Z} \to [0,\infty]$ allows us to write the evolution equation in the form

$$0 \in \partial_{\dot z}\mathcal{R}(z,\dot z) + \overline{\partial}_z\mathcal{I}(t,z) \quad \subset \boldsymbol{Z}^*, \tag{1}$$

where $\overline{\partial}_z$ denotes a suitable subgradient of $\mathcal{I}(t,\cdot)$, while $\partial_{\dot z}\mathcal{R}(z,\cdot)$ denotes the convex subdifferential of $\mathcal{R}(z,\cdot)$. The generalized gradient system $(\boldsymbol{Z},\mathcal{I},\mathcal{R})$ is rate independent if $\mathcal{R}(z,\cdot)$ is positively homogeneous of degree 1, since this implies $\partial_v\mathcal{R}(z,\alpha v)=\partial_v\mathcal{R}(z,v)$ for all $\alpha>0$. We then call $(\boldsymbol{Z},\mathcal{I},\mathcal{R})$ a rate-independent system, shortly RIS. Hence, system (1) is necessarily nonsmooth. In fact, the convex subdifferential $\partial_v\mathcal{R}(z,\cdot):\boldsymbol{Z}\rightrightarrows\boldsymbol{Z}^*$ is not continuous and set-valued.

However, the main difference to the usually studied generalized gradient flows is that $\mathcal{R}(z,\cdot)$ has at most linear growth, and we cannot guarantee continuity of the solutions $z:[0,T]\to\boldsymbol{Z}$. Since we can guarantee the absolute continuity needed in (1) only under strong convexity assumptions (cf. [MiR07], we mainly discuss the question, how the strong differential form should be weakened to allow for solutions with jumps. For full details we refer to the survey [Mie09] or the papers [MRS09b, MRS09a, MiZ09].

To motivate the main structures of the different solution concepts for RIS, we start from the Fenchel equivalence ($\mathcal{R}^*(z,\cdot)$ is the Legendre transform of $\mathcal{R}(z,\cdot)$)

$$\eta\in\partial_v\mathcal{R}(z,v)\quad\Longleftrightarrow\quad v\in\partial_\eta\mathcal{R}^*(z,\eta)\quad\Longleftrightarrow\quad \mathcal{R}(z,v)+\mathcal{R}^*(z,\eta)\le\langle\eta,v\rangle.$$

While the statement on the left-hand side of this equivalence is a force balance, the statement on the right-hand side is given in terms of energy rates. Using $-\eta=\xi(t)\in\overline{\partial}_z\mathcal{I}(t,z(t))$ and a chain rule, we find that (1) is equivalent to the scalar, upper energetic inequality

$$\begin{aligned}\mathcal{I}(T,z(T))+\textstyle\int_0^T\mathcal{R}(z(t),\dot z(t))+\mathcal{R}^*(z(t),-\xi(t))\,\mathrm{d}t&\\ \le\mathcal{I}(0,z(0))+\textstyle\int_0^T\partial_t\mathcal{I}(t,z(t))\,\mathrm{d}t.&\end{aligned}\tag{2}$$

The particularity of RIS is that $\mathcal{R}^*(z,-\xi)$ only takes the two values 0 and ∞, viz. $\mathcal{R}^*(z,-\xi)=0$ if and only if $0\in\partial_v\mathcal{R}(z,0)+\xi$. Thus, the energetic inequality (2) can be rewritten in terms of two conditions

(3a) local stability $\qquad 0\in\partial_v\mathcal{R}(z(t),0)+\overline{\partial}_z\mathcal{I}(T,z(t))$ a.e. in $[0,T]$,

(3b) energy inequality $\qquad \begin{aligned}&\mathcal{I}(T,z(T))+\mathrm{Diss}_{\mathcal{R}}(z,[0,T])\\ &\qquad\le\mathcal{I}(0,z(0))+\textstyle\int_0^T\partial_t\mathcal{I}(t,z(t))\,\mathrm{d}t,\end{aligned}$

where $\mathrm{Diss}_{\mathcal{R}}(z,[r,t])=\int_r^t\mathcal{R}(z(s),\dot z(s))\,\mathrm{d}s$ is the energy dissipated in $[r,t]$.

The local stability condition is a purely static concept and does not involve any time dependence, which shows that RIS are very close to static systems. Relation (3b) is a simple scalar energy inequality, which in fact should hold as an identity and also for all times $t\in[0,T]$ and not just for $t=T$. In all the different solution concepts discussed below we have these two different principles, namely (i) a static stability condition and (ii) an energy inequality. However, a crucial point in the definitions of solutions to RIS is always that the stability condition and the energy inequality interact in such a way that the stability condition implies a lower energy estimate on all subintervals of $[0,T]$, which together with the upper energy estimate (3b) provides energy balance on all subintervals.

Local solutions were introduced in [ToZ09] and are characterized by (3a) and the upper energy estimate (3b) but for each subinterval $[r,t] \subset [0,T]$. This notion is still quite general and all solutions consider here fall into this class.

Energetic solutions (also called *irreversible quasistatic evolutions* in [DaT02, DFT05], and surveyed in [Mie05]) ask for an energy equality (E), where the dissipation is formulated in terms of a dissipation distance $\mathcal{D} : \boldsymbol{Z} \times \boldsymbol{Z} \to [0,\infty]$. Moreover, the local stability (3a) is replaced by a global stability conditions (S), namely $\mathcal{I}(t,z(t)) \leq \mathcal{I}(t,\widetilde{z}) + \mathcal{D}(z(t),\widetilde{z})$ for all $\widetilde{z} \in \boldsymbol{Z}$.

Parametrized solutions are obtained in the vanishing-viscosity limit in

$$0 \in \partial_{\dot z}\mathcal{R}(z^\varepsilon, \dot z^\varepsilon) + \varepsilon \mathbb{V} \dot z^\varepsilon + \overline{\partial}_z \mathcal{I}(t,z^\varepsilon) \quad \subset \boldsymbol{Z}^*, \tag{4}$$

after an arclength parametrization. Taking the limit $\varepsilon \to 0$ directly in $z^\varepsilon : [0,T] \to \boldsymbol{Z}$ is difficult because of the formation of jumps, i.e. fast transitions on time intervals of length $1/\varepsilon$. We use the arclength parametrization $\zeta^\varepsilon = (\tau_\varepsilon, Z_\varepsilon) : [0,S^\varepsilon] \to \mathbb{R} \times \boldsymbol{Z}$ such that $\tau'_\varepsilon(s) + \|Z'_\varepsilon(s)\|_{\mathbb{V}} = 1$ and $z^\varepsilon(\tau_\varepsilon(s)) = Z_\varepsilon(s)$ a.e. The limit $\zeta = (t,Z)$ for $\varepsilon \to 0$ is called parametrized solution and satisfies the limit problem

$$0 \in \partial_{\dot Z}\mathcal{R}(Z(s),Z'(s)) + \partial\mathcal{C}(Z'(s)) + \overline{\partial}_Z \mathcal{I}(\tau(s),Z(s)), \quad \tau'(s) + \|Z'(s)\|_{\mathbb{V}} = 1,$$

where $\mathcal{C}(v) = 0$ for $\|v\|_{\mathbb{V}} = \langle \mathbb{V}v, v\rangle^{1/2} \leq 1$ and ∞ otherwise. Existence results for parabolic situations are established in [MiZ09].

BV solutions $\widetilde{z} : [0,T] \to Z$ are in principle defined as projections of the parametrized solutions, i.e. there exists a parametrized solution $\zeta = (\tau, Z)$ such that $(t,\widetilde{z}(t)) = (\tau(s(t)), Z(s(t))$ for some monotone $s : [0,T] \to [0,S]$. However, it is important to have an independent characterization which can be obtained via the *vanishing-viscosity contact potential*

$$\boldsymbol{p}(v,\xi) := \inf\big\{ \mathcal{R}_\varepsilon(v) + \mathcal{R}^*_\varepsilon(\xi) \,\big|\, \varepsilon > 0 \big\} \text{ with } \mathcal{R}_\varepsilon(v) = \Psi(v) + \frac{\varepsilon}{2}\langle \mathbb{V}v, v\rangle.$$

This allows us to define a supplemented dissipation distance via

$$\Delta(t,z_0,z_1) := \inf\big\{ \textstyle\int_{r=0}^1 \boldsymbol{p}(\dot y(r), -\mathrm{D}\mathcal{I}(t,y(r)))\,\mathrm{d}r \,\big|\, y \in \mathrm{W}^{1,1}([0,1];\boldsymbol{Z}),\ y(0)=z_0,\ y(1)=z_1 \big\}.$$

Note that $\Delta(t,z_0,z_1) \geq \Psi(z_1 - z_0) \geq \|z_1 - z_0\|_{\boldsymbol{X}}$ for some Banach space $\boldsymbol{X}$.

A function $\widetilde{z} : [0,T] \to \boldsymbol{Z}$ is called a *BV solution* of the RIS $(\boldsymbol{Z}, \mathcal{I}, \Psi, \mathbb{V})$, if $\widetilde{z} \in \mathrm{BV}([0,T];\boldsymbol{X}) \cap \mathrm{L}^\infty([0,T];\boldsymbol{Z})$ and the following holds:

(5a) local stability $\forall\, t \in C(z) :\ 0 \in \partial\Psi(0) + \overline{\partial}_z \mathcal{I}(t,z)$;

(5b) energy balance
$\forall\, t \in [0,T]$: $\mathcal{I}(t,z(t)) + \mathrm{Diss}_{\boldsymbol{p},\mathcal{I}}(z,\ [0,t]) = \mathcal{I}(0,z(0)) + \int_0^t \partial_\tau \mathcal{I}(\tau, z(\tau))\,\mathrm{d}\tau$,

where $C(\widetilde{z}) \subset [0,T]$ denotes the continuity points of $\widetilde{z} : [0,T] \to \boldsymbol{X}$ and $\mathrm{Diss}_{\boldsymbol{p},\mathcal{I}}$ is a special variation of defined via Ψ at continuity points and Δ at jump points, see [MRS09a, Mie09]. These works contain first convergence results of the viscous approximations z^ε towards BV solutions. Moreover, for viscous time-incremental problems of the form

$$z_k^{\varepsilon,\delta} \in \operatorname{Argmin} \mathcal{I}(k\delta, z) + \delta\mathcal{R}_\varepsilon\Big(\frac{1}{\delta}(z - z_{k-1}^{\varepsilon,\delta})\Big),$$

where $\delta > 0$ is the time-step, it is shown that the piecewise affine interpolants $\widehat{z}^{\varepsilon,\delta} : [0,T] \to \boldsymbol{Z}$ converge to BV solutions if ε, δ, and δ/ε tend to 0. If instead, ε/δ goes to 0, then the limits are energetic solutions.

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On a mesoscopic many-body Hamiltonian describing elastic shears and dislocations

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(joint work with Stephan Luckhaus)

We assume that low-energy states in a mono-atomic crystalline material are given by approximately linear deformations of a ground state lattice (here a simple Bravais-lattice $\mathcal{L}_G := \{Gz : z \in \mathbb{Z}^d\}$, for some $G \in GL^+(d, \mathbb{R})$). Hence we construct a "mesoscopic" many-body interaction potential acting on finite systems of particles (a Hamiltonian in the language of statistical mechanics and of this report) that is able to describe deformed crystals with defects. More precisley the