Homogenization for rate-independent systems

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This paper is devoted to the homogenization for a class of rate-independent systems described by the *energetic formulation*. The associated nonlinear partial differential system has periodically oscillating coefficients, but has the form of a standard evolutionary variational inequality. Thus, the model applies to standard linearized elastoplasticity with hardening.
Using the recently developed methods of *two-scale convergence*, *periodic unfolding* and the new introduced one, *periodic folding*, we show that the homogenized problem can be represented as a two-scale limit which is again an energetic formulation, but now involving the macroscopic variable in the physical domain as well as the microscopic variable in the periodicity cell.

Our aim is to provide homogenization results for evolutionary variational inequalities of the type:

$$\forall v \in X: \ \langle Ay - \ell(t), v - \dot{y}\rangle + \mathcal{R}(v) - \mathcal{R}(\dot{y}) \geq 0,$$

which have several equivalent formulations. The one that we will use in our work is the so called *energetic formulation* for rate-independent hysteresis problem associated with the energy functional $\mathcal{E}$ and dissipation functional $\mathcal{R}$.
Let us consider: $\Omega \subset \mathbb{R}^d$ a nonempty, connected open bounded set with Lipschitz boundary Γ, $Y \subset \mathbb{R}^d$ unit periodicity cell, $u : \Omega \to \mathbb{R}^d$ displacement, $z : \Omega \to \mathbb{R}^m$ internal variable. For every $\delta > 0$, define the following two functionals.
Energy functional $\mathcal{E}_\delta$:

$$\mathcal{E}_\delta(t, u, z) = \int_\Omega \frac{1}{2}\langle \mathbb{C}\left(\frac{x}{\delta}\right)(\varepsilon(u) - \mathbb{B}\left(\frac{x}{\delta}\right)z), \varepsilon(u) - \mathbb{B}\left(\frac{x}{\delta}\right)z\rangle + \frac{1}{2}\langle \mathbb{A}\left(\frac{x}{\delta}\right)z, z\rangle - u(x) f_{\text{ext}}(t, x)\,\mathrm{d}x,$$

$$\text{where } \varepsilon(u) = \frac{1}{2}(\nabla u + \nabla u^\mathsf{T}) \in \mathbb{R}^{d\times d}_{\text{sym}} := \{\sigma \in \mathbb{R}^{d\times d} \,|\, \sigma = \sigma^\mathsf{T}\}.$$

Dissipation functional $\mathcal{R}_\delta$:

$$\mathcal{R}_\delta(\dot{z}) = \int_\Omega \rho\left(\frac{x}{\delta}, \dot{z}(x)\right)\,\mathrm{d}x.$$

The tensors $\mathbb{C}, \mathbb{A}, \mathbb{B}$ defined on $\mathbb{R}^d$ are Y-periodic and take values in: $\mathbb{C}(y) \in$ Sym 4th order tensor, $\mathbb{B}(y) \in \mathrm{Lin}(\mathbb{R}^m, \mathbb{R}^{d\times d}_{\text{sym}})$, $\mathbb{A}(y) \in \mathbb{R}^{m\times m}_{\text{sym}}$. We will work under the following hypotheses.
Assumptions for $\mathbb{C}, \mathbb{A}, \mathbb{B}$:
For all $y \in \mathbb{R}^d$ and $z \in \mathbb{R}^m$,

$$\frac{1}{C}|\varepsilon|^2 \leq \langle \mathbb{C}(y)\varepsilon, \varepsilon\rangle \leq C|\varepsilon|^2, \quad \frac{1}{C}|z|^2 \leq \langle \mathbb{A}z, z\rangle \leq C|z|^2, \quad ||\mathbb{B}(y)||_{\mathbb{R}^{d\times d}} \leq C,$$

with $C > 0$.
Assumptions for ρ:

$$(\mathrm{H}_\rho) \quad \begin{cases} \rho : \mathbb{R}^d \times \mathbb{R}^m \to [0, \infty), \\ \rho(\cdot, v) \text{ Lebesgue measurable }, Y - \text{periodic } \forall v \in \mathbb{R}^m, \\ \rho(y, \cdot)\ 1- \text{ homogeneous and convex for a.e. } y \in \mathbb{R}^d, \\ \frac{1}{C}|v| \leq \rho(y, v) \text{ for a.e. } y \in \mathbb{R}^m \text{ and every } v \in \mathbb{R}^m, \\ |\rho(y, v) - \rho(y, v')| \leq C|v - v'| \text{ for a.e. } y \in \mathbb{R}^d \text{ and all } v, v' \in \mathbb{R}^m. \end{cases}$$

Let us denote by $\mathcal{F}$ the space $\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d \times \mathrm{L}^2(\Omega)^m$.
We call (u_δ, z_δ) an ***energetic solution*** of the problem associated with $\mathcal{E}_\delta$ and $\mathcal{R}_\delta$, if for all $t \in [0, T]$ the *stability condition* (S^δ) and the *energy balance* (E^δ) hold:

$$(\mathrm{S}^\delta) : \mathcal{E}_\delta(t, u_\delta(t), z_\delta(t)) \leq \mathcal{E}_\delta(t, u, z) + \mathcal{R}_\delta(z - z_\delta(t))\ \forall (u, z) \in \mathcal{F}.$$

$$(\mathrm{E}^\delta) : \mathcal{E}_\delta(t, u_\delta(t), z_\delta(t)) + \int_0^t \mathcal{R}_\delta(\dot{z}_\delta(s))\,\mathrm{d}s = \mathcal{E}_\delta(0, u_\delta(0), z_\delta(0)) - \int_0^t \int_\Omega \dot{f}_{\text{ext}}(s, x) \cdot u(x)\,\mathrm{d}x\,\mathrm{d}s.$$

We state now an existence and uniqueness result for (S^δ) & (E^δ).

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Proposition 0.1 *Let* $f_{\mathrm{ext}} \in \mathrm{C}^{\mathrm{Lip}}([0,T],(\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d)^*)$. *Then for all* $\delta > 0$ *and all stable* $(u^0_\delta, z^0_\delta) \in \mathcal{F}$ $\exists!\,(u_\delta, z_\delta) \in \mathrm{C}^{Lip}([0,T],\mathcal{F})$ *solution for* (S^δ) & (E^δ) *with* $(u_\delta(0), z_\delta(0)) = (u^0_\delta, z^0_\delta)$.
Moreover, we obtain δ*-independent Lipschitz bounds for solutions.*

We formulate the problem $(\mathbb{S})$ & $(\mathbb{E})$. Our final aim is to prove that $(\mathbb{S})$ & $(\mathbb{E})$ is the two-scale homogenized problem for (S^δ) & (E^δ).
Let

$$\mathcal{W}_{\mathrm{per}}(Y) := \{f \in \mathrm{H}^1_{\mathrm{per}}(Y) \mid \int_Y f(y)\,\mathrm{d}y = 0\},$$

$$\mathcal{H} = \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d \times \mathrm{L}^2(\Omega;\mathcal{W}_{\mathrm{per}}(Y))^d, \quad \mathcal{Z} = \mathrm{L}^2(\Omega;\mathrm{L}^2(Y))^m = \mathrm{L}^2(\Omega\times Y)^m, \quad \mathcal{Q} = \mathcal{H}\times\mathcal{Z}.$$

For all $(U = (U_0, U_1), Z)$ in Q define the two-scale functionals $\mathbb{E}$ and $\mathbb{R}$.

$$\mathbb{E}(t,U,Z) = \int_\Omega\int_Y \frac{1}{2}\langle\mathbb{C}(y)(\widehat{\varepsilon}(U) - \mathbb{B}(y)Z), \widehat{\varepsilon}(U) - \mathbb{B}(y)Z\rangle + \frac{1}{2}\langle\mathbb{A}(y)Z, Z\rangle - U_0(x)f_{\mathrm{ext}}(t,x)\,\mathrm{d}y\,\mathrm{d}x,$$

where $\widehat{\varepsilon}(U) = \varepsilon_x(U_0) + \varepsilon_y(U_1)$, which means $\widehat{\varepsilon}(U)(x,y) = \varepsilon_x(U_0(\cdot))(x) + \varepsilon_y(U_1(x,\cdot))(y)$.

$$\mathbb{R}(\dot{Z}) = \int_\Omega\int_Y \rho(y,\dot{Z}(x,y))\,\mathrm{d}y\,\mathrm{d}x.$$

The energetic formulation for the two-scale homogenized problem $(\mathbb{S})$ & $(\mathbb{E})$ reads:
For all $t \in [0,T]$, the stability condition $(\mathbb{S})$ and the energy balance $(\mathbb{E})$ hold.

$$(\mathbb{S}) : \mathbb{E}(t,U(t),Z(t)) \le \mathbb{E}(t,\widetilde{U},\widetilde{Z}) + \mathbb{R}(\widetilde{Z} - Z(t))\ \forall(\widetilde{U},\widetilde{Z}) \in \mathcal{H}\times\mathcal{Z},$$

$$(\mathbb{E}) : \mathbb{E}(t,U(t),Z(t)) + \int_0^t \mathbb{R}(\dot{Z}(s))\,\mathrm{d}s = \mathbb{E}(0,U(0),Z(0)) - \int_0^t\int_\Omega \dot{f}_{\mathrm{ext}}(s,x)\cdot U_0(x)\,\mathrm{d}x\,\mathrm{d}s.$$

The following result shows the existence and uniqueness of solutions for the problem $(\mathbb{S})$ & $(\mathbb{E})$.

Proposition 0.2 *Let* $f_{\mathrm{ext}} \in \mathrm{C}^{\mathrm{Lip}}([0,T],(\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d)^*)$. *Then for all* $(U^0, Z^0) \in \mathcal{Q}$ *stable,* $(\mathbb{S})$ & $(\mathbb{E})$ *has a unique solution* $(U,Z) \in \mathrm{C}^{Lip}([0,T],\mathcal{Q})$ *with* $(U(0), Z(0)) = (U^0, Z^0)$.

We introduce now the definition for *two-scale convergence* and *two-scale cross-convergence*.

Definition 0.3 A sequence $\{v_\delta\}_\delta$ in $\mathrm{L}^2(\Omega)$ is said to ***two-scale converge*** to a limit $v_0 = v_0(x,y)$ belonging to $\mathrm{L}^2(\Omega\times Y)$, if, for any function $\psi = \psi(x,y) \in \mathrm{C}^\infty_0(\Omega;\mathrm{C}^\infty_{\mathrm{per}}(Y))$, we have

$$\lim_{\delta\to 0}\int_\Omega v_\delta(x)\psi(x,\frac{x}{\delta})\,\mathrm{d}x = \int_\Omega\int_Y v_0(x,y)\psi(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

Definition 0.4 A sequence $(u_\delta, z_\delta)_\delta \subset \mathcal{F}$ is called ***two-scale cross-convergent*** to $(U = (U_0,U_1), Z) \in \mathcal{H}\times\mathcal{Z}$, if and only if all $(u_\delta)_\delta$, $(\nabla u_\delta)_\delta$, and $(z_\delta)_\delta$ are two-scale convergent to U_0, $\nabla U_0 + \nabla_y U_1$, and Z, respectively.
We then write $(u_\delta, z_\delta) \overset{\mathrm{c2w}}{\rightharpoonup} (U,Z)$.

We can now formulate our main result which states that $(\mathbb{S})$ & $(\mathbb{E})$ is the two-scale homogenized problem for (S^δ) & (E^δ).

Theorem 0.5 *Let* $(u_\delta, z_\delta)_\delta$ *be the solution for* (S^δ) & (E^δ). *Assume the following conditions for the initial data:*

$$\forall\delta > 0,\ (u^0_\delta, z^0_\delta)\ \textit{stable}\ , (u^0_\delta, z^0_\delta) \overset{c2w}{\rightharpoonup} (U^0, Z^0) \in \mathcal{Q},$$

$$\mathcal{E}_\delta(0, u^0_\delta, z^0_\delta) \to \mathbb{E}(0, U^0, Z^0).$$

Then $(u_\delta, z_\delta)_\delta$ *two-scale cross-converges to* (U,Z), *where* (U,Z) *is the unique solution for the two-scale homogenized problem* $\mathbb{S}$ & $\mathbb{E}$, *with* $(U(0), Z(0)) = (U^0, Z^0)$.

References

[1] G. Allaire, Homogenization and two-scale convergence, SIAM J. Math. Anal. **23**, 1482–1518 (1992).

[2] D. Cioranescu, A. Damlamian, R. De Arcangelis, Homogenization of nonlinear integrals via the periodic unfolding method, C.R. Acad. Sci. Paris **Ser. I 339**, 77–82 (2004).

[3] A. Mielke, F. Theil, V. Levitas, A variational formulation of rate-independent phase transformations using an extremum principle, Arch. Rational Mech. Anal. **162**, 137–177 (2002).

[4] A. Mielke, Evolution for rate-independent systems, in Handbook of differential equations (Evolutionary Equations Vol. II), edited by C. Dafermos and E. Feireisl, (Elsevier B.V. 2005), 461–559.

[5] G. Nguetseng, A general convergence result for a functional related to the theory of homogenization, SIAM J. Math. Anal. **20**, 608–623 (1989).