PAMM · Proc. Appl. Math. Mech. **9**, 43 – 46 (2009) / **DOI** 10.1002/pamm.200910012

A Model for the Evolution of Laminates

Klaus Hackl[1], **Sebastian Heinz**[*2], and **Alexander Mielke**[2,3]

[1] Institut für Mechanik, Ruhr-Universität Bochum, Universitätsstr. 150, D-44801 Bochum, Germany
[2] Weierstraß-Institut für Angewandte Analysis und Stochastik, Mohrenstr. 39, D-10117 Berlin, Germany
[3] Humboldt-Universität zu Berlin, Institut für Mathematik, D-10099 Berlin, Germany

We study the time evolution of a generalized standard material in elastoplasticity. Of our particular interest are the formation and the evolution of microstructure. Our aim is to prove the existence of solutions. This is a challenging task, since the presence of microstructure comes along with a lack of convexity and, hence, compactness arguments cannot be applied to prove the existence of solutions. In order to overcome this problem, we will incorporate information on the microstructure into the internal variable, which is still compatible with the notion of generalized standard materials. More precisely, we shall allow such forms of microstructure that are given by simple laminates. We will consider a model for the evolution of these laminates and we will state a result on the existence of solutions to the time-incremental minimization problem.

1 Microstructure in Elastoplasticity

We are going to show one way of incorporating microstructure into a model of elastoplasticity. Therefore, we start with a model in finite-strain elastoplasticity and show how it can be transformed into a model that describes the evolution of microstructure.

1.1 Finite-Strain Elastoplasticity

Let $[0,T] \subset \mathbb{R}$ be a nonempty time interval and let $\Omega \subseteq \mathbb{R}^3$ be a Lipschitz domain – the reference configuration of an elastoplastic body. Then the time evolution is determined by two functions defined on the set $[0,T] \times \Omega$: the deformation $y = y(t,x)$ and the plastic strain $P = P(t,x)$. In terms of standard generalized materials, P serves as the internal variable.

Let us also fix two real numbers $q_P > q_Y > 3$. From now on, we study pairs (y, P) so that $y(t)$ is contained in the Sobolev space $\mathrm{W}_0^{1,q_Y}(\Omega, \mathbb{R}^3)$ (with prescribed homogeneous Dirichlet boundary conditions: $y = \mathrm{id}$ on $\partial\Omega$) and $P(t)$ is contained in the Lebesgue space $\mathrm{L}^{q_P}(\Omega, \mathbb{R}^{3\times3})$ for every $t \in [0,T]$. Moreover, the plastic strain is supposed to be an element of the special linear group

$$P(t,x) \in \mathrm{SL}(3) = \{P \in \mathbb{R}^{3\times3} \mid \det(P) = 1\} \tag{1}$$

for every time $t \in [0,T]$ and almost every material point $x \in \Omega$.

We assume a multiplicative split of the spatial gradient ∇y of the deformation into the elastic strain F_{el} and the plastic strain P, so that

$$\nabla y = F_{\mathrm{el}} P. \tag{2}$$

Hence, we will often write $\nabla y P^{-1}$ instead of F_{el}. As one necessary condition to avoid interpenetration of matter, we require

$$\nabla y(t,x) \in \mathrm{GL}^+(3) = \{F \in \mathbb{R}^{3\times3} \mid \det(F) > 0\} \tag{3}$$

for every time $t \in [0,T]$ and almost every material point $x \in \Omega$. Together with (1) and (2), this implies that $F_{\mathrm{el}} \in \mathrm{GL}^+(3)$.

1.2 Energy

Throughout this paper, all topological objects and properties are understood with respect to the underlying Euclidean structure unless specified otherwise. In particular, $|.|$ denotes the Euclidean norm.

We associate an energy $\mathcal{E}$ to every time $t \in [0,T]$, every deformation y and plastic strain P via the formula

$$\mathcal{E}(t,y,P) = \int_\Omega \big[W(\nabla y(t,x)P(t,x)^{-1}) + H(P(t,x))\big]\mathrm{d}x - \langle L(t), y(t)\rangle.$$

* Corresponding author: e-mail: heinz@wias-berlin.de, Phone: +49 (0)30 20372 541, Fax: +49 (0)30 2044975

© 2009 Wiley-VCH Verlag GmbH & Co. KGaA, Weinheim

The energy consists of three parts. The stored elastic energy is given by a function $W\colon \mathrm{GL}^+(3) \to \mathbb{R}$. We assume W to be coercive, meaning, there exist positive constants $w_1, w_2, w_3 > 0$ so that

$$W(F_{\mathrm{el}}) \geq w_1 |F_{\mathrm{el}}|^{q_F} + w_2 \det(F_{\mathrm{el}})^{-\gamma} - w_3 \tag{4}$$

holds for all $F_{\mathrm{el}} \in \mathrm{GL}^+(3)$. Here the exponent q_F is given by the formula

$$\frac{1}{q_Y} = \frac{1}{q_F} + \frac{1}{q_P}. \tag{5}$$

In addition, W has to fulfill a convexity condition: There exists a convex real valued function $\mathbb{W}$ so that

$$W(F_{\mathrm{el}}) = \mathbb{W}(\mathbb{M}(F_{\mathrm{el}})), \quad F_{\mathrm{el}} \in \mathrm{GL}^+(3). \tag{6}$$

The quantity $\mathbb{M}(F_{\mathrm{el}})$ denotes the vector of all minors of the matrix F_{el} in a fixed order. By (6), the function W becomes polyconvex (see Ball [1]).

A second term is due to plastic hardening. The hardening function $H\colon \mathrm{SL}(3) \to \mathbb{R}$ is assumed to be convex. In addition, we require that there exist positive constants $h_1, h_2 > 0$ so that

$$H(P) \geq h_1 |P|^{q_P} - h_2 \tag{7}$$

holds for all $P \in \mathrm{SL}(3)$.

The third term measures the work done with respect to a time-dependent external loading $L = L(t, x)$.

1.3 Dissipation & Balance Equations

In the classical sense, a pair (y, P) is said to be a solution of the model in finite-strain elastoplasticity if it meets a given initial condition $(y(0), P(0)) = (y_0, P_0)$ and if the following two equations are fulfilled for every $t \in [0, T]$: the equation of elastic equilibrium

$$\mathrm{D}_y \mathcal{E}(t, y(t), P(t)) = 0 \tag{8}$$

and the plastic flow rule (Biot's equation)

$$0 \in \partial_{\dot{P}} \mathcal{R}(P(t), \dot{P}(t)) + \mathrm{D}_P \mathcal{E}(t, y(t), P(t)) \tag{9}$$

where $\partial_{\dot{P}} \mathcal{R}$ denotes the subgradient (with respect to the second variable) of the dissipation potential $\mathcal{R}$ given by

$$\mathcal{R}(P(t), \dot{P}(t)) = \int_\Omega \sigma_{\mathrm{yield}} |\dot{P}(t,x) P(t,x)^{-1}| \mathrm{d}x.$$

Here we assumed von Mises plasticity with yield stress $\sigma_{\mathrm{yield}} > 0$. Clearly, the equations (8) and (9) require a certain differentiability of the energy $\mathcal{E}$. This issue is not further discussed here, since we will concentrate on the time-incremental minimization problem only.

The form of the dissipation potential $\mathcal{R}$ induces a dissipation distance D for plastic strains $P_0, P_1 \in \mathrm{SL}(3)$ given by

$$D(P_0, P_1) = \inf \int_0^1 |\dot{P}(s) P(s)^{-1}| \mathrm{d}s$$

where the infimum is taken over all sufficiently smooth paths $P\colon [0,1] \to \mathrm{SL}(3)$ so that $P(0) = P_0$ and $P(1) = P_1$. Such a dissipation is studied, for example, in Gürses, Mainik, Miehe, and Mielke [5].

1.4 Reformulation for Laminates

Now we are going to replace the plastic strain P by the new internal variable Λ in order to model a microstructural state in each macroscopic material point $x \in \Omega$. Experimental pictures often show microstructure with a laminate structure. This is why, we consider the set $\mathcal{L}$ of all probability measures over $\mathrm{GL}^+(3) \times \mathrm{SL}(3)$ that are simple laminates. An element $\Lambda \in \mathcal{L}$ can be written in the form

$$\Lambda = \alpha \delta_{A,Q} + (1-\alpha) \delta_{B,R}$$

where $Q, R \in \mathrm{SL}(3)$, $A = I + (1-\alpha)\, a \otimes n$ and $B = I - \alpha\, a \otimes n$ for some $\alpha \in [0,1]$, $a, n \in \mathbb{R}^3$, $|n| = 1$, and I being the identity matrix in $\mathbb{R}^{3\times 3}$.

The new internal variable $\Lambda \in \mathcal{L}$ encodes the micro-fluctuations of the deformation gradient ∇y as well as the information about the plastic strain. Without writing the dependence on t and x explicitly, the new energy reads

$$\tilde{\mathcal{E}}(y, \Lambda) = \int_\Omega \Big[\alpha\big(W(\nabla y A Q^{-1}) + H(Q)\big) + (1-\alpha)\big(W(\nabla y B R^{-1}) + H(R)\big)\Big] \mathrm{d}x - \langle L, y \rangle.$$

1.5 Metric Structure on Laminates

In order to define a topology on the set $\mathcal{L}$ of simple laminates, we shall use the Wasserstein distance d_{W} with respect to a suitably chosen distance on the underlying set $\mathrm{GL}^+(3) \times \mathrm{SL}(3)$. By means of integration, we define a distance of simple-laminate fields

$$\mathcal{D}_r(\Lambda_0, \Lambda_1) = \left[\int_\Omega d_{\mathrm{W}}(\Lambda_0(x), \Lambda_1(x))^r \mathrm{d}x \right]^{1/r}.$$

We fix the exponent $r > 1$ large enough so that the energy $\tilde{\mathcal{E}}$ becomes lower semicontinuous. In particular, we want to achieve that $\tilde{\mathcal{E}}$ is jointly lower semicontinuous in (y, Λ) with respect to the weak convergence in $\mathrm{W}_0^{1,q_Y}(\Omega, \mathbb{R}^3)$ for the y-part and with respect to $\mathcal{D}_r$ for the Λ-part.

2 Time-Incremental Minimization

Let (y_0, Λ_0) be an initial condition and $0 = t_0 < t_1 < \ldots < t_n = T$ a finite partition of the time interval $[0, T]$. Then the time-incremental minimization problem is to solve iteratively

$$(y_l, \Lambda_l) \in \mathrm{Argmin}\, [\tilde{\mathcal{E}}(t_l, y, \Lambda) + \mathcal{D}_{\mathrm{HK}}(\Lambda_{l-1}, \Lambda)], \quad l = 1, 2, \ldots, n. \tag{10}$$

Note that we can use the dissipation $\mathcal{D}_{\mathrm{HK}}$ introduced by Hackl and Kochmann [4] instead of $\mathcal{D}_r$. The proof of the existence of solutions remains unaltered, since $\mathcal{D}_{\mathrm{HK}}$ is lower semicontinuous with respect to $\mathcal{D}_r$.

The problem (10) cannot be solved in general. In fact, more complicated microstructure than simple laminates could occur (see, for example, Carstensen, Hackl and Mielke [2]). We circumvent this problem by regularization.

2.1 Regularization

We regularize the energy $\tilde{\mathcal{E}}$ and set $\tilde{\mathcal{E}}_{\mathrm{reg}} = \tilde{\mathcal{E}} + \mathcal{G}$ with the help of the following double integral

$$\mathcal{G}(\Lambda) = \int_\Omega \int_\Omega \frac{d_{\mathrm{W}}(\Lambda(x), \Lambda(y))^r}{|x - y|^{3+\theta r}} \mathrm{d}x \mathrm{d}y.$$

This choice of a regularization term has the following advantage: As long as we take $0 < \theta < 1/r$, the term $\mathcal{G}$ is weak enough to allow for certain discontinuities. More precisely, there is a subset $\mathcal{L}_*$ of discontinuous simple-laminate fields so that $\mathcal{G}(\Lambda)$ stays finite for every $\Lambda \in \mathcal{L}_*$. We will see that $\mathcal{G}$ is still strong enough to imply compactness.

2.2 Compactness Result

We can prove a compactness result, as the main argument for the existence of solutions to the time-incremental problem.

Lemma 2.1 *Let $(y_{l,1}, \Lambda_{l,1}), (y_{l,2}, \Lambda_{l,2}), \ldots$ be an infimizing sequence for the regularized energy $\tilde{\mathcal{E}}_{\mathrm{reg}}$ at time $t_l \in [0, T]$. Then there exists a subsequence (not relabeled), a deformation y_l and a simple-laminate field Λ_l so that all the following conditions are fulfilled:*

(i) $y_{l,k} \to y_l$ *weakly in* $\mathrm{W}_0^{1,q_Y}(\Omega, \mathbb{R}^3)$ *as* $k \to \infty$

(ii) $\Lambda_{l,k} \to \Lambda_l$ *with respect to* $\mathcal{D}_r$

(iii) *in particular,* $\Lambda_{l,k}(x) \to \Lambda_l(x)$ *pointwise with respect to* d_{W} *for almost every* $x \in \Omega$

(iv) $\mathcal{G}(\Lambda_l) \leq \liminf_{k\to\infty} \mathcal{G}(\Lambda_{l,k})$.

Proof. (Sketch) We use coercivity conditions that rely on an analysis of (4) and (7). In order to do so, we follow ideas from Mainik and Mielke [6]. The rest of the proof is achieved with the help of tools from measure theory. □

2.3 Existence of Time-incremental Solutions

Now we are in the position to state the theorem on the existence of solutions to the time-incremental minimization problem with regularized energy $\tilde{\mathcal{E}}_{\mathrm{reg}} = \tilde{\mathcal{E}} + \mathcal{G}$.

Theorem 2.2 *The regularized time-incremental minimization problem*

$$(y_l, \Lambda_l) \in \mathrm{Argmin}\, [\tilde{\mathcal{E}}_{\mathrm{reg}}(t_l, y, \Lambda) + \mathcal{D}_{\mathrm{HK}}(\Lambda_{l-1}, \Lambda)], \quad l = 1, 2, \ldots, n.$$

with initial condition (y_0, Λ_0) admits solutions.

Proof. (Sketch) We follow the strategy of the direct methods in the calculus of variations. Fix a positive integer $l \in \{1, 2, \ldots, n\}$. Assume that $(y_0, \Lambda_0), \ldots, (y_{l-1}, \Lambda_{l-1})$ is a solution for the times $t_0, \ldots, t_{l-1}$. Let $(y_{l,1}, \Lambda_{l,1}), (y_{l,2}, \Lambda_{l,2}), \ldots$ be an infimizing sequence for the regularized energy $\tilde{\mathcal{E}}_{\mathrm{reg}}$ at time t_l. Every subsequence is also an infimizing sequence. Based on a result by Eisen [3], we can prove that the energy $\tilde{\mathcal{E}}$ is jointly lower semicontinuous with respect to the weak convergence in $\mathrm{W}_0^{1,q_Y}(\Omega, \mathbb{R}^3)$ for the y-part and with respect to $\mathcal{D}_r$ for the Λ-part. Moreover, $\mathcal{D}_{\mathrm{HK}}$ is lower semicontinuous with respect to $\mathcal{D}_r$. In view of the compactness result Lemma 2.1, there exists a pair (y_l, Λ_l) so that

$$[\tilde{\mathcal{E}}_{\mathrm{reg}}(t_l, y_l, \Lambda_l) + \mathcal{D}_{\mathrm{HK}}(\Lambda_{l-1}, \Lambda_l)] \leq [\liminf_{k\to\infty} \tilde{\mathcal{E}}_{\mathrm{reg}}(t_l, y_{l,k}, \Lambda_{l,k}) + \mathcal{D}_{\mathrm{HK}}(\Lambda_{l-1}, \Lambda_{l,l})].$$

Hence, the pairs $(y_0, \Lambda_0), \ldots, (y_l, \Lambda_l)$ form a solution of the time-incremental minimization problem for the times $t_0, \ldots, t_l$. The rest of the proof is by induction. □

References

[1] J. M. Ball. Convexity conditions and existence theorems in nonlinear elasticity. *Arch. Rational Mech. Anal.*, **63**, 337–403 (1976/77).

[2] C. Carstensen, K. Hackl, and A. Mielke. Non-convex potentials and microstructures in finite-strain plasticity. *Proc. Royal Soc. London Ser. A*, **458**, 299–317 (2002).

[3] G. Eisen. A selection lemma for sequences of measurable sets, and lower semicontinuity of multiple integrals. *Manuscripta math.*, **27**, 73–79 (1979).

[4] K. Hackl, D. Kochmann, in preparation.

[5] E. Gürses, A. Mainik, C. Miehe, and A. Mielke. Analytical and numerical methods for finite-strain elastoplasticity. In R. Helmig, A. Mielke, and B. Wohlmuth (eds.), *Multifield Problems in Solid and Fluid Mechanics* (Springer Verlag, Berlin, 2006), 443–481.

[6] A. Mainik, A. Mielke. Global existence for rate-independent gradient plasticity at finite strain. *J. Nonlinear Sci. (in press). WIAS-preprint* 1299.