On rate independent models for crack propagation

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We model the evolution of a single crack as a rate-independent process based on the Griffith criterion. Three approaches are presented, namely a model based on global energy minimization, a model based on a local description involving the energy release rate and a refined local model which is the limit problem of regularized, viscous models. Finally we present an example which sheds light on the different predictions of the models.

1 Griffith criterion and crack evolution models

In this note we concentrate on quasistatic, rate independent models for crack propagation based on the Griffith fracture criterion. In order to make the mathematical and modeling issues visible we restrict our discussion to a very simplified situation, namely we consider a two dimensional model problem, where a single crack may evolve along a prescribed straight line. It is intrinsic to rate independent models that discontinuous evolutions might occur, i.e. the function $s : [0,T] \to \mathbb{R}$ describing the position of the crack tip might develop jumps. Such discontinuities are not accounted for in the usual crack propagation models. It is the purpose of this note to discuss different approaches to complete these models in such a way that "physically meaningful" discontinuities are predicted.

Let $\Omega \subset \mathbb{R}^2$ denote the body with boundary $\partial\Omega$, which is divided into a part Γ_D, where the displacements are prescribed, and into a part Γ_N, where the surface forces are imposed. Let $(0,0)^\top \in \partial\Omega$ and let $L>0$ such that for all $s\in[0,L]$ we have $C_s=\{\, x\in\mathbb{R}^2;\ x=(\sigma,0)^\top,\ \sigma\in(0,s]\,\}\subset\Omega$. The line C_s describes a crack of length s. Moreover, $\Omega_s = \Omega\backslash C_s$ is the domain with crack C_s, see Fig. 1(a). For given time t, crack tip position s and displacements $u : \Omega_s \to \mathbb{R}^2$ the deformation energy takes the form

$$\mathcal{E}(t,u,s) = \int_{\Omega_s} \tfrac{1}{2}(\mathbf{C}\varepsilon(u)) : \varepsilon(u)\,\mathrm{d}x - \int_{\Gamma_N} h(t)\cdot u\,\mathrm{d}s.$$

Here, $h(t){:}\Gamma_N\to\mathbb{R}^2$ describes the applied surface forces, $\varepsilon(u)$ is the linearized strain tensor and $\mathbf{C}$ denotes the elasticity tensor. For a given crack length s the set of admissible displacements satisfying non-penetration conditions on C_s is given by $V(\Omega_s) = \{\, u \in \mathrm{H}^1(\Omega_s,\mathbb{R}^2)\,;\ u|_{\Gamma_D} = 0,\ [u]\cdot\mathbf{n} \geq 0 \text{ on } C_s \,\}$. For fixed t and s the energy $\mathcal{E}(t,\cdot,s)$ has a unique minimizer with respect to $V(\Omega_s)$. We denote by $\mathcal{I} : [0,T]\times(0,L) \to \mathbb{R},\ \mathcal{I}(t,s) = \min_{v\in V(\Omega_s)} \mathcal{E}(t,v,s)$ the corresponding reduced energy functional.

The Griffith criterion states that a crack is stationary if the energy, which would be released at a small crack extension, is less than the energy which is needed to create the new surface. In this context the energy release rate $\mathcal{G}(t,s)$ is defined as

$$\mathcal{G}(t,s) = -\tfrac{\mathrm{d}}{\mathrm{d}s}\mathcal{I}(t,s).$$

For general strictly convex elastic energy densities it is proved in [1] that $\mathcal{G}$ belongs to $\mathrm{C}^0([0,T]\times(0,L))$. Formulas for $\mathcal{G}$ can be found e.g. in [2, 3] (quadratic case) or in [1, 4] (general convex energy density, curved crack path, polyconvex case). Assuming that the dissipated energy is related with the fracture toughness $\kappa \in \mathrm{C}^0([0,L])$, $\kappa > 0$, the Griffith criterion suggests the following version of a quasistatic, rate independent model for crack propagation in Karush–Kuhn–Tucker form:

The function $s : [0,T] \to \mathbb{R}$ is non-decreasing, $\dot{s}(t) = \frac{\mathrm{d}}{\mathrm{d}t}s(t)$ exists for all $t \in [0,T]$ and for all $t \in [0,T]$ we have
(a) $\kappa(s(t)) - \mathcal{G}(t,s(t)) \geq 0$ (local stability), (1)
(b) $\dot{s}(t)\big(\kappa(s(t)) - \mathcal{G}(t,s(t))\big) = 0$ (complementarity condition).

The problem is that the assumption that $\dot{s}$ exists everywhere is too strong. If t_*, s_* are reached with $\kappa(s_*){-}\mathcal{G}(t_*,s_*){=}0$ and $\kappa(s_*{+}\epsilon){-}\mathcal{G}(t_*{+}\delta,s_*{+}\epsilon){<}0$ for $\delta\in(0,\delta_0)$, $\epsilon\in(0,\epsilon_0)$, then every continuous non-decreasing function $s{:}[t_*,t_*{+}\delta_0]{\to}(0,L)$ with $s(t_*){=}s_*$ violates condition (a). Thus, this model is not satisfactory and has to be refined by allowing for discontinuous solutions and by adding further conditions for the discontinuities. Let $\mathrm{BV}([0,T],\mathbb{R})$ denote the space of functions with bounded variation. For $s \in \mathrm{BV}([0,T])$ the set $J(s) \subset [0,T]$ is the jump set and consists of the discontinuity points of s.

Definition 1.1 A function $s \in \mathrm{BV}([0,T])$ is a *local solution* (LS) to the crack problem if it is non-decreasing and satisfies
(a) $t \in [0,T]\backslash J(s) \ \Rightarrow\ \kappa(s(t)) - \mathcal{G}(t,s(t)) \geq 0$,
(b) Energy inequality: For all $0 \leq t_1 < t_2 \leq T$ we have $\mathcal{I}(t_2,s(t_2)) + \int_{s(t_1)}^{s(t_2)} \kappa(\sigma)\mathrm{d}\sigma \leq \mathcal{I}(t_1,s(t_1)) + \int_{t_1}^{t_2} \partial_t\mathcal{I}(t,s(t))\mathrm{d}t$.

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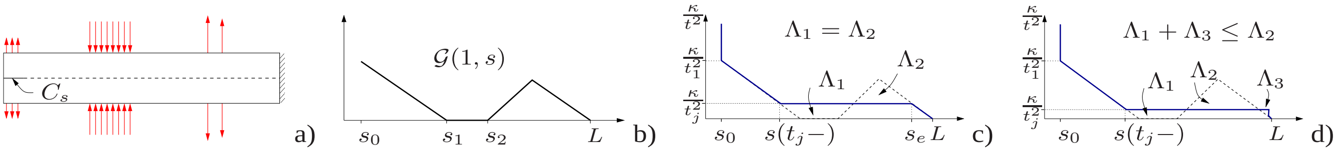

Fig. 1 (a) Domain with surface loading h_1; (b) Graph of $\mathcal{G}(1,\cdot)$; (c) Global energetic solution (thick line); (d) A local solution.

Note that (a) and (b) of Definition 1.1 imply that the complementarity condition (1)(b) is satisfied for almost every $t \in [0,T]$. The model described in Definition 1.1 allows for a great variety of solutions. By either a global minimization criterion or a criterion based on a vanishing viscosity approach, one may select particular local solutions.

Definition 1.2 $s \in \mathrm{BV}([0,T])$ is a *global energetic solution* (GES) if it is non-decreasing and satisfies for all $t \in [0,T]$
(a) Global stability: For all $\tilde{s} \in [s(t), L]$ we have $\mathcal{I}(t, s(t)) \leq \mathcal{I}(t, \tilde{s}) + \int_{s(t)}^{\tilde{s}} \kappa(\sigma)\mathrm{d}\sigma$,
(b) Energy equality: $\mathcal{I}(t, s(t)) + \int_{s(0)}^{s(t)} \kappa(\sigma)\mathrm{d}\sigma = \mathcal{I}(0, s(0)) + \int_0^t \partial_t \mathcal{I}(\tau, s(\tau))\mathrm{d}\tau$.

The global energetic formulation is a general concept for modeling rate independent problems, see e.g. [5] for a survey. The existence of GES follows from this general framework. Note that every GES is a special LS.

A further selection principle is based on the vanishing viscosity method. In this approach one introduces the artificial viscosity $\nu > 0$ (Theorem 1.3) and studies the behavior of solutions if the viscosity tends to zero (Theorem 1.4).

Theorem 1.3 *[1] For every $\nu > 0$ exists a non-decreasing function $s^\nu \in \mathrm{H}^1([0,T];\mathbb{R})$ satisfying for a.e. $t \in [0,T]$*
(a^ν) $\kappa(s^\nu(t)) + \nu\dot{s}^\nu(t) - \mathcal{G}(t, s^\nu(t)) \geq 0$,
(b^ν) $\big(\kappa(s^\nu(t)) + \nu\dot{s}^\nu(t) - \mathcal{G}(t, s^\nu(t))\big)\dot{s}^\nu(t) = 0$.

Theorem 1.4 *[1] There exist $s \in \mathrm{BV}([0,T])$ and a subsequence $\nu \searrow 0$ with $s^\nu \overset{*}{\rightharpoonup} s$ in $\mathrm{BV}([0,T])$ and $s^\nu(t) \to s(t)$ for every $t \in [0,T]$. Moreover, s is a local solution and satisfies in addition*
(a) $\kappa(s(t)) - \mathcal{G}(t, s(t)) \geq 0$ for every $t \in [0,T]\backslash J(s)$,
(b) if $\kappa(s(t)) - \mathcal{G}(t, s(t)) > 0$, then $\dot{s}(t) = 0$,
(c) $\forall\, t \in J(s)$, $\forall s_* \in [s(t_-), s(t_+)]$ we have $\kappa(s_*) - \mathcal{G}(t, s_*) \leq 0$.

Definition 1.5 $s \in \mathrm{BV}([0,T])$ is a *local energetic solution* (LES) if it is non-decreasing and satisfies (a)–(c) of Thm. 1.4.

2 Example

As an example we consider a rectangular domain as in Fig. 1(a), where the crack may follow the dashed line. We choose a symmetric loading $h(t,x) = th_1(x)$ and assume that κ is constant. With this loading it follows that $\mathcal{I}(t,s) = t^2\mathcal{I}(1,s)$ and $\mathcal{G}(t,s) = t^2\mathcal{G}(1,s)$. It is assumed that h_1 is chosen in such a way that the energy release rate $\mathcal{G}(1,s)$ has the shape indicated in Fig. 1(b), i.e. there exist $0 < s_0 < s_1 < s_2 < L$ such that $\mathcal{G}(1,\cdot)$ is strictly decreasing on (s_0, s_1), $\mathcal{G}(1,\cdot) = 0$ on (s_1, s_2) and $\mathcal{G}(1,s) > 0$ for $s > s_2$. Thanks to the non-penetration conditions, the region with $\mathcal{G} = 0$ can be generated by compressing the body. The stability criteria of Definitions 1.1–1.5 imply that for all $t \in [0,T]\backslash J(s)$ we have $\kappa/t^2 \geq \mathcal{G}(1, s(t))$.

In the case of a GES, for jump times t_j condition (b) of Definition 1.2 implies that $\int_{s(t_j-)}^{s(t_j+)} \big(\frac{\kappa}{t_j^2} - \mathcal{G}(1,\sigma)\big)\mathrm{d}\sigma = 0$. Thus, starting with $s(0) = s_0$ the crack does not grow until time t_1 with $\kappa/t_1^2 = \mathcal{G}(1, s_0)$. For $t > t_1$ it grows continuously until the first time t_j is reached, for which there exists $s_e > s(t_j)$ such that $\int_{s(t_j-)}^{s_e} \big(\frac{\kappa}{t_j^2} - \mathcal{G}(1,\sigma)\big)\mathrm{d}\sigma = 0$, i.e. the areas Λ_1 and Λ_2 in Fig. 1(c) are equal. After the jump from $s(t_j-)$ to $s(t_j+) = s_e$ the crack will proceed continuously.

The behavior of the local energetic model is completely different. Definition 1.5 implies that along jumps from $s(t_j-)$ to $s(t_j+)$ at jump time t_j, the horizontal line κ/t_j^2 has to lie below the graph of $\mathcal{G}(1,\cdot)$ in the region $(s(t_j-), s(t_j+))$. Due to the particular shape of $\mathcal{G}$ it is not possible to jump through the region (s_1, s_2). Thus, starting with $s(0) = s_0$ and using the local energetic model, the crack does not grow until time t_1 with $\kappa/t_1^2 = \mathcal{G}(1, s_0)$. For $t > t_1$ the crack grows continuously with $\kappa/t^2 = \mathcal{G}(1, s(t))$ and will never exceed the length s_1. We refer to Ref. [1] for an example with a discontinuous LES.

The GES and LES are special local solutions. A further local solution, which is neither a LES nor a GES, is plotted in Fig. 1(d). The condition $\Lambda_1 + \Lambda_3 \leq \Lambda_2$ is a consequence of the energy inequality from Definition 1.1.

This example shows that different choices of the jump criteria lead to completely different predictions of the crack models. It is a modeling issue to find those criteria which fit best to reality.

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