Some existence results in finite-strain plasticity

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Most theories of finite-strain elastoplasticity are based on Kröner and Lee's assumption of the multiplicative decomposition $\mathbf{F} = \mathbf{F}_{\mathrm{elast}}\mathbf{P}$, where $\mathbf{F} = \mathrm{D}\boldsymbol{\varphi}$ is the gradient of the deformation $\boldsymbol{\varphi} : \Omega \to \mathbb{R}^d$. The plastic tensor $\mathbf{P}$ and additional hardening variables $p \in \Pi$ are taken to be internal parameters. Moreover, the modeling is usually done in the rate-independent setting. This note concerns the implications of these two basic axioms.

Right from the beginning we emphasize that $\mathbf{F}$ and $\mathbf{P}$ should not be considered as elements of the linear space $\mathbb{R}^{d\times d}$ but rather as elements of Lie groups, namely

$$\mathbf{F} \in \mathrm{GL}^+(d) = \{\mathbf{A} \in \mathbb{R}^{d\times d} \mid \det \mathbf{A} > 0\} \text{ and } \mathbf{P} \in \mathrm{SL}(d) = \{\mathbf{A} \in \mathbb{R}^{d\times d} \mid \det \mathbf{A} = 1\}.$$

Thus, the geometric nonlinearities of finite-strain plasticity can be understood in the sense of Lie groups.

The constitutive laws of associate plasticity are given in terms of a stored energy density $W(x, \mathbf{F}, \mathbf{P}, p)$ and a dissipation potential $R(\mathbf{P}, p, \dot{\mathbf{P}}, \dot{p})$. The basic axioms of multiplicative plasticity (cf. [Mie02, Mie03, GA04]) lead to the following special form of the constitutive laws

$$\begin{aligned} W(x, \mathbf{F}, \mathbf{P}, p) &= \widetilde{W}(x, \mathbf{F}_{\mathrm{elast}}, p) \text{ with } \mathbf{F}_{\mathrm{elast}} = \mathbf{F}\mathbf{P}^{-1} \in \mathrm{GL}^+(d) \text{ and} \\ R(\mathbf{P}, p, \dot{\mathbf{P}}, \dot{p}) &= \widetilde{R}(x, p, \boldsymbol{\xi}, \dot{p}) \text{ with } \boldsymbol{\xi} = \dot{\mathbf{P}}\mathbf{P}^{-1} \in \mathrm{sl}(d) = \mathrm{T}_{\mathbf{1}}\mathrm{SL}(d). \end{aligned}$$

Rate-independence means 1-homogeneity of R in the rates, i.e., $\widetilde{R}(x, p, \gamma\boldsymbol{\xi}, \gamma\dot{p}) = \gamma\widetilde{R}(x, p, \boldsymbol{\xi}, \dot{p})$ for $\gamma \geq 0$. The dissipation potential R is in one-to-one correspondence with the elastic domains $\mathbb{E}(x, p)$ (whose boundary is the yields surface) via Legendre-Fenchel transform in $(\boldsymbol{\xi}, \dot{p})$, namely $\chi_{\mathbb{E}(x,p)} = \mathcal{L}\widetilde{R}(x, p, \cdot)$.

The classical plasticity equations consist of the elastic equilibrium problem and the flow rule $0 \in \partial_{\dot{\mathbf{P}},\dot{p}} R + \mathrm{D}_{\mathbf{P},p} W$. A weaker form of these differential form is the *energetic formulation* which is solely base on the *energy functional*

$$\mathcal{E}(t, \boldsymbol{\varphi}, \mathbf{P}, p) = \int_\Omega \widetilde{W}(x, \mathbf{F}_{\mathrm{elast}}, p)\,\mathrm{d}x - \langle \ell(t), \boldsymbol{\varphi}\rangle$$

and the *dissipation distance* $\mathcal{D}((\mathbf{P}_0, p_0), (\mathbf{P}_1, p_1)) = \int_\Omega D(x, (\mathbf{P}_0, p_0), (\mathbf{P}_1, p_1))\,\mathrm{d}x$, where $D : \Omega\times(\mathrm{SL}(d)\times\Pi)^2 \to [0, \infty]$ is defined via

$$\begin{aligned} D(x, (\mathbf{P}_0, p_0), (\mathbf{P}_1, p_1)) = \inf \Big\{ \int_{s=0}^{1} R(x, p, \dot{\mathbf{P}}\mathbf{P}^{-1}, \dot{p})\,\mathrm{d}s \,\Big|\, &(\mathbf{P}(0), p(0)) = (\mathbf{P}_0, p_0), \\ &(\mathbf{P}(1), p(1)) = (\mathbf{P}_1, p_1),\ (\mathbf{P}, p) \in \mathrm{C}^1([0,1]; \mathrm{SL}(d)\times\Pi), \Big\} \end{aligned}$$

The calculation of D is a difficult task as it involves the geodesic curves on $\mathrm{SL}(d)\times\Pi$ with respect to the Riemannian or Finslerian metric R. For some special cases, like von Mises plasticity this can be done, see [Mie02, HMM03].

By $\mathcal{F} \subset \mathrm{W}^{1,p}(\Omega; \mathbb{R}^d)$ we denote the set of kinematically admissible deformation and by $\mathcal{Z}$ the set of all internal states $(\mathbf{P}, p) : \Omega \to \mathrm{SL}(d)\times\Pi$. A function $(\boldsymbol{\varphi}, \mathbf{P}, p)$:

$[0,T] \to \mathcal{F}\times\mathcal{Z}$ is called *energetic solution* for the above problem, if for all $t \in [0,T]$ the global stability condition (S) and the energy balance (E) hold:

$$\text{(S)}\ \forall\,(\widetilde{\boldsymbol{\varphi}},\widetilde{\mathbf{P}},\widetilde{p}) \in \mathcal{F}\times\mathcal{Z}\colon \mathcal{E}(t,\boldsymbol{\varphi}(t),\mathbf{P}(t),p(t)) \le \mathcal{E}(t,\widetilde{\boldsymbol{\varphi}},\widetilde{\mathbf{P}},\widetilde{p})+\mathcal{D}(\mathbf{P}(t),p(t),\widetilde{\mathbf{P}},\widetilde{p}),$$

$$\begin{aligned}\text{(E)}\quad \mathcal{E}(t,\boldsymbol{\varphi}(t),\mathbf{P}(t),p(t)) &+ \mathrm{Diss}_{\mathcal{D}}((\mathbf{P},p),[0,T]) \\ &= \mathcal{E}(0,\boldsymbol{\varphi}(0),\mathbf{P}(0),p(0)) + \int_0^t \partial_s\mathcal{E}(s,\boldsymbol{\varphi}(s),\mathbf{P}(s),p(s))\,\mathrm{d}s\end{aligned}$$

The solvability of this weak formulation is still an open problem, except for a few special cases in space dimension 1, see [Mie04b].

However, it is quite natural to consider a fully implicit time incremental problem (IP) and for some simpler material models the convergence of solutions of (IP) for step size going to 0 to solutions of (S) & (E) is established, see [MTL02, Mie05b, FM05]. For finite-strain plasticity already the solvability for (IP) is a major problem under current investigation, since formation of microstructure is to be expected in many cases, see [OR99, ORS00, CHM02].

$$\text{(IP)}\qquad (\boldsymbol{\varphi}_k,\mathbf{P}_k,p_k) \in \operatorname*{Argmin}_{(\widetilde{\boldsymbol{\varphi}},\widetilde{\mathbf{P}},\widetilde{p})\in\mathcal{F}\times\mathcal{Z}} \ \mathcal{E}(t_k,\widetilde{\boldsymbol{\varphi}},\widetilde{\mathbf{P}},\widetilde{p}) + \mathcal{D}(\mathbf{P}_k,p_k,\widetilde{\mathbf{P}},\widetilde{p}).$$

The major observation is that the incremental problem consists of k successive minimization steps, which was first observed in [OR99].

In [Mie04b] an existence result is established under the assumption that the so-called *condensed potential*

$$W_p^{\mathrm{cond}}(\mathbf{F}) = \min_{\widetilde{\mathbf{P}},\widetilde{p}} \ W(\mathbf{F}\widetilde{\mathbf{P}}^{-1},\widetilde{p})+D(\mathbf{1},p,\widetilde{\mathbf{P}},\widetilde{p})$$

is polyconvex. This assumption is very hard to check but an example for dimension $d = 2$ was established with the help of [Mie05a]. Imposing suitable coercivity assumptions, which show that exponential hardening is needed, it is then shown that (IP) has solutions.

To avoid the difficult assumptions on W^{cond} it is possible to introduce regularizing terms into $\mathcal{E}$ via $\mathcal{E}^{\mathrm{reg}} = \mathcal{E} + \int_\Omega \kappa|(\mathrm{curl}\mathbf{P})\mathbf{P}^{\mathsf{T}}|^{q_C}\,\mathrm{d}x$. This case is analyzed in [MM05] via $\mathcal{A}$–quasiconvexity and a special identity for the minors of the product $\mathrm{D}\boldsymbol{\varphi}\mathbf{P}^{-1}$. Assuming that the semicondensed potential

$$(\mathbf{F}_{\mathrm{elast}},\mathbf{P}) \mapsto \min_{\widetilde{p}} W(\mathbf{F}_{\mathrm{elast}},\widetilde{p}) + D(\mathbf{P}_0,p_0,\mathbf{P},\widetilde{p})$$

is polyconvex and coercive with suitable exponents, it is possible to show that (IP) is solvable.

In many situations without regularization (IP) does not have solutions. In these situations one needs to relax the problem to find effective equations or one needs to find evolution equations for the associated microstructure, which often can be described by (sequential) laminates. In the mechanics literature this is described, e.g., in [AMO03, ML03, BCHH04]. For an attempt to provide a mathematical underpinning to these procedures we refer to [Mie04a, KMR05] and for the complete analysis in a very special case see [CT05].

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