# Exercise Sheet 13

**Exercise 52: Rank-one affinity.** Consider a rank-one affine function $f : \mathbb{R}^{m\times 2} \to \mathbb{R}$.

(a) Using the notation $A = (a_1, a_2) \in \mathbb{R}^{m\times 2}$ with $a_1, a_2 \in \mathbb{R}^m$ show that $f$ can be written in the form

$$f(a_1, a_2) = \beta + b_1 \cdot a_1 + b_2 \cdot a_2 + a_1 \cdot (Ba_2),$$

where $\beta \in \mathbb{R}$, $b_1, b_2 \in \mathbb{R}^m$ and $B \in \mathbb{R}^{m\times m}$.

(b) Conclude that $B$ is skew-symmetric.

(c) Show that there is $\boldsymbol{\beta} \in \mathbb{R}^{\tau(m,2)}$ such that $f(A) = \beta + \langle \boldsymbol{\beta}, T(A)\rangle$.

**Exercise 53: Quadratic densities.** Assume that $f : \mathbb{R}^{m\times d} \to \mathbb{R}$ is quadratic, i.e. $f(A) = (MA){:}A$ for some $M \in \mathrm{Lin}(\mathbb{R}^{m\times d}; \mathbb{R}^{m\times d})$. Show the following equivalences:

(a) $f$ is convex $\iff$ $f(A) \geq 0$ for all $A$.

(b) $f$ is polyconvex $\iff$ $\exists\, \boldsymbol{\beta} \in \mathbb{R}^{\tau_2(m,d)}\ \forall\, A \in \mathbb{R}^{m\times d} :\ f(A) \geq \langle \boldsymbol{\beta}, T_2(A)\rangle$.

(c) $f$ is rank-one convex $\iff$ $\forall\, \xi \in \mathbb{R}^m,\ \eta \in \mathbb{R}^d :\ f(\xi\otimes\eta) \geq 0$.

**Exercise 54: Lower semi-continuity implies quasiconvexity.**
Consider $f \in \mathrm{C}(\mathbb{R}^{m\times d}; \mathbb{R})$ with $|f(A)| \leq C(1+|A|)^p$. For a bounded domain $\Omega \subset \mathbb{R}^d$ consider the functional $I : \mathrm{W}^{1,p}(\Omega; \mathbb{R}^m) \to \mathbb{R}$ with $I(u) = \int_\Omega f(\nabla u(x))\,\mathrm{d}x$. The aim is to show "$\Longrightarrow$" in the equivalence:

$$I \text{sequentially weakly lower semi-continuous} \iff f \text{ quasiconvex.} \qquad ((*))$$

(a) Consider sequences of the form $v_n : x \to Ax + \frac{1}{n} w(nx)$ where $w \in \mathrm{W}^{1,p}_{\mathrm{loc}}(\mathbb{R}^d; \mathbb{R}^m)$ is periodic. Show $v_n \rightharpoonup v : m \mapsto Ax$.

(b) Establish "$\Longrightarrow$" in (*).

(c) For $f \in \mathrm{C}(\overline{\Omega} \times \mathbb{R}^m \times \mathbb{R}^{m\times d})$ define $I : \mathrm{W}^{1,p}(\Omega; \mathbb{R}^m) \to \mathbb{R};\ u \mapsto \int_\Omega f(x, u(x), \nabla u(x))\,\mathrm{d}x$. Derive the analog of "$\Longrightarrow$" in (*).

**Exercise 55: Cofactor and adjugate matrix.** (Deutsch: Kofaktor- und adjunkte Matrix) Consider a matrix $A \in \mathbb{R}^{d\times d}$. Then, $\mathrm{cof}\, A \in \mathbb{R}^{d\times d}$ is defined such that $(\mathrm{cof}\, A)_{ij}$ is $(-1)^{i+j} M_{ij}$, where $M_{ij}$ is the determinant of the $(d{-}1) \times (d{-}1)$ matrix obtained after deleting column $i$ and row $j$. Moreover, $\mathrm{adj}(A) = \mathrm{cof}(A)^\mathsf{T}$.

(a) For $f(A) = \det A$ show $\mathrm{D}f(A)[B] = \mathrm{cof}(A){:}B = \mathrm{tr}(\mathrm{adj}(A)B)$.

(b) Prove the formula $\mathrm{cof}(A)A^\mathsf{T} = \mathrm{adj}(A)A = \det(A)I$. Relate this to Cramer's rule and to Euler's formula $qf(A) = \langle \mathrm{D}f(A), A\rangle$ for $q$-homogeneous functions.

(c) For $d = 2$ and $d = 3$ show $\det(A{+}B) = \det A + \mathrm{cof}(A){:}B + \det B$ and $\det(A{+}B) = \det A + \mathrm{cof}(A){:}B + A : \mathrm{cof}(B) + \det B$, respectively.

**Please turn in written solutions on February 8, 2010.**