Multidimensional Calculus of Variations
Winter Term 2009/10
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Exercise Sheet 12

Exercise 48: Subdifferentials. Calculate the subdifferentials of the following functions:

(a) $I : X \to \mathbb{R};\ u \mapsto \|u\|_X^2$, where X is a Hilbert space.

(b) $I : \mathrm{L}^p(\Omega) \to \mathbb{R}; u \mapsto \|u\|_{\mathrm{L}^p}^\alpha$ for $p \in [1, \infty]$ and $\alpha > 1$.

(c) $I : \mathrm{L}^2(]0,1[) \to \mathbb{R}_\infty; \mapsto \frac{1}{2}\|u'\|_{\mathrm{L}^2}^2$ if $u \in \mathrm{H}^1(]0,1[)$ and ∞ else.

Exercise 49: Nonempty subdifferentials. Consider a lower semiconvex function $I : X \to \mathbb{R}_\infty$ on a Banach space X.

(a) Show that the domain $\mathrm{dom}I$ is convex.

(b) Assume that I is continuous at one point. Show that

$$A(I) := \mathrm{interior}(\mathrm{dom}I)$$

is nonempty and that I is continuous at all $u \in A$.

(c) Show that the set $B = \{\, (u, \alpha) \in X \times \mathbb{R} \,|\, u \in A,\ \alpha > I(u) \,\}$ is open and convex.

(d) For all $u \in A$ construct one $\eta \in \partial I(u)$.

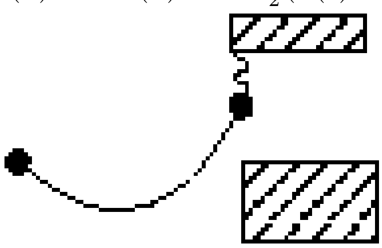
Exercise 50: Normal cones. Consider a closed convex set $C \in \mathbb{R}^m$. For $p \in]1, \infty[$ and a bounded domain $\Omega \in \mathbb{R}^d$ consider $X = \mathrm{L}^p(\Omega; \mathbb{R}^m)$ and $K = \{\, u \in X \,|\, u(x) \in C \text{ a.e. in } \Omega \,\}$. Show that the normal cones $\mathrm{N}_K(u)$ are given by

$$\mathrm{N}_K(u) = \{\, \eta \in \mathrm{L}^{p^*}(\Omega; \mathbb{R}^m) \mid \eta(x) \in \mathrm{N}_C(u(x)) \text{ a.e. in } \Omega \,\}.$$

Exercise 51: Variational inequality. We consider a string which is fixed rigidly on the left end and which is elastically supported on the right end under the constraint that the support is in an interval.

Characterize the minimizer of $I(u) = \int_0^\ell \frac{a}{2}u'(x)^2 + bu(x)\,\mathrm{d}x + \frac{k}{2}(u(\ell) - u_0)^2$ under the constraints $u(\ell) \in [u_{\mathrm{low}}, u_{\mathrm{upp}}]$.

Choose the mathematical setup and discuss under what conditions the constraint give rise to forces.



Please turn in written solutions on February 1, 2010.