Exercise Sheet 11

Exercise 44: Catenary The catenary is the curve which one can see as the form of a hanging chain. The chain has given length L and minimizes the gravitational potential I_{pot}. If the curve is described as graph of a function u over the horizontal interval $[0,\ell]$, then $I_{\text{pot}}(u) = \int_0^\ell \rho u(x)\sqrt{1+u'(x)^2}\,\mathrm{d}x$ has to be minimized subject to $J(u) = \int_0^1 \sqrt{1+u'(x)^2}\,\mathrm{d}x = L$. Derive the form of the catenary u.

Exercise 45: Eigenvalue problem. Consider the two bilinear forms on $\mathrm{H}^1_0(]0,1[)$:

$$\langle u,v\rangle_0 = \int_0^1 \rho(x)u(x)v(x)\,\mathrm{d}x, \quad \langle u,v\rangle_1 = \int_0^1 \alpha(x)u'(x)v'(x)+\beta(x)u(x)v(x)\,\mathrm{d}x,$$

where α, β, and ρ are sufficiently smooth positive functions.

(a) Write the associated eigenvalue problem $\langle u,v\rangle_1 = \lambda\langle u,v\rangle_0$ in terms of differential operators, i.e.

$$au'' + bu' + cu = \lambda u.$$

Discuss which triples (a,b,c) can occur.

(b) Solve the eigenvalue problem '$-u'' - 2u' = \lambda u$, $u(0) = u(1) = 0$ explicitly and check the orthogonality conditions for the eigenfunctions.

Exercise 46: Subdifferentials. Consider $X = \mathrm{L}^2(\Omega)$ and $I(u) = \int_\Omega |u(x)|^4\,\mathrm{d}x$.

(a) Show that I is strictly convex and lower semicontinuous, but not continuous.

(b) Calculate $\partial I(u) \subset X'$ explicitly.
Hint: Show that for $u,v \in \mathrm{L}^4(\Omega)$ the directional derivative $\mathrm{D}I(u)[v]$ is well defined. Try to extend $\mathrm{D}I(u)$ to a mapping in X'.

(c) For general lsc and convex functions J we know that for all $u \in X$ and $\varepsilon > 0$ there exists $\eta_\varepsilon \in X'$ such that $J(u+w) \geq J(u)+\langle\eta_\varepsilon, w\rangle$ for all $w \in X$. Construct such η_ε explicitly for I from above.

Exercise 47: Sum rule for subdifferentials. Consider general convex functions I and J on a reflexive Banach space X.

(a) Show that for all $u \in X$ we have $\partial I(u) + \partial J(u) \subset \partial(I+J)(u)$.

(b) Give a counterexample to equality in (a) with i and J lower semicontinuous.

(c) Show equality in (a) under the additional assumption that I is lower semicontinuous and J is continuous at u.
Hint: Nontrivial application of a separation theorem.

Please turn in written solutions on January 25, 2010.