Exercise Sheet 10

Exercise 40: Extending the domain of definition.
Consider BANACH spaces X and Y with continuous embedding $Y \subset X$. Moreover, assume that the functional $I : Y \to \mathbb{R}_\infty$ is weakly lower semicontinuous and coercive.

(a) Show that the extension $\widetilde{I} : X \to \mathbb{R}_\infty$ of I with $\widetilde{I}(x) = \infty$ for $x \in X \setminus Y$ is also weakly lower semicontinuous and coercive.

(b) Provide an example for X, Y, and I such that $I : X \to \mathbb{R}_\infty$ is weakly lower semicontinuous while $\widetilde{I} : Y \to \mathbb{R}_\infty$ is not.

Exercise 41: Weak closedness. Consider a closed subset C of $\mathbb{R}^m$ and a bounded domain $\Omega \subset \mathbb{R}^d$. Define $\mathcal{C}_p = \{ u \in \mathrm{L}^p(\Omega;\mathbb{R}^m) \mid u(x) \in C \text{ a.e. } \}$ for $p \in [1, \infty]$.

(a) Show that $\mathcal{C}_p$ is weakly closed if C is convex.

(b) Show that weak closedness of $\mathcal{C}_p$ implies convexity of C.

Exercise 42: Ground state in quantum mechanics. The spatial density distribution of the electron in a hydrogen atom is given in terms of the density $\rho : \Omega \to [0, \infty[$, where for simplicity we let $\Omega = B_R(0) \subset \mathbb{R}^3$. The wave function $\psi : \Omega \to \mathbb{C} \simeq \mathbb{R}^2$ defines the density ρ via $\rho(x) = |\psi(x)|^2$. The ground state is the minimizer of the total energy $I(\psi) = I_{\text{kin}}(\psi) + I_{\text{Coul}}(\psi)$ under the constraint $\int_\Omega |\psi(x)|^2 \,\mathrm{d}x = \int_\Omega \rho(x) \,\mathrm{d}x = 1$.

The kinetic energy $I_{\text{kin}}(\psi) = \int_\Omega \mu |\nabla\psi(x)|^2 \,\mathrm{d}x$ and the Coulomb-interaction energy $I_{\text{Coul}}(\psi) = \int_\Omega -\frac{\gamma}{|x|} |\psi(x)|^2 \,\mathrm{d}x$ are given via positive physical constants μ and γ.

(a) Show that a ground state exists for all $R \in]0, \infty[$.
Hint: Show and use $(x \mapsto 1/|x|) \in \mathrm{L}^2(\Omega)$ and $\|u\|_{\mathrm{L}^4} \leq C \|u\|_{\mathrm{L}^2}^{1/4} \|u\|_{\mathrm{H}^1}^{3/4}$.

(b) Show that there is a real and nonnegative ground state, i.e. $\psi(x) = \operatorname{Re}\psi(x) \geq 0$ a.e.

(c) For the physical case $R = \infty$ show that a solution of the form $\psi(x) = \alpha \mathrm{e}^{-\beta|x|}$ satisfies the EULER–LAGRANGE equations.

Exercise 43: Parametrized version of the isoperimetric problem.
Consider all parametrized curves $u : [0, 1] \to \mathbb{R}^2$ starting and ending at 0, i.e. $u \in X = \mathrm{C}_0^1([0, 1]; \mathbb{R}^2)$.

The length of the curve is given by $I(u)$ and the oriented enclosed area by $J(u)$:

$$I(u) = \int_0^1 \sqrt{\dot{u}_1(t)^2 + \dot{u}_2(t)^2} \,\mathrm{d}t, \qquad J(u) = \int_0^1 u_1(t)\dot{u}_2(t) \,\mathrm{d}t.$$

(a) Show that J really gives the enclosed area if u defines a simple curve.

(b) Determine the necessary conditions for a minimizer of I under the constraint $J(u) = A$.

(c) Show that minimizers $u \in \mathrm{C}_0^1([0, 1]; \mathbb{R}^2)$ with $|\dot{u}(t)| > 0$ are circles.

Please turn in written solutions on January 18, 2010.