Exercise Sheet 7

Exercise 28: Mazur's lemma and lower semicontinuity.
Let X be a reflexive Banach space.

(a) Prove Mazur's lemma: for each sequence $(u_n)_{n\in\mathbb{N}}$ with $u_n \rightharpoonup u$ in X, there exists a sequence $(v_m)_{m\in\mathbb{N}}$ such that $v_m \to u$ in X (strongly) and $v_m \in \operatorname{conv}\{\, u_n \mid n \in \mathbb{N}\,\}$.

(b) Assume that $I : X \to \mathbb{R}_\infty$ is convex and strongly lower semi-continuous. Use (a) to show weak lower semi-continuity.

Exercise 29: Sobolev embeddings. Let $\Omega = B_1(0) \subset \mathbb{R}^d$.

(a) Consider the function $u : \Omega \to \mathbb{R}$ with $u(x) = |x|^\alpha$ for $x \neq 0$ and $u(0) = 0$. For which p do we have $u \in \mathrm{L}^p(\Omega)$ and $u \in \mathrm{W}^{1,p}(\Omega)$?

(b) Consider the function $u(x) = \big(1 - \log|x|\big)^\beta$ with $\beta \in \mathbb{R}$. For which β and $p \in [1,\infty]$ do we have $u \in \mathrm{W}^{1,p}(\Omega)$?

(c) For the case $d = 2$ give a function $u \in H^1(\Omega) \setminus \mathrm{L}^\infty(\Omega)$.

Exercise 30: Embedding operators. Consider two reflexive Banach spaces X and Y. A bounded linear mapping $E : X \to Y$ is called embedding if it is injective.

(a) Define the adjoint mapping $E^* : Y^* \to X^*$ via $(E^*\eta)(x) = \eta(Ex)$ for all $\eta \in Y^*$ and all $x \in X$. Show that E^*Y^* is dense in X^* if E is an embedding.

(b) Assume that EX is dense in Y and conclude that E^* is an embedding.

(c) Provide an example of an embedding E such that E^* is not an embedding. Give a non-injective mapping $F : X \to Y$ such that F^* is an embedding.

Exercise 31: Poincaré's inequality. Consider the trapezoidal domain Ω_δ and the boundary part $\Gamma_\delta = \{\, x \in \overline{\Omega} \mid x_1 = 0 \,\}$ with $\Omega_\delta = \{\, x \in \mathbb{R}^2 \mid x_1 \in \,]\delta, 1[\,,\ |x_2| < x_1 \,\}$.

(a) For $\delta > 0$ show that $(\Omega_\delta, \Gamma_\delta)$ satisfies a Poincaré's inequality in $\mathrm{W}^{1,p}(\Omega_\delta)$ for all $p \in [1,\infty]$. Give an explicit estimate for the constant.

(b) Now consider the case $\delta = 0$, which implies $\Gamma_0 = \{0\}$. Show that Poincaré's inequality holds if and only if $p > d = 2$.

Submission of written solutions on 14th of December 2009.