

Exercise Sheet 6

MAZUR's separation theorems or

Geometric form of the HAHN-BANACH theorem:

Let A and B be nonempty convex subsets of a Banach space X with $A \cap B = \emptyset$.

- (1) If A is closed and B is open, then there exists $\ell \in X^* \setminus \{0\}$ and $\alpha \in \mathbb{R}$ such that $\langle \ell, a \rangle \leq \alpha \leq \langle \ell, b \rangle$ for all $a \in A$ and $b \in B$.
- (2) If A is compact and B is closed, then there exists $\ell \in X^*$ and $\alpha, \beta \in \mathbb{R}$ such that $\langle \ell, a \rangle \leq \alpha < \beta \leq \langle \ell, b \rangle$ for all $a \in A$ and $b \in B$.

Exercise 24: Estimates for convex functions. Consider a convex and lower semicontinuous functional $I : X \rightarrow \mathbb{R}_\infty$.

- (a) Show that for all u with $I(u) < \infty$ and all $\varepsilon > 0$ there exists $\xi \in X^*$ such that $I(u+v) \geq I(u) - \varepsilon + \langle \xi, v \rangle$ for all $v \in X$.
- (b) Show that for all u with $I(u) = \infty$ and all $M \in \mathbb{R}$ there exists $\xi \in X^*$ such that $I(u+v) \geq M + \langle \xi, v \rangle$ for all $v \in X$.
- (c) Conclude from (a) and (b) that I is weakly lower semicontinuous.

Hint for (a) and (b): Apply a suitable version of MAZUR's theorem to the epigraph of I : $\text{epi}I = \{ (u, s) \in X \times \mathbb{R} \mid s \geq I(u) \}$.

Exercise 25: Distance functional. For a reflexive Banach space X consider $M \subset X$ and $u_* \in X \setminus M$. Define

$$I : X \rightarrow \mathbb{R}_\infty; u \mapsto \begin{cases} \|u_* - u\| & \text{for } u \in M, \\ \infty & \text{otherwise.} \end{cases}$$

- (a) Show that I is lower semicontinuous if M is closed.
- (b) Deduce convexity of I from convexity of M .
- (c) Assume that M is convex and closed. Establish the existence of a minimizer of I . Construct an example where the minimizer is not unique and X is infinite dimensional.
- (d) Consider $X = \ell^2$ and $M = \{ u \in \ell^2 \mid \sum_{n=1}^{\infty} (1 - \frac{1}{n}) u_n^2 = 1 \}$. Show that M is closed and that $\|u\| > 1$ for all $u \in M$. Find $u_* \in X$ such that I has no minimizer.

Exercise 26: Uniform convexity I. We consider function $f : \mathbb{R} \rightarrow [0, \infty[$ and the associated functional $I : u \mapsto \int_{\Omega} f(u(x)) dx$.

- (a) Assume $f \in C^2(\mathbb{R})$ and $f''(u) \geq \gamma > 0$ for all u . Find $c > 0$ and $p, q \in [1, \infty]$ such that

$$I\left(\frac{1}{2}(u_0+u_1)\right) \leq \frac{1}{2}I(u_0)+I(u_1) - c\|u_0-u_1\|_{L^p(\Omega)}^q \text{ for all } u \in L^p(\Omega).$$

- (b) Consider the same problem as in (a) for $f(u) = u^4$.
- (c) Consider general $f(u) = |u|^p$ for general $p \geq 2$.

(please turn)

Exercise 27: Uniform convexity II. Consider $f(u) = (1+u^2)^{\alpha/2}$ with $\alpha \in]1, 2[$.

(a) Find $c > 0$ such that

$$f\left(\frac{1}{2}(u+w)\right) \leq \frac{1}{2}f(u) + \frac{1}{2}f(w) - c_1 \frac{(u-w)^2}{(1+u^2+w^2)^{(2-\alpha)/2}} \quad \text{for all } u, w \in \mathbb{R}.$$

Hint: Use $0 < f''(u) \leq C(1+u^2)^{(\alpha-2)/2}$.

(b) Derive from (c) the uniform convexity estimate

$$I\left(\frac{1}{2}(u+w)\right) \leq \frac{1}{2}I(u) + \frac{1}{2}I(w) - c_2 \frac{\|u-w\|_{L^p(\Omega)}^\beta}{(1+I(u)+I(w))^\gamma}$$

for suitable β , γ , and p .

Hint: Use a reverse HÖLDER estimate on the last term in (a).

(c) Show that the condition in (b) is enough to guarantee strong convergence in $L^p(\Omega)$ of infimizing sequences for functionals of the form $u \mapsto I(u) - \langle \ell, u \rangle$.

Submission of written solutions on 7th of December 2009.