Exercise Sheet 5

Exercise 20: Convexity of integrands and functionals. For functionals $I(u) = \int_\Omega f(u(x), \nabla u(x))\,\mathrm{d}x$ the convexity of $f : \mathbb{R}^m \times \mathbb{R}^{m\times d} \to \mathbb{R}$ is sufficient for convexity of I but not necessary. Consider the one-dimensional case $\Omega =]0, \ell[$ and $f(u, A) = \frac{1}{2}|A|^2 + (Bu)\cdot A + \frac{1}{2}|u|^2$, $u, A \in \mathbb{R}^m$.

(a) Show that convexity of f is equivalent to $\|B\| := \sup\{\, |Bu| \mid |u| \le 1 \,\} \le 1$.

(b) Show that convexity of $I : X = \mathrm{C}^1_0([0,\ell]; \mathbb{R}^m) \to \mathbb{R}$ holds, if $\|B - B^\mathsf{T}\| \le 2$.

(c) Improve the result in (b) in dependence of ℓ by using a Poincaré inequality.

Exercise 21: Null Lagrange functions. A function $N \in \mathrm{C}^2(\overline{\Omega} \times \mathbb{R}^m \times \mathbb{R}^{m\times d}; \mathbb{R})$ is called a *null* Lagrange *function* (NLF), if the associated Euler–Lagrange equation

$$-\mathrm{div}\big(\partial_A N\big) + \partial_u N \equiv 0$$

holds identically for all $u \in \mathrm{C}^2(\Omega; \mathbb{R}^m)$.

(a) Consider the case $d = m = 2$ and show that $N(x, u, A) = \det A$ is a NLF.

(b) For $d = 1$ show that for all $\phi \in \mathrm{C}^2(\mathbb{R} \times \mathbb{R}^m)$ the function $N(x, u, A) = \partial_x \phi(x, u) + \partial_u \phi(x, y) \cdot A$ is NLF.

(c) Show that in (b) there are no other NLF. *Hint: Test with polynomials* $u \in \mathrm{C}^2(\mathbb{R}; \mathbb{R}^m)$.

(d) Classify all NLF for the case $m = 1$ and $d \ge 2$.

Exercise 22: Convexity and lower semicontinuity. We consider functionals $I : \mathbb{R}^n \to \mathbb{R}_\infty$ with the property $I(u) = 0$ for $|u| < 1$ and $I(u) = \infty$ for $|u| > 1$, where $|\cdot|$ denotes the Euclidian norm.

(a) Consider $n = 1$. First, characterize all I of the above form that are lower semicontinuous. Second, characterize all I of the above form that are convex. Third, study the intersection of the two sets.

(b) Repeat the characterizations in (a) for $n \ge 2$.

(c) Repeat the characterizations in (a) for $\mathbb{R}^2$ with the norm $|(u_1, u_2)| = |u_1| + |u_2|$.

Exercise Supporting hyperplanes. In the Hilbert space $\ell^2 = \{\, (u_n)_{n\in\mathbb{N}} \mid \sum_{n\in\mathbb{N}} u_n^2 < \infty \,\}$ consider the functional $I : \ell^2 \times [0, \infty];\ u \mapsto \sum_{n\in\mathbb{N}} \frac{n^2}{2} u_n^2 - b_n u_n$, where $b \in \ell^2$ is fixed.

(a) Show the uniform convexity condition $I(\frac{1}{2}(u+v)) \le \frac{1}{2}I(u) + \frac{1}{2}I(v) - c\|u-v\|^2$.

(b) Conclude convergence of infinimizing sequences and existence of a minimizer u_*.

(c) Find $w \in \ell^2$ such that there exists no $\xi \in \ell^2$ satisfying $I(u) \ge I(w) + \langle \xi, u-w \rangle$ for all $u \in \ell^2$.

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