Exercise Sheet 4

Exercise 16: Convexity. Consider functions $f : \mathbb{R}^n \to \mathbb{R}_\infty := \mathbb{R} \cup \{\infty\}$. Convexity is defined via

$$\forall\,\theta \in [0,1]\ \forall\,u,v \in \mathbb{R}^n: \quad f(\theta u+(1-\theta)v) \leq (1-\theta)f(u) + \theta f(v).$$

(a) Show that this for continuous functions this definition is equivalent to convexity as defined in the course.

(b) Give a convex function $f : \mathbb{R} \to \mathbb{R}_\infty$ which has a point u such that $f(u) < \infty$ and there exists no a with $f(v) \geq f(u) + a \cdot (v-u)$ for all v.

(c) Assume $f : \mathbb{R}^n \to \mathbb{R}$ (no value ∞ allowed). Deduce the continuity of f.

Exercise 17: Jensen's inequality. Assume that $f : \mathbb{R}^m \to \mathbb{R}_\infty$ is convex.

(a) Consider $u_1, \ldots, u_K$, $\theta_k \geq 0$ with $\theta_1 + \cdots \theta_K = 1$. Show

$$f(\theta_1 u_1+\cdots+\theta_K u_K) \leq \sum_{k=1}^{K} \theta_k f(u_k).$$

(b) For a piecewise constant $u : \Omega \to \mathbb{R}^m$ show

$$f\Big(\frac{1}{\mathrm{vol}(\Omega)} \int_\Omega u(x)\,\mathrm{d}x\Big) \leq \frac{1}{\mathrm{vol}(\Omega)} \int_\Omega f(u(x))\,\mathrm{d}x.$$

(c) Show that (b) also holds for $u \in \mathrm{C}^0(\Omega)$ or $\mathrm{L}^1(\Omega)$.

Exercise 18: Determinants. Show that the function $f : \mathbb{R}^{d\times d} \to \mathbb{R}$ with $f(A) = \det A$ is rank-one affine, i.e. for all $A \in \mathbb{R}^{d\times d}$, $\xi, \eta \in \mathbb{R}^d$ the function $t \mapsto f(A+t\xi \otimes \eta)$ is affine in t. (Hint: Consider first $\eta = e_1$):

Exercise 19: Quasiconvexity. The definition of quasiconvexity involves the unit ball $B_1(0) \subset \mathbb{R}^d$. Show that in the defintion of quasiconvexity any open bounded domain $\Omega \subset \mathbb{R}^d$ can be used without changing the definition.

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