Calculus of Variations
Winter term 2009/10
Prof. Alexander Mielke
9th of November 2009

Exercise Sheet 3

Exercise 11: Noether's theorem for rotationally invariant systems. The density $f \in \mathrm{C}^2([\alpha,\beta]\times\mathbb{R}^m\times\mathbb{R}^m;\mathbb{R})$ with $m \geq 2$ defines the functional $I(u) = \int_\alpha^\beta f(t,u(t),\dot{u}(t))\,\mathrm{d}t$. For the rotation matrix $R_\varphi \in \mathbb{R}^{m\times m}$ with

$$R_\varphi(y_1,y_2,y_3,...,y_m)^\mathsf{T} = (\cos\varphi\, y_1 - \sin\varphi\, y_2, \sin\varphi\, y_1 + \cos\varphi\, y_2, y_3, ..., y_m)^\mathsf{T},\ \ \phi\in\mathbb{R},\ \ y\in\mathbb{R}^m,$$

the density f satisfies the rotational symmetry $f(t,R_\varphi u, R_\varphi A) = f(t,u,A)$ for all t,u,A,φ.
(a) Show that along solutions $u:[\alpha,\beta]\to\mathbb{R}^m$ of the Euler–Lagrange equation we have conservation of the moment of momentum (Drehimpulserhaltung):

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[u_1(t)\partial_{A_2}f(t,u(t),\dot{u}(t)) - u_2(t)\partial_{A_1}f(t,u(t),\dot{u}(t))\right] = 0.$$

(Hint: Calculate first $\frac{\mathrm{d}}{\mathrm{d}\varphi}f(t,R_\varphi u,R_\varphi A)\big|_{\varphi=0}$.)
(b) Now consider $R_\varphi = \mathrm{e}^{\varphi B} \in \mathbb{R}^{m\times m}$ for a general $B\in\mathbb{R}^{m\times m}B$ with $B = -B^\mathsf{T}$ and assume the symmetry $f(t,R_\varphi u,R_\varphi A) = f(t,u,A)$. Which quantity $J(u,\dot{u})$ is now conserved?

Exercise 12: Vector analysis and Eshelby's tensor. Consider a domain $\Omega\subset\mathbb{R}^d$.
(a) For tensor fields $A\in\mathrm{C}^1(\Omega;\mathbb{R}^{m\times d})$ and $B\in\mathrm{C}^1(\Omega;\mathbb{R}^{n\times m})$ derive the product rule

$$\operatorname{div}(B\,A) = C{:}A + B\operatorname{div}A \qquad \text{with } (C{:}A)_\nu = \textstyle\sum_{\mu=1}^m\sum_{j=1}^d \partial_{x_j}B_{\nu\mu}A_{\mu j}.$$

(b) For a volume density $f\in\mathrm{C}^2(\Omega\times\mathbb{R}^m\times\mathbb{R}^{m\times d})$ the Eshelby's tensor is defined via

$$\mathbb{E}(x,u,A) = A^\mathsf{T}\partial_A f(x,u,A) - f(x,u,A)I_d \in \mathbb{R}^{d\times d}$$

Show that a solution $u\in\mathrm{C}^2(\Omega)$ of the Euler–Lagrange equation associated to f satisfies the identity

$$\operatorname{div}\mathbb{E}(\cdot,u,\nabla u)(x) + \partial_x f(x,u(x),\nabla u(x)) = 0 \in \mathbb{R}^d.$$

Exercise 13: Weak and strong local minimizers. Consider $M = \mathrm{C}^1([a,b];\mathbb{R})$, functions $g,h\in\mathrm{C}^2(\mathbb{R};\mathbb{R})$, and the functional $I: M\to\mathbb{R}$ defined via

$$I(u) = \int_a^b g(u'(x)) + h(u(x))\,\mathrm{d}x.$$

(a) Derive the associated Euler–Lagrange equation. Which conditions guarantee that solutions of the form $u(x) = u^* = \text{const}$ exist?

(b) Assume that $u(x) = u^* = \text{const}$ is a stationary point of I. Show that the conditions $h''(u^*) > 0$ and $g''(0) > 0$ are sufficient to imply that I has a strict weak local minimizer in u^*.

(c) Assume now that $g(A) \geq 0 = g(0)$ for all $A \in \mathbb{R}^{1\times 1}$ and that u^0 is a local minimizer of h. Show that the constant function $\overline{u} : x \mapsto u^0$ is a strong local minimizer. What additional conditions imply that $\overline{u}$ is a global minimizer?

Exercise 14: Second variation Consider the functional $I : \mathrm{C}^1(\overline{\Omega};\mathbb{R}^m) \to \mathbb{R}$ with $I(u) = \int_\Omega f(x, u(x), \nabla u(x))\,\mathrm{d}x$, where $f \in \mathrm{C}^2(\overline{\Omega}\times\mathbb{R}^m\times\mathbb{R}^{m\times d})$. For $\gamma_1, \gamma_2 > 0$ assume the estimates

$$\int_\Omega \partial_A^2 f(x, u_0(x), \nabla u_0(x))[\nabla w, \nabla w]\,\mathrm{d}x \geq \gamma_1 \int_\Omega |\nabla w|^2\,\mathrm{d}x, \tag{1}$$

$$\mathrm{D}^2 I(u_0)[w,w] \geq \gamma_2 \int_\Omega |w|^2\,\mathrm{d}x. \tag{2}$$

(a) Use (1) and suitable estimates for $\partial_A\partial_u f$ and $\partial_u^2 f$ to find C^* such that

$$\mathrm{D}^2 I(u_0)[w,w] \geq \gamma_1 \int_\Omega |\nabla w|^2\,\mathrm{d}x - C^*|w|^2\,\mathrm{d}x \text{ for all } w.$$

(b) Combine (2) and (1) to find $\gamma_3 > 0$, such that

$$\mathrm{D}^2 I(u_0)[w,w] \geq \gamma_3 \int_\Omega |\nabla w|^2 + |w|^2\,\mathrm{d}x \quad \text{for all } w \in C^1(\overline{\Omega};\mathbb{R}^m).$$

Exercise 15: Anisotropic elasticity theory. The functional $I : \mathrm{C}^1(\overline{\Omega};\mathbb{R}^d) \to \mathbb{R};\ u \mapsto \int_\Omega f(\nabla u)\,\mathrm{d}x$ is defined via

$$f(A) = \frac{\lambda}{2}(\mathrm{spur}A)^2 + \frac{\mu}{4}\left|A + A^T\right|^2 + \frac{\delta}{2}A_{11}^2.$$

(a) Establish the formula $\partial_A^2 f(A)[B,B] = 2f(B)$ for all $A, B \in \mathbb{R}^{d\times d}$.

(b) For which $\lambda, \mu, \delta \in \mathbb{R}$ do we have $f(A) \geq 0$ for all $A \in \mathbb{R}^{d\times d}$ (which is equivalent to convexity)? Try first to solve the case $d = 2$.
(Hint: For testing the positivity it essentially suffices to consider diagonal matrices.)

(c) For which $\lambda, \mu, \delta \in \mathbb{R}$ does f satisfy the LEGENDRE–HADAMARD condition? Try first to solve the case $d = 2$.
(Hint: Write $\partial_A^2 f(x,u,A)[\xi\otimes\eta, \xi\otimes\eta] \geq 0$ in the form $\mathbb{B}(\eta)\xi\cdot\xi \geq 0$ with $\mathbb{B}(\eta) \in \mathbb{R}^{d\times d}$.)

Submission of written solutions on 16th of November 2009.