Calculus of Variations
Winter term 2009/10
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2nd of November 2009

Exercise Sheet 2

Exercise 6. Construct a function $I \in \mathrm{C}^1(\mathbb{R}^2;\mathbb{R})$ with the following properties:

(i) For all straight line $\gamma(t) = tv$, $v \in \mathbb{R}^2$ the restriction of I has a STRICT local minimum.

(ii) $x = 0$ is not a local minimizers of I, i.e. $\forall\, \varepsilon > 0\ \exists\, y \in B_\varepsilon(0):\ I(y) < I(0)$.

Hint: Look for I in the form $I(x) = |x|^2 - g(x)$ where $g(x) = 0$ outside the region $0 < x_2 < x_1^2$.

Exercise 7: Lemma of du Bois–Reymond.
Consider $T = \{\, v \in \mathrm{C}^1([\alpha,\beta];\mathbb{R}^m) \mid v(\alpha) = v(\beta) = 0 \,\}$ and $f,\ g \in \mathrm{C}^0([\alpha,\beta];\mathbb{R}^m)$.

(a) Show that $\int_\alpha^\beta g(x)\cdot v'(x)\,\mathrm{d}x = 0$ for all $v \in T$ implies that g is constant on $[\alpha,\beta]$. (Hint: Construct a $v \in T$ with $v'(x) = g(x) - \gamma$.)

(b) Now assume $\int_\alpha^\beta [f(x)\cdot v(x) + g(x)\cdot v'(x)]\,\mathrm{d}x = 0$ for all $v \in T$. Conclude $g \in \mathrm{C}^1([\alpha,\beta];\mathbb{R}^m)$ and $g'(x) = f(x)$ for all $x \in [\alpha,\beta]$.
(Note that we gain smoothness of g without imposing it.)

Exercise 8: Explicitly solvable problems.
(a) Let $M = \{\, u \in \mathrm{C}^1([0,\pi]) \mid u(0) = 0 \,\}$ and $I : M \to \mathbb{R}$ with

$$I(u) = \int_0^\pi [(u'(x))^2 + 2u(x)\cos(x) + x^{27}]\,\mathrm{d}x + 4u(\pi).$$

Derive the Euler–Lagrange equation and calculate all critical points.

(b) Determine all critical points of the functional $I(u) = \int_1^2 x^3(u'(x))^2\,\mathrm{d}x$ subject to the boundary conditions $u(1) = 0$ and $u(2) = 3$. What are the extremal properties of these critical points.

Exercise 9: Euler-Lagrange equation. Consider the domain $\Omega \in \mathbb{R}^2$, the set $M = \mathrm{C}^2(\overline{\Omega},\mathbb{R})$, and the functional $I(u) : M \to \mathbb{R}$ defined via

$$I(u) = \int_\Omega \left[\frac{1}{2}\left(\nabla u(x)\cdot\begin{pmatrix} 9 & 5 \\ 3 & 2 \end{pmatrix}\nabla u(x) + \frac{\pi^2}{17}u^2\right) - \frac{1}{3}u^3\right]\mathrm{d}x + \int_{\partial\Omega} 9\sin u\,\mathrm{d}a.$$

Derive the associated Euler–Lagrange equation including boundary condition.

please turn

Exercise 10. Minimal surface of revolution. Consider $I : M \to \mathbb{R}$ with

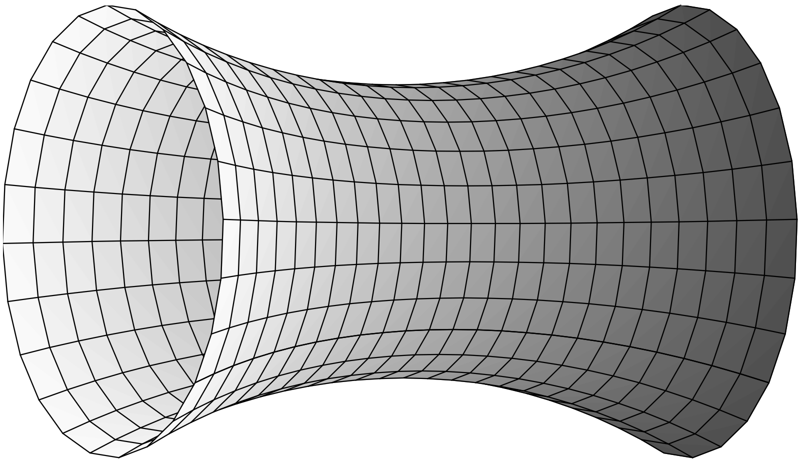
$$I(u) = \int_0^\ell 2\pi u(x)\sqrt{1+u'(x)^2}\,\mathrm{d}x,\ \ M = \{\, u \in \mathrm{C}^1([0,\ell]) \mid u(x) \geq 0,\ u(0) = r_0,\ u(\ell) = r_\ell \,\}.$$

Solutions of the EULER–LAGRANGE equation have the form $u(x) = U(c,d,x) = c\cosh\left(\frac{x-d}{c}\right)$.

(a) Consider the case $r_0 = r_\ell$ and show that we may choose $d = \ell/2$. Consider c as free parameter, which determines ℓ and the solution. Discuss the number of solutions for different values of ℓ. For these solutions plot (use a computer!) the value of $I(u)$ in dependence of ℓ (of the curve $c \mapsto (I(u_c), \ell_c)$).

(b) Consider arbitrary $r_0 > 0$ and $r_\ell > 0$. Derive the estimate $i(r_0, r_1, \ell) := \inf\{\, I(u) \mid u \in M \,\} \leq \pi(r_0^2 + r_\ell^2)$ via suitable sequences. Compare with (a).

(c) Provide a good lower bound for $i(r_0, r_0, \ell)$ by using $u_\mathrm{m} = \min\{\, u(x) \mid x \in [0,\ell] \,\}$ and the estimate $\sqrt{1+u'^2} \geq \max\{1, |u'|\}$. (Hint: Use $|u'|\,\mathrm{d}x = |\mathrm{d}u|$ and minimize w.r.t. u_m.)



Submission of written solutions on 9th of November 2009.